Conformal geometry and hydroelastic waves

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Introduction

We consider the motion of ideal incompressible liquid under elastic sheet.

Our goal is to investigate the question on existence and properties of nontrivial periodic stationary solutions to this problem which, in particular, describe the propagation wave of finite magnitude on the surface of the ice ocean.

The governing equations have the Hamiltonian structure and the problem can be reduced to the problem on existence of critical points of the *total energy* functional involved the elastic energy of the sheet, kinetic energy of fluid, and gravitational energy of the fluid.

This problem is close to the variational problems of conformal geometry and includes the joint minimisation of a harmonic mapping, the Willmore functional and the Dirichlet integral.

Problem formulation

We shall assume that the flow occupies a domain G in the space $\mathbb{R}^3$ of points $x = (x^1, x^2, x^3)$ bounded by the surface ∂G satisfying the following conditions.

H.1 The surface $S = \partial G$ is bounded from below and above,

$$\sup_{x \in S} |x^3| < \infty.$$

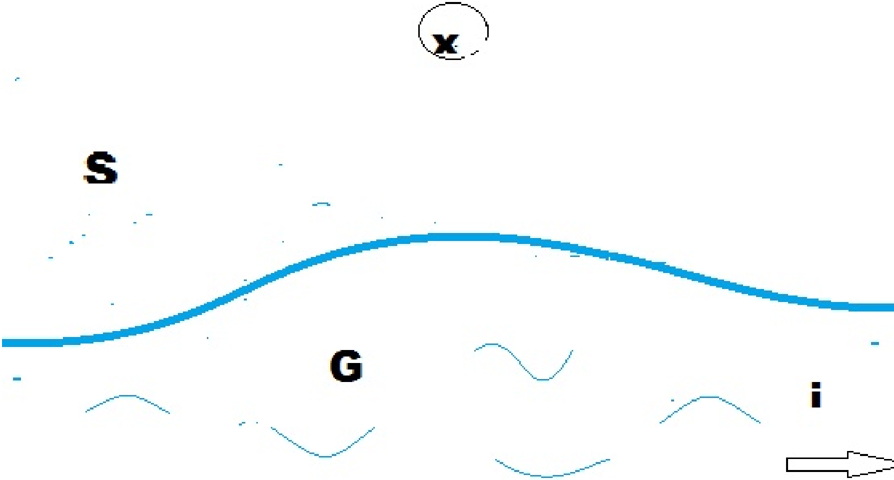
H.2 The domain G contains a half space,

H.3 The flow domain G and the elastic sheet S satisfy the periodicity conditions

$$G + n\mathbf{i} + m\mathbf{j} = G, \quad S + n\mathbf{i} + m\mathbf{j} = S \quad \text{for all} \quad (m, n) \in \mathbb{Z}^2,$$

The vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ along with $\mathbf{k} = (0, 0, 1)$ form the canonical basis in $\mathbb{R}^3$.

Problem formulation



Problem formulation

The flow is irrotational, periodic, satisfies the kinematic condition at the free surface, and tends to the uniform flow at infinity. This means that the velocity field $\mathbf{v}$ satisfies the conditions

$$\begin{gathered}
\operatorname{curl} \mathbf{v}(x) = 0, \quad \operatorname{div} \mathbf{v}(x) = 0 \text{ in } G, \\
\mathbf{v}(x + n\,\mathbf{i} + m\,\mathbf{j}) = \mathbf{v}(x) \text{ in } G, \\
\mathbf{v}(x) \cdot \mathbf{n}(x) = 0 \text{ on } S, \\
\mathbf{v}(x) \to \mathbf{i} \text{ as } x^3 \to -\infty.
\end{gathered}$$

Here $\mathbf{n} = (n^1, n^2, n^3)$ is the unit outward normal to S.

