A model with chaotic scattering and reduction of wave packets

Italo Guarneri

Dynamics in Siberia - in the 90th anniversary of B.V. Chirikov
Feb 27, 2018

Center for Nonlinear and Complex Systems - Universita' dell'Insubria a Como

When Boris talks...

...people listen !!

Dynamical Stability of Quantum "Chaotic" Motion in a Hydrogen Atom

G. Casati,[a] B. V. Chirikov, I. Guarneri,[b] and D. L. Shepelyansky
Institute of Nuclear Physics, 630090 Novosibirsk, Union of Soviet Socialist Republics
(Received 23 January 1986)

A simple numerical reversibility test which proves useful in exposing the chaotic nature of classical dynamical systems is applied to the quantum model of a hydrogen atom in a microwave field. The remarkable result is that, in spite of some apparent chaotic features, the quantum motion proves to be perfectly stable in contrast to the high instability of the classical chaotic motion.

VOLUME 56, NUMBER 23 PHYSICAL REVIEW LETTERS 9 JUNE 1986

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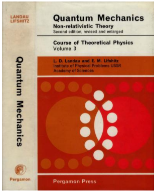
Yet, on account of the stable character of "quantum chaos," quantum dynamics must now be acknowledged a much more deterministic character than classical dynamics. Of course, the quantum measurement process remains irreversible and inevitably statistical, so that we see here an additional reason to distinguish the measurement process from the proper quantum dynamics. As a matter of fact, the latter is much more stable and less chaotic than classical

"...the infamous measurement process" (JS Bell)



"...the very nature of this process involves a far-reaching principle of irreversibility."

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Smilansky's "irreversible" model.

"...is a striking example of a problem which, in spite of its seeming simplicity, exhibits many unexpected effects". A particle in a line (coordinate x) coupled to a linear harmonic oscillator (coordinate q) by point interaction:

$$\mathcal{H} = -\frac{1}{2}\frac{\partial^2}{\partial x^2} - \frac{1}{2}\frac{\partial^2}{\partial q^2} + \frac{1}{2}\omega^2 q^2 + \alpha\, q\, \delta(x - x_0)\,. \tag{1}$$

$\alpha > 0$ a parameter, x_0 a fixed point.

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Theorem

(M Solomyak 04; SN Naboko, M Solomyak 06) If $\alpha < \omega$, the spectrum of $\mathcal{H}$ is: pure absolutely continuous in $[\frac{1}{2}\omega, +\infty)$; pure point, and finite, below $\frac{1}{2}\omega$.

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If $\alpha > \omega$, there is no point spectrum and the ac spectrum acquires an additional component with multiplicity 1 that coincides with $\mathbb{R}$.

Boxing the particle

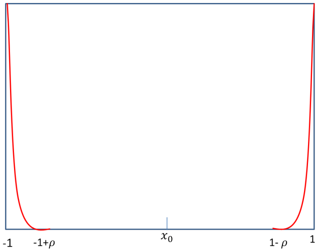
Let the particle be confined within a 1-dimensional box $I = (-1, 1)$, with hard-wall, smooth impenetrable wall, or periodic boundary conditions.

$$H^{(\rho)} = -\frac{1}{2}\frac{d^2}{dx^2} + W(x) \ ,$$

$$W(x) = V(x+1) + V(1-x)$$

$$V(x) = \begin{cases} \sigma(1/x^2 - 1/\eta^2), & \text{for} |x| < \eta; \\ 0, & \text{for} |x| > \eta. \end{cases} \tag{2}$$

$$-1 + \eta < x_0 < 1 - \eta \ , \quad \sigma > 3/8 \ . \tag{3}$$



Variants

Theorem

(IG 2011,2018) If $\alpha < \omega$, the spectrum of $\mathcal{H}$ is pure point. If $\alpha > \omega$, then for all choices of x_0 except countably many exceptions at most the spectrum of $\mathcal{H}$ is absolutely continuous and coincides with $\mathbb{R}$.

Solomyak&Naboko's proof still works over the threshold, with minor adaptations .

Other variants studied in several papers by Exner and coworkers.

Dynamics of Smilansky's model

Theorem

(IG 2011,2018). Let $\alpha > \omega$ *and* ψ *in the absolutely continuous subspace of* $\mathcal{H}$*. Then, as* $t \to \infty$*:*

- *the probability distribution of the position* x *in the state* $e^{-i\mathcal{H}t}\psi$ *weakly converges to* $\delta(x - x_0)$*,*
- *the expectation value of the energy of the oscillator exponentially diverges with the rate* $2\sqrt{\alpha^2 - \omega^2}$ *.*

Band formalism.

