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Symplectic geometry of linear Hamiltonian systems

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1. Linear Hamiltonian systems

$$\Gamma \dot{x} = -Px, \qquad x \in \mathbb{R}^r,$$

$(\,\cdot\,,\cdot\,)$ is the scalar product, $\Gamma^* = -\Gamma$, $P^* = P$,

$\det \Gamma \neq 0 \quad \Longrightarrow \quad r$ is even $(r = 2n)$, $\det P \neq 0$,

$\Omega(\xi, \eta) = (\Gamma \xi, \eta)$ is the symplectic structure,

$H = \dfrac{1}{2}(Px, x)$ is a Hamiltonian function (first integral),

$v(x) = -\Gamma^{-1} Px$ is a Hamiltonian vector field

$$i_v \Omega = -dH,$$

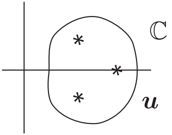
$$\Gamma = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix} \qquad \text{or} \qquad P = \mathrm{diag}(\pm 1, \ldots, \pm 1)$$

i^+, i^- are the indexes of inertia of the quadratic form H,

$\det(\Gamma^{-1}P + \lambda E) = 0$, u is the degree of unstability

$$u \leqslant \min(i^-, i^+),$$

$$u \equiv i^- \pmod 2$$



Example. $\ddot{x} + G\dot{x} + Wx = 0, \quad x \in \mathbb{R}^n$

$G^* = -G$ is the matrix of gyroscopic forces,

$W^* = W$, $\frac{1}{2}(Wx, x)$ the potential energy

$$\underbrace{\begin{pmatrix} G & E \\ -E & 0 \end{pmatrix}}_{\Gamma} \begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = -\underbrace{\begin{pmatrix} W & 0 \\ 0 & E \end{pmatrix}}_{P} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad y = \dot{x}$$

$H = \dfrac{1}{2}(y, y) + \dfrac{1}{2}(Wx, x)$ is the total energy.

Scaling. $G \mapsto NG, \ N \to \infty$

(a) $\tau = Nt, \qquad \dfrac{dy}{d\tau} + Gy = 0, \qquad y \in \mathbb{R}^n$

(b) $\tau = \dfrac{t}{N}, \qquad G\dfrac{dx}{d\tau} = -Wx, \qquad x \in \mathbb{R}^n$

2. Quadratic integrals

Theorem 1. *The Hamilton equations admit a family of quadratic integrals*

$$\Phi_m = \frac{1}{2}\big(P(P^{-1}\Gamma)^m x, (P^{-1}\Gamma)^m x\big), \qquad m \in \mathbb{Z},$$

moreover

1) *all the Φ_m are non-singular quadratic forms and their signatures coincide with the signature of $\Phi_0 = H$,*

2) *Φ_m are in pairwise involution (with respect to the symplectic structure Ω).*

Theorem 2. *If the spectrum of a linear Hamiltonian system does not contain multiple eigenvalues, then the first integrals* $\Phi_0, \Phi_{-1}, \ldots, \Phi_{-n+1}$ *are independent.*

Remark. If there exist q distinct squares of the eigenvalues, then

$$\operatorname{rank} \frac{\partial(\Phi_0, \Phi_{-1}, \ldots, \Phi_{-n+1})}{\partial(x_1, \ldots, x_{2n})} \geqslant q.$$

3. Integral cones and singular subspaces

$K = \{x \in \mathbb{R}^{2n} \colon \Phi_m(x) = 0,\ m \in \mathbb{Z}\}$ is the integral cone.

Subspace Π is the singular subspace of K

if $\Pi \subset K$.

$$\dim \Pi \leqslant \min(i^-, i^+).$$

Theorem 3. *For every $a \in K$ there is a singular subspace $\Pi(a)$, $a \in \Pi(a)$, such that*

1) $\Pi(a)$ *contains the sequence* $\Lambda^k a$, $k \in \mathbb{Z}$; $\Lambda = P^{-1}\Gamma$,
2) $\dim \Pi(a) = \operatorname{rank}[\dots, \Lambda^k a, \dots]$,
3) $\dim \Pi(a) \leqslant \min(i^-, i^+)$,
4) $\Pi(a)$ *is an invariant subspace,*
5) $\Pi(a)$ *is an isotropic subspace.*

$$\Lambda \mapsto M = \Lambda^{-1} = \Gamma^{-1} P$$

Isotropic subspace: $(\Gamma\xi, \eta) = 0$ for all $\xi, \eta \in \Pi(a)$

Theorem 4. *If for some $a \in K$ the dimension of $\Pi(a)$ is odd, then the equilibrium $x = 0$ is unstable.*

4. Estimate for the degree of unstability

Theorem 5. *If the spectrum does not contain multiple eigenvalues, then*

1) *the cone K is the sum of 2^k subspaces with identical dimension, where k is the total number of real pairs and complex tetrads of eigenvalues,*

2) *among these subspaces there is an u-dimensional unstable subspace.*

Corollary 1. $u = \max_{a \in K} \dim \Pi(a)$.