Problem formulation

Introduce the flow potential $\varphi(x) = x^1 + \Phi(x)$. The function Φ satisfies the equations and boundary conditions

$$\begin{aligned} \Delta\Phi(x) = 0 \quad &\text{in} \quad G, \\ \Phi(x + n\,\mathbf{i} + m\,\mathbf{j}) = \Phi(x) \quad &\text{in} \quad G, \\ \nabla\Phi(x) \cdot \mathbf{n}(x) + n^1(x) = 0 \quad &\text{on} \quad S, \\ \nabla\Phi(x) \to 0 \quad &\text{as} \quad x_3 \to -\infty. \end{aligned} \tag{1}$$

Recall that the fluid pressure p for irrotational flow is given by

$$p = -\frac{1}{2}|\nabla\varphi|^2 - gx^3 + \text{const.} \quad \text{in} \quad G,$$

where unessential constant can be taken 0

Renormalized flow energy

Set

$$\Omega = G \cap \{|x^i| \leq 1\}, \quad \Sigma = S \cap \{|x^i| \leq 1\}, \quad i = 1, 2.$$

. The reduced kinetic energy $\mathcal{K}_f(\Sigma, \Phi)$ of the flow per period is defined by the equality

$$\mathcal{K}_f = \frac{1}{2} \int\limits_{\Omega} |\nabla \Phi|^2 \, dx + \int\limits_{\Sigma} \Phi n^1 \, d\Sigma + \frac{1}{2} \int\limits_{\Sigma} x^3 n^3 \, d\Sigma. \qquad (2)$$

Renormalized flow energy

The potential (gravity) energy per period is defined by

$$\mathcal{G}_f(\Sigma) = \lambda \int_{\Sigma} (x^3)^2 n^3 \, d\Sigma. \tag{3}$$

Elastic energy

There are numerous approaches toward elastic shell theory proposed in the literature, too numerous to list there, we refer to S. Antman *Nonlinear problem of elasticity. Second edition*, Springer, New York,(2005).
G. Ciarlet, *An introduction to differential geometry with applicationa to elasticity*, J. Elasticity, vol. 78/79,no.1-3, (2005).
for introduction to the problem and references.

Elastic energy

We mentioned only the two ways for mathematical modeling of dynamics of elastic shells. The first is obtaining a mathematical model of a shell by reducing a three-dimensional model via physically reasonable assumptions on the kinematics to two-dimensional model: We refer to
G. Friesecke, R. James, S. Muller, *A Theorem on Geometric ridigity and the derivation of nonlinear plate theory from three-dimensional elasticity*(2002)
H. Le Dret, A. Raolt, *The nonlinear membrain model as variational limit of nonlinear three-dimensional elasticity* (1995,1996) for rigorous results and complete list of references.

Elastic energy

The second is *intrinsic approach* which takes the shell to be the two-dimensional medium with extrinsic directors, so-called Cosserat theory of shells, see Antman (2005), P. Naghdi (1972), and P.Neff(2004) for the foundations of the Cosserat theory.

Bending energy

Following Friesecke, James, and Muller we can take the bending energy of the elastic shell per period in the form

$$E_b(\Sigma) = \frac{1}{24} \int_{\Sigma} \left(2\mu |\mathbf{A}|^2 + \frac{\lambda\mu}{\mu + \lambda/2} |\mathrm{Tr}\ \mathbf{A}|^2 \right) d\Sigma.$$

Here $\mathbf{A}$ is the second fundamental for of S, and the constants λ, μ are defined by the stored energy of the 3D elastic material. Recall that

$$|\mathbf{A}|^2 = k_1^2 + k_2^2 \quad \text{and} \quad \mathrm{Tr}\ \ \mathbf{A} = (k_1 + k_2),$$

where k_i are the principal curvatures of S.

Bending energy. Willmore functional

It follows from the Gauss-Bonnet theorem that

$$\int_{\Sigma} k_1 k_2 \, d\Sigma = \int_{S/\mathbb{Z}^2} k_1 k_2 \, dS = 0.$$

Recall that the mean curvature $H = (k_1 + k_2)/2$.

$$E_b = c_b \int_{S} |\mathbf{A}|^2 \, d\Sigma \equiv 4c_b \int_{\Sigma} |H|^2 \, d\Sigma, \; c_b = \frac{1}{24}(2\mu + \frac{\lambda\mu}{\mu + \lambda/2}). \tag{4}$$

Hence the bending energy with the accuracy up to a constant coincides with the Willmore functional

$$W(\Sigma) = \int_{\Sigma} H^2 \, d\Sigma.$$

Stretching energy

The formulae for the membrane energy was derived by H. Le Dret, A. Raolt (1995,1996). It was shown that, in contrast to the case of the bending energy, the membrane energy is very sensitive to the structure of the three-dimensional energy density. The results are essentially simplified for the case of linear elasticity.