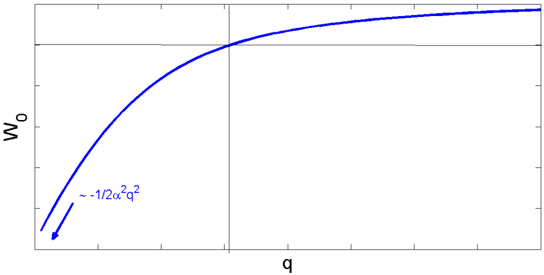
The particle is moving in a box I parametrized by $x \in [-L, +L]$.

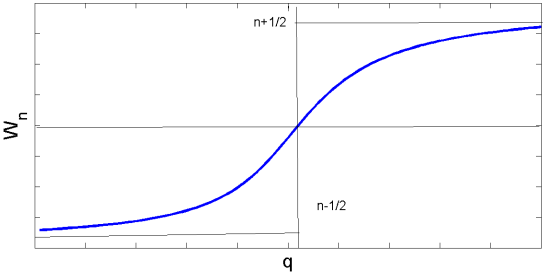
$$\mathcal{H} \;=\; \int_{\mathbb{R}}^{\oplus} dq \; H_\alpha(q) \;+\; \mathbb{I} \otimes H^{\mathrm{osc}} \;,$$

$$H_\alpha(q) \;=\; -\frac{1}{2}\frac{d^2}{dx^2} \;+\; \alpha q \; \delta(x) \;.$$

$$\psi(x,q) \;=\; \sum_{n=0}^{+\infty} Q_n(q) \; \phi_{q,n}(x) \;,$$

$$H_\alpha(q)\phi_{q,n} \;=\; W_n(q)\phi_{q,n} \;, \quad \phi_{q,n} \in L^2(I) \;. \tag{4}$$


~ -1/2α2q2
W0
q


n+1/2
n-1/2
Wn
q

Band formalism, II

$$
\begin{aligned}
(\psi,\ \mathcal{H}\psi) \ &= \ (\psi,\ \mathbb{I} \otimes H_\omega^{(\mathrm{osc})}\ \psi) + \\
&+ \sum_{n=0}^{+\infty} \int_{\mathbb{R}} dq\ W_n(q) |Q_n(q)|^2 \\
&> \ (\psi,\ \mathbb{I} \otimes \tilde{H}_{\alpha,\omega}\ \psi) + \\
&+ \sum_{n=1}^{+\infty} \int_{\mathbb{R}} dq\ (n - 1/2) |Q_n(q)|^2 \ .
\end{aligned}
$$

$$
\tilde{H}_{\alpha,\omega} \ = \ -\frac{1}{2}\frac{d^2}{dq^2} \ + \ \frac{1}{2}\omega^2 q^2 \ + \ W_0^-(q)
$$

$$
W_0^-(q) = \tfrac{1}{2}(1 - \mathrm{sign}(q)) W_0(q)
$$

Spectral Expansion

Theorem

For $\Psi(E)$ a function on $\mathbb{R}$ compactly supported away from exceptional points, and for any $n \in \mathbb{N}_0$, define $\psi_n(x) = \int dE \, \Psi(E) u_n(x, E)$. Then:

1) $\{\psi_n(x)\}_{n \in \mathbb{N}_0} \in \mathfrak{H}$,

2) the map $\imath : \Psi \mapsto \{\psi_n(x)\}_{n \in \mathbb{N}_0}$ extends to a unitary isomorphism of $L^2(\mathbb{R})$ onto an absolutely continuous subspace of the Hamiltonian $\mathcal{H}$,

3)$\forall t \in \mathbb{R}$, $\imath\left(e^{-iEt}\Psi\right) = e^{-i\mathcal{H}t}\imath(\Psi)$.

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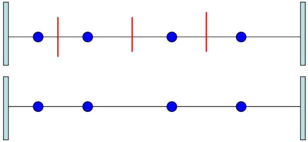
Multi-Oscillator case: Wave operators



Above: $\mathcal{H}^{(b)}$. Below: $\mathcal{H}$.
Wave operators:

$$\Omega^{\pm} = \lim_{t \to \pm\infty} e^{i\mathcal{H}t} e^{-i\mathcal{H}^{(b)}t}$$

(Evans, Solomyak, 2005) When $I = (-\infty, +\infty)$ the wave operators exist and are complete.