Corollary 2. *For all* $a \in K$

$$u \geqslant \dim \Pi(a).$$

5. Conditions for stability

Theorem 6. *If $K = \{0\}$ then the equilibrium $x = 0$ is stable. If the equilibrium $x = 0$ is stable and there are no coinciding eigenvalues, then $K = \{0\}$.*

6. Systems with gyroscopic forces

$$\ddot{x} + G\dot{x} + Wx = 0, \qquad x \in \mathbb{R}^n$$

$$\Phi_1 = \frac{1}{2}\big(Gx + \dot{x},\ W^{-1}(Gx + \dot{x})\big) + \frac{1}{2}(x, x)$$

(V. Lahadanov, 1975)

$$\Phi_{-1} = \frac{1}{2}\big(Wx + G\dot{x},\ Wx + G\dot{x}\big) + \frac{1}{2}(W\dot{x}, \dot{x})$$

(V. Kozlov, 1997)

$$\Phi_2 = \frac{1}{2}\big(W^{-1}((GW^{-1}G - E)x + GW^{-1}\dot{x}), \\ (GW^{-1}G - E)x + GW^{-1}\dot{x}\big) \\ + \frac{1}{2}(W^{-2}(Gx + \dot{x}), Gx + \dot{x}),$$

$$\Phi_{-2} = \frac{1}{2}(W(Wx + G\dot{x}), Wx + G\dot{x}) \\ + \frac{1}{2}\big(GWx + (-W + G^2)\dot{x},\ GWx + (-W + G^2)\dot{x}\big).$$

$$G = 0 \quad \Longrightarrow \quad \Phi_m = \frac{1}{2}[(W^m\dot{x}, \dot{x}) + (W^{m+1}x, x)], \qquad m \in \mathbb{Z}.$$

If all eigenvalues of the operator W are distinct, then the integrals $\Phi_0 = H$, $\Phi_1, \ldots, \Phi_{n-1}$ are independent.

7. Linear systems with a quadratic integral

$\dot{x} = Ax, \;\; x \in \mathbb{R}^n; \quad f = \dfrac{1}{2}(Px, x)$ is the first integral

$\det A \neq 0$ and $\det P \neq 0$

$\Omega(\xi, \eta) = (\Gamma\xi, \eta), \quad \Gamma = PA^{-1}$ is a symplectic structure

Theorem 7. $\Phi_m = \dfrac{1}{2}(PA^{-m}x, A^{-m}x), \;\; m \in \mathbb{Z},$ *are quadratic integrals, and*

1) Φ_m *are nondegenerate quadratic forms and their signatures coincide with the signature of* $f = \Phi_0$*,*

2) Φ_m *are in involution (with respect to the symplectic structure* Ω*).*

Theorem 8. *If $f(x)$ is the first integral of the linear system, then for all $m \in \mathbb{Z}$ the functions*

$$x \mapsto f(A^m x)$$

are also the first integrals of the system.

8. Finite-dimensional quantum systems

$i\hbar \dfrac{\partial \psi}{\partial t} = \widehat{H}\psi$ **is the Schrödinger equation**

$\hbar = 1,\ \ \psi \in \mathbb{C}^n,\ \ \langle \cdot\,,\cdot \rangle$ **is the Hermitian scalar product**

$\widehat{H}$ **is the Hermitian operator**

Quadratic integrals: $\langle \widehat{H}\psi, \psi \rangle$ **and** $\langle \psi, \psi \rangle (= 1)$

Let $\psi = x + iy\ \ (x, y \in \mathbb{R}^n)$ **and** $\widehat{H} = A + iB$

$$A^* = A \ \text{ and } \ B^* = -B$$

$*$ **denotes conjugation with respect to the 'standard' scalar product in** $\mathbb{R}^n$**.**

Schrödinger equation:

$$\dot{x} = Bx + Ay, \quad \dot{y} = -Ax + By \ \text{ or}$$

$$\underset{\substack{\parallel \\ \Gamma}}{\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix}^{\cdot} = -\underset{\substack{\parallel \\ P}}{\begin{pmatrix} A & -B \\ B & A \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{is the Hamiltonian system}$$

$$H(x,y) = \frac{1}{2}(Ax,x) + (Bx,y) + \frac{1}{2}(Ay,y)$$

$$\langle \psi, \psi \rangle = (x,x) + (y,y)$$

Let $\widehat{\Lambda}\colon \mathbb{C}^n \to \mathbb{C}^n$ be an Hermitian operator

and $[\widehat{H}, \widehat{\Lambda}] = \widehat{H}\widehat{\Lambda} - \widehat{\Lambda}\widehat{H} = 0.$

Then $\langle \widehat{\Lambda}\psi, \psi \rangle$ is the first integral of the Schrödinger equation.