Stretching energy

Assume that the membrane is defined in the reference frame by the equality

$$x = \mathbf{r}(X), \quad X \in D.$$

. Then

$$E_m = \int_D C^{\alpha\beta} g_{\alpha\beta}\, dX^1 dX^2, \tag{5}$$

where $g_{\alpha\beta} = \partial_{X^\alpha}\mathbf{r}\partial_{X^\beta}\mathbf{r}$ are the coefficient of the first fundamental form, $(C^{\alpha\beta})_{2\times 2}$ is a constant positive matrix.

Stretching energy

Periods in the reference frame

$$\mathbf{l}_1 = (T, 0), \quad \mathbf{l}_2 = (-\alpha, T^{-1}), \quad T > 0. \tag{6}$$

Vectors $n\mathbf{l}_1 + m\mathbf{l}_3$ define the lattice Γ. The dual lattice is defined by the vectors

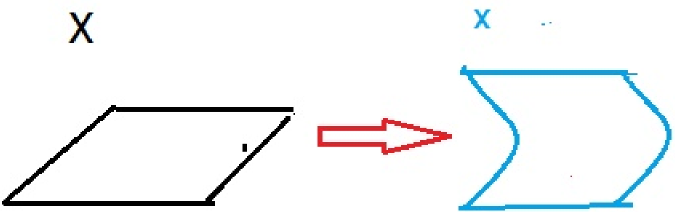
$$\mathbf{l}_1^* = (T^{-1}, \alpha), \quad \mathbf{l}_2^* = (0, T). \tag{7}$$

$$S: \ x = (\mathbf{l}_1^* \cdot X)\mathbf{i} + (\mathbf{l}_2^* \cdot X)\mathbf{j} + \mathbf{U}(X^1, X^2),$$

where $\mathbf{U}$ is Γ periodic. We take the stretching energy in the form

$$E_m = c_m \int\limits_{\mathbb{R}^2/\Gamma} \left((\partial_1 \mathbf{r})^2 + (\partial_2 \mathbf{r})^2 \right) dX^1 dX^2, \quad \partial_i = \partial_{X_i}. \tag{8}$$

Problem formulation



Problem formulation

The total energy $\mathcal{E}$ is defined as follows

$$
\begin{aligned}
\mathcal{E}(\mathbf{r}, \Phi, \alpha, T) &= E_m + E_b + \mathcal{K}_f + \mathcal{G}_f\,, \\
E_m &= c_m \int\limits_{\mathbb{R}^2/\Gamma} \left(|\partial_1 \mathbf{r}|^2 + |\partial_2 \mathbf{r}|^2\right) dX, \\
E_b &= c_b \int\limits_{\Sigma} |\mathbf{A}|^2 \, d\Sigma = 4 c_b \int\limits_{\Sigma} |H|^2 \, d\Sigma, \\
\mathcal{K}_f &= \frac{1}{2} \int\limits_{\Omega} |\nabla \Phi|^2 + \int\limits_{\Sigma} \Phi n^1 \, d\Sigma + \frac{1}{2} \int\limits_{\Sigma} x^3 n^3 \, d\Sigma, \\
\mathcal{G}_f &= \frac{\lambda}{2} \int\limits_{\Sigma} (x^3)^2 n^3 \, d\Sigma.
\end{aligned}
\tag{9}
$$

The problem is to find surface S, Φ, Γ, and $\mathbf{r}(X)$, which minimize $\mathcal{E}$ in the class of periodic surfaces and potentials

Variations. Equations and boundary conditions

Let S^t, $t \in (-1,1)$ be a family of oriented surfaces in $\mathbb{R}^3$ by the equations $x = \mathbf{r}(X) + t\phi(X)\mathbf{n}(X)$with a smooth functions $\mathbf{u}, \phi \in C^\infty(\mathbb{R}^2/\Gamma)$. Then

$$\frac{d}{dt}E_b(S^t)\Big|_{t=0} = 4c_b \int_\Sigma (\Delta H + 2H(H^2 - K))\phi \, d\Sigma. \qquad (10)$$

$$\frac{d}{dt}E_m(S^t)\Big|_{t=0} = -4c_m \int_\Sigma H\phi \, d\Sigma,$$

$$\frac{d}{dt}(\mathcal{K}_f + \mathcal{G}_f)(S^t)\Big|_{t=0} = \int_\Sigma (\frac{1}{2}|\nabla\Phi + +\mathbf{i}|^2 + \lambda x^3)\phi \, d\Sigma.$$

Remark. Identity (10) was proved by Blaschke and Willmore.