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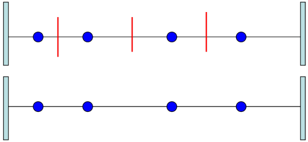

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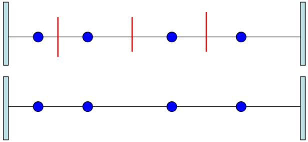

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$\mathcal{H} - \mathcal{H}^{(b)} = \mathcal{W}$ is the confining potential. Cook's theorem applies because $\int_0^\infty dt\ \|\mathcal{W} e^{-i\mathcal{H}t}\psi\| < +\infty$ due to exponentially fast sinking of $e^{-i\mathcal{H}t}\psi$ into the interaction points.

Reduction

As $t \to +\infty$ the state $\psi(t) = e^{-i\mathcal{H}t}\,\psi$ comes closer and closer to the state

$$e^{-i\mathcal{H}^{(b)}t}\,\Omega^{+\dagger}\,\psi$$

a superposition of states $\psi_j(t) \in L^2(B_j) \otimes L^2(\mathbb{R}^N)$, $j = 1, \dots, N$, each with the particle in a different box B_j, and the corresponding j-th oscillator exponentially excited.

To detect interferences, an observable must conserve non-negligible matrix elements between oscillator states that separate exponentially fast.

Collapse

If $\alpha > \omega$ then, for any ψ in the absolutely continuous subspace of $\mathcal{H}$, the probability distribution of the particle converges weakly as $t \to \pm\infty$ to a superposition of δ functions supported in the interaction points:

$$\int \cdots \int_{\mathbb{R}^N} dq_1 \ldots dq_N \, |\psi(x, q_1, \ldots, q_N, t)|^2 \underset{t \to \pm\infty}{\longrightarrow} \sum_{j=1}^{N} \gamma_j^{\pm} \, \delta(x - x_j) \; ,$$

where:

$$\gamma_j^{\pm} \; = \; \| P_j \, \Omega_{\pm}(\mathcal{H}^{(b)}, \mathcal{H}) \psi \|^2 \; , \tag{5}$$

and P_j denotes projection onto $\mathfrak{H}_j = L^2(B_j) \otimes L^2(\mathbb{R}^N)$, ($B_j$: j-th box).

Band formalism

Born-Oppenheimer-like description: oscillator dynamics described by the "ground-band Hamiltonian"

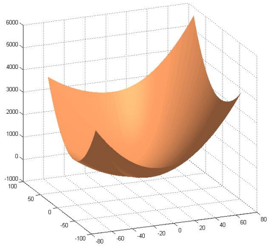
$$-\frac{1}{2}\frac{\partial^2}{\partial q_1^2} + \ldots - \frac{1}{2}\frac{\partial^2}{\partial q_N^2} + \frac{1}{2}\omega^2(q_1^2 + \ldots + q_N^2) + W(q_1, \ldots, q_N)$$

where $W(q_1, \ldots)$ is the ground state energy of the particle Hamiltonian

$$-\frac{1}{2}\frac{d^2}{dx^2} + \alpha \sum_1^N q_j \delta(x - x_j)$$

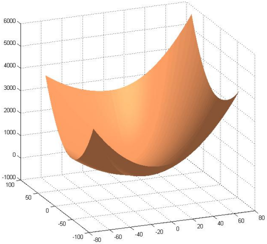
parametrically dependent on $q_1, \ldots, q_N$.

"Phase transition"

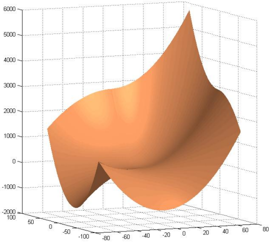


$$\alpha \; < \; \omega$$

"Phase transition"



$\alpha < \omega$



$\alpha > \omega$

A Regular Variant

$$H(x, q_1, \ldots, q_N, p, p_1, \ldots p_N) = \tfrac{1}{2} p + \tfrac{1}{2} \sum_{j=1}^{N} (p_j^2 + \omega^2 q_j^2) +$$

$$-\alpha a \pi^{-1} \sum_{j=1}^{N} q_j^2 e^{-a^2 q_j^2 (x - x_j)^2} \qquad (6)$$

In the case when $N = 1$ a Smilansky-like quantum spectral transition across a critical value of α (D.Barseghyan, P.Exner 2016) .

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This variant has a classical version !

(IG 2018).