Let $\widehat{\Lambda} = C + iD$. Then $C^* = C$, $D^* = -D$ and

$$2\langle \widehat{\Lambda}\psi, \psi \rangle = (Cx,x) + 2(Dx,y) + (Cy,y), \qquad \begin{pmatrix} C & -D \\ D & C \end{pmatrix}$$

Theorem 9. *The Schrödinger equation admits the family of integrals*

$$\Theta_m = \langle \widehat{H}^m \psi, \psi \rangle, \qquad m \in \mathbb{Z},$$

moreover the functions $\dots, \Theta_1, \Theta_3, \Theta_5, \dots$ *are in pairwise involution (with respect to the standard symplectic structure in* $\mathbb{R}^{2n}$*).*

Theorem 10. *If there are no equal (real) eigenvalues of Hamiltonian operator* $\widehat{H}$*, then the first integrals* $\Theta_1, \dots, \Theta_{2n-1}$ *are functionally independent.*

9. Мультигамильтоновость линейных систем с квадратичным инвариантом

$$\dot{x} = Ax, \quad f = \frac{1}{2}(Bx, x); \quad BA + A^*B = 0; \quad \det A \neq 0, \quad \det B \neq 0;$$

$$\Gamma = BA^{-1}, \quad \Gamma^* = -\Gamma; \qquad \Omega_m(\xi, \eta) = (\Gamma_m \xi, \eta), \quad \Gamma_m = {A^*}^m \Gamma A^m$$

$$f_m = \frac{1}{2}(B_m x, x), \quad B_m = {A^*}^m B A^m; \qquad m \in \mathbb{Z}$$

Теорема 11. *При каждом $m \in \mathbb{Z}$ линейная система гамильтонова относительно симплектической структуры Ω_m с функцией Гамильтона f_m.*

$$i_v \Omega_m = df_m \quad \Longleftrightarrow \quad \Gamma_m \dot{x} = B_m x.$$

Теорема 12. *Для любых целых k, s, m квадратичные формы f_k и f_s находятся в инволюции относительно симплектической структуры Ω_m.*

$$B_k \Gamma_m^{-1} B_s = B_s \Gamma^{-1} B_k \quad \text{для всех целых } k,\ s,\ m.$$

Замечание. Линейная система гамильтонова относительно непрерывного семейства симплектических структур

$$\sum \mu_m \Omega_m, \qquad \mu_m \in \mathbb{R},$$

с квадратичными гамильнонианами

$$\sum \mu_m f_m.$$

Единственное условие: $\det\Big(\sum \mu_m \Gamma_m\Big) \neq 0.$

Теорема 13. *Если у оператора A нет кратных собственных значений, то квадратичные формы $f_0 = f$, f_1, ..., f_{n-1} функционально независимы.*

10. Линейные отображения с квадратичным инвариантом

$x \mapsto Fx, \quad x \in \mathbb{R}^r; \quad (Bx, x)$ – инвариант
$F^*BF = B.$

Условия невырожденности:

1) оператор F не имеет собственных значений $\rho = \pm 1$,
2) $\det B \neq 0$.

Тогда отображение будет симплектическим: оно сохраняет симплектическую структуру

$$\Omega(\xi, \eta) = (\Gamma\xi, \eta), \qquad \Gamma = F^*B - BF.$$

$$(F^* - E)B(F + E) = \Gamma \quad \Longrightarrow \quad \det \Gamma \neq 0.$$

$\Longrightarrow$ r четно; $r = 2n$

$$A=(F+E)(F-E)^{-1}=(F-E)^{-1}(F+E),$$
$$\widetilde{\Omega}(\xi,\eta)=(\widetilde{\Gamma}\xi,\eta), \qquad \widetilde{\Gamma}=BA^{-1}.$$
$$\widetilde{\Omega}_m(\xi,\eta)=(\widetilde{\Gamma}_m\xi,\eta) \qquad \widetilde{\Gamma}_m=A^{*m}\widetilde{\Gamma}A^m$$
$$f_m=(B_m x,x), \qquad B_m=A^{*m}BA^m$$

Теорема 14. *Линейное отображение $x \mapsto Fx$ симплектическое относительно всех симплектических структур $\widetilde{\Omega}_m$, $m \in \mathbb{Z}$, и допускает семейство инвариантов f_p, $p \in \mathbb{Z}$, причем для любых целых k, s, m квадратичные формы f_k и f_s находятся в инволюции относительно $\widetilde{\Omega}_m$. Если, кроме того, оператор F не имеет кратных собственных значений, то квадратичные формы $f_0, f_1, \ldots, f_{n-1}$ функционально независимы.*

$F \mapsto A$ – преобразование Кэли.