Variations. Equations and boundary conditions

Thus we get the following conditions on the free boundary

$$c_b(\Delta H + 2H(H^2 - K)) - c_m H + \frac{1}{2}|\nabla\Phi + \mathbf{i}|^2 + \lambda x^3 = 0 \quad \text{on} \quad S.$$

$$(\nabla\Phi + \mathbf{i}) \cdot \mathbf{n} = 0 \quad \text{on} \quad S.$$

Variations. Equations and boundary conditions

The tangential variations of E_b, $\mathcal{K}_f$, $\mathcal{G}_f$ equal zero. The tangential variation of E_m gives

$$\left(\frac{\partial}{\partial X^1} + i\frac{\partial}{\partial X^2}\right)\left(g_{11} - g_{22} + 2ig_{12}\right) = 0.$$

Hence it is Γ-periodic holomorphic function and hence

$$g_{11} - g_{22} = \text{const.}, \quad g_{12} = \text{const.}$$

The variation with respect to Γ gives

$$g_{11} = g_{22} := e^{2f}, \quad g_{12} = 0.$$

The optimal reference frame defines the conformal parametrization of S.

Geometry.First fundamental form

Let a mapping $x = \mathbf{r}(X)$, $X = (X^1, X^2) \in \mathbb{R}^2$ define the parametrization of the surface $S \subset \mathbb{R}^n$. For any function $u(X)$ set

$$\partial_{X^i} u := \partial_i u.$$

The first fundamental form of S is the quadratic form

$$ds^2 = \mathbf{g}(dX^1, dX^2) = g_{\alpha\beta} dX^\alpha dX^\beta,$$

where the quantities

$$g_{\alpha\beta} = \partial_\alpha \mathbf{r} \cdot \partial_\beta \mathbf{r} \tag{11}$$

Geometry.Conformal coordinates

The parametrization $x = \mathbf{r}(X)$, $X \in \mathbb{R}^2$, is conformal if and only if

$$g_{11} = g_{22}, \quad g_{12} = 0.$$

In this case we will write

$$g_{11} = g_{22} = e^{2f},$$

$$\mathbf{e}_i = e^{-f}\partial_1 \mathbf{r}, \quad \mathbf{e}_2 = e^{-f}\partial_2 \mathbf{r}, \quad \mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 \equiv \mathbf{n}.$$

The triplet $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ can be regarded as moving orthogonal frame.

Geometry. Second fundamental form

The second fundamental form $\mathbf{A}$ of the surface S is defined as follows

$$\mathbf{A}(dX) = b_{11}(dX^1)^2 + 2b_{12}dX^1dX^2 + b_{22}(dX^2)^2$$

$$b_{ij} = \partial_i\partial_j\mathbf{r}(X)\cdot\mathbf{n}(X)$$

The principal curvatures k_1, k_2 of S are the eigenvalues of the matrix $\{b_{ij}\}$

$$H = \frac{1}{2}(k_1 + k_2), \quad |\mathbf{A}|^2 = k_1^2 + k_2^2.$$

Geometry. Willmore functional

If $M \subset \mathbb{R}^3$ is 2D surface, then the Willmore functional is defined by the equality

$$W(M) = \int_M H^2 \, dM$$

The Willmore functional is invariant with respect to any conformal transform of M. In particular, for every spherical surface $\partial B(R, a) = \{|x - a| = R\}$ we have

$$W(\partial B(a, R)) = 4\pi.$$

Geometry. Willmore functional

If 2D closed surface $M \subset \mathbb{R}^n$ and

$$W(M) < 8\pi,$$

then M contains no self intersections. (P.Li, S. Yau 1982).

Among all surfaces M of genus 1 (tori), there is an analytical surface which minimize the Willmore functional (L. Simon 1993). See also E.Kuwert and R. Schätzle (2013).