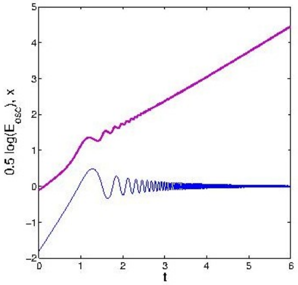
Let $\alpha_{\rm cr} = \omega^2\pi/2$. Then for any N: if $\alpha < \alpha_{\rm cr}$, $H \geq 0$ with compact energy hypersurfaces $H = E > 0$.

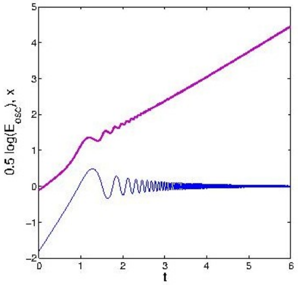
if $\alpha > \alpha_{\rm cr}$, H is unbounded, with non-compact energy hypersurfaces.

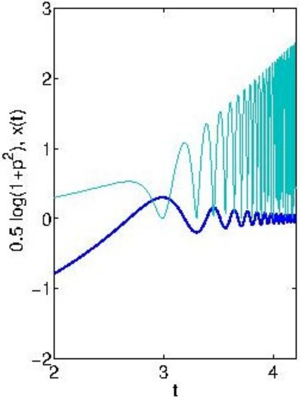
Special trajectories

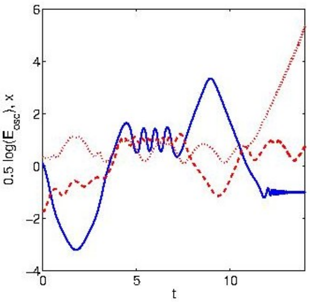
The particle is at rest at $x = x_n$. All oscillators except the n-th one are also at rest. The n-th oscillator moves according to the Hamiltonian $H_n = \frac{1}{2}(p_n^2 + (\omega^2 - 2\alpha\pi^{-1})q_n^2)$.
When $\alpha > \alpha_{\rm cr}$ this oscillator is inverted .
Linearized dynamics around such an orbit when $\alpha > \alpha_{\rm cr}$: as $t \to \infty$,

$$\delta x(t) \sim ce^{-kt}\cos(a_0 e^{2kt} + 2a_1 kt + \phi)\ , \quad k = \omega(\alpha/\alpha_{\rm crit} - 1)^{1/2}$$

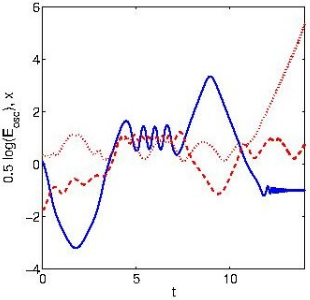

0.5 log(E_{osc}), x
t
5
4
3
2
1
0
−1
−2
0
1
2
3
4
5
6


0.5 log(E_{osc}), x
t


0.5 log(1+p^2), x(t)
t



6
4
2
0
−2
−4
0
5
10
t
0.5 log(E_osc), x


0.5 log(E_osc), x
t
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4
2
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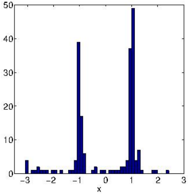
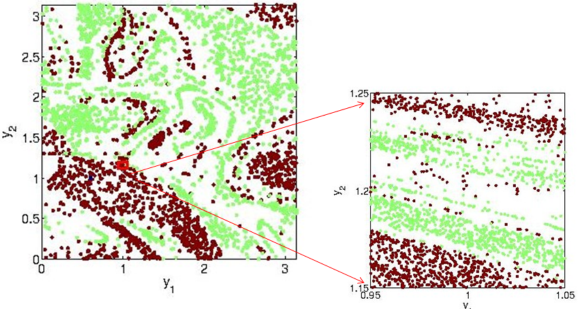
−2
−4
0
5
10


50
40
30
20
10
0
−3
−2
−1
0
1
2
3
x

Irregular Scattering

2 oscillators at $x = \pm 1$, box $(-2, 2)$. Each point represents an initial condition picked at random from a bounded 2-dim submanifold of the $H = 5$ energy surface. Brown (Green) markers if at time $t = 5$ the particle was found within 0.15 of $x = 1$ $(x = -1)$.



Conclusion

- Irreversibility in Smilansky's model is not associated with any kind of relaxation to equilibrium, but rather with an extreme form of irregular (chaotic) scattering.
- On the quantum side, such dynamics mimic a measurement process.
- To embed this idealization into realistic problems requires - at least - a generalization to higher dimension.