Geometry. Bi-Lipschitz parametrization

The following is a modification of general results by Helein (2002), which, in its turn, is a generalization of the Toro theorem (1995)and the results by Muller & Sverak (1995). Let $x = \mathbf{r}(X)$ be a conformal, Γ-periodic parametrization of S. Let

$$g_{11}(X) = g_{22}(X) = e^{2f(X)}, \quad \mathbf{e}_i = e^{-f}\partial_i \mathbf{r}, \tag{12}$$

$$\int_\Sigma |\mathbf{A}|^2 \, d\Sigma = \int_{\mathbb{R}^2/\Gamma} |d\mathbf{e}_3|^2 < 16\pi/3 - \delta,$$

Then

$$\|df\|_{L^2(\mathbb{R}^2/\Gamma)} \leq \left(\frac{8\pi}{3}\right)^{1/2} - \sqrt{\delta/2},$$

$$\|d\mathbf{e}_1\|^2_{L^2(\mathbb{R}^2/\Gamma_k)} + \|d\mathbf{e}_2\|^2_{L^2(\mathbb{R}^2/\Gamma_k)} \leq 16\pi,$$

$$\|f\|_{C(\mathbb{R}^2)} \leq C(M, \delta).$$

Geometry. Bi-Lipschitz parametrization

For any points $x' = \mathbf{r}(X')$ and $x'' = \mathbf{r}(X'')$, $Y', Y'' \in \mathbb{R}^2$ we define the distance as

$$\rho(x', x'') = \inf_{\mathcal{C}} \int_C e^f \, ds, \tag{13}$$

where the infinum is taken over the set of all rectifiable curves in the plane of points X connected points X' and X''. Since the function f is bounded we conclude from this that

$$C^{-1}\rho(x', x'') \leq |X' - X''| \leq C\rho(x', x'') \tag{14}$$

Geometry. Periodic surfaces. Preventing self- intersections

Let S be a periodic surface with the Γ- periodic conformal parametrization

$$x = \mathbf{r}(X) \equiv (\mathbf{l}_1^* \cdot X)\mathbf{i} + (\mathbf{l}_2 \cdot X)\mathbf{j} + \mathbf{U}(X),$$

Introduce 2D surface $M \in \mathbb{R}^5$ defined in the parametric form by the relations

$$z = \mathbf{u}(X), \quad z \in R^5, \quad \mathbf{u} = (u^1, u^2, u^3, u^4, u^5),$$

$$\begin{aligned} u^1(X) &= \frac{1}{2\pi}\cos 2\pi r^1(X), \quad u^2(X) = \frac{1}{2\pi}\sin 2\pi r^1(X), \\ u^3(X) &= \frac{1}{2\pi}\cos 2\pi r^2(X), \quad u^4(X) = \frac{1}{2\pi}\sin 2\pi r^2(X), \\ & \qquad\qquad\qquad\qquad u^5 = r^3(X), \end{aligned} \tag{15}$$

The mapping $\mathbf{u} : \mathbb{R}^2/\Gamma \to \mathbb{R}^5$ defines the parametrization of a compact torus $M \in \mathbb{R}^5$.

Geometry. Periodic surfaces. Preventing self- intersections

The surface S does not contain intersections if and only if M does not contain self-intersections. Calculations give.

$$\int_M |\mathbf{H}_M|^2 \, dM = 2\pi^2 + \int_\Sigma H^2 \, d\Sigma.$$

Hence S has no self-intersections if

$$\int_\Sigma H^2 \, d\Sigma < 8\pi - 2\pi^2 \text{ or equivalently } \int_\Sigma |\mathbf{A}|^2 \, d\Sigma < 32\pi - 8\pi^2.$$

Bounds for the bending energy

Non-self-intersection:

$$\int_{\Sigma} |\mathbf{A}|^2 \, d\Sigma < 32\pi - 8\pi^2,$$

Existence of the conformal bi-Lipschitz parametrization:

$$\int_{\Sigma} |\mathbf{A}|^2 \, d\Sigma < 16\pi/3$$

The total bound for the bending energy is

$$\int_{\Sigma} |\mathbf{A}|^2 \, d\Sigma < 16\pi/3.$$

Compactness

Corollary. If $\Sigma : x = \mathbf{r}(X)$, $X \in \mathbb{T}^2$, satisfies

$$\int_\Sigma |\mathbf{A}|^2 \, d\Sigma \equiv \int_{\mathbb{T}^2} |d\mathbf{n}|^2 \, dx < \infty,$$

then Σ admits Bi-Lipschjtz conformal parametrization. Let

$$\Sigma_k : x = \mathbf{r}_k(X), X \in \mathbb{T}^2, \quad \int_{\mathbb{T}^2} |d\mathbf{n}_k|^2 \, dx \le M. \tag{1}$$

$\Sigma_k \to \Sigma$ weakly if $\mathbf{r}_k \to \mathbf{r}$ uniformly in $\mathbb{T}$.
If Σ_k satisfy (1) and there is $r > 0$, $\delta > 0$ such that

$$\int_{B(r,X_0)} |d\mathbf{n}_k|^2 \, dx \le 8\pi/3 - \delta \quad \text{for all} \quad X_0 \in \mathbb{T},$$

then the sequence Σ_k is weakly compact.

Compactness

Let $\{\Sigma\}$ has a common bound

$$\Sigma : x = \mathbf{r}(X), X \in \mathbb{T}^2, \quad \int_{\mathbb{T}^2} |d\mathbf{n}|^2\, dx \leq M. \qquad (1)/$$

If in addition there are $r > 0$, $\delta > 0$ such that

$$\int_{B(r,X_0)} |d\mathbf{n}|^2\, dx \leq 8\pi - \delta \quad \text{for all} \quad X_0 \in \mathbb{T},$$

then for some $\nu \in (0,1)$ and $\varsigma > 2$

$$\|\mathbf{r}\|_{C^\nu(\mathbb{T})} \leq c, \quad \|e^f\|_{L^\varsigma(\mathbb{T})} \leq c \qquad (2).$$

Bounds for the bending energy

The optimal energy bound

$$\int_{\Sigma} |\mathbf{A}|^2 \, d\Sigma < 32\pi - 8\pi^2,$$

Modified energy functional

. Choose an arbitrary $N > 0$ and introduce the convex functions $\Psi_b : [0, 16\pi/3) \to \mathbb{R}$, $\Psi_m : [1, N) \to \mathbb{R}$ such that

$$\Psi_b''(s) \geq 0, \quad \Psi_b'(s) > 0, \Psi_b(0) = 0, \quad \Psi_b(s) \to \infty \text{ as } s \nearrow 16\pi/3.$$

$$\Psi_m''(s) \geq 0, \quad \Psi_m'(s) > 0, \Psi_m(1) = 1, \quad \Psi_m(s) \to \infty \text{ as } s \nearrow N.$$

Modified energy functional

Define the modified energy functional $\mathcal{J}$ as follows

$$\mathcal{J}(S, \mathbf{r}, \Phi, \Gamma) = \Psi_b(E_b) + \Psi_m(E_m) + \mathcal{K}_f + \mathcal{G}_f,$$

where

$$E_b = \int_\Sigma |\mathbf{A}|^2 \, dx, \quad E_m = \int_{\mathbb{R}^2/\Gamma} (|\partial_1 \mathbf{r}|^2 + |\partial_2 \mathbf{r}|^2) dX$$

Variational problem. Existence of solutions

Denote by $\mathfrak{S}$ the class of all Lipschitz periodic surfaces S, the lattices Γ, Γ-periodic conformal mappings $\mathbf{r} : \mathbb{R}^2 \to S$, and potentials Φ with $\nabla\Phi \in L^2(\Omega)$ such that

$$\int_\Sigma x^3 \, d\Sigma = 0.$$

Then there is $(S, \mathbf{r}, \Phi, \Gamma)$ such that

$$\mathcal{J}(\mathbf{r}, \Phi, \Gamma) = \min_{\mathfrak{S}} \mathcal{J}.$$

Moreover, for every $\lambda > 0$ there is $\delta > 0$ with the following property. If $0 < \Psi_b'(0), \Psi_m'(1) < \delta$, then the minimal surface S is not flat.

Remark

If S is the flat surface $(\{x^3 = 0\})$, then $J = 1$. If

$$S : x^3 = \varepsilon \cos nx^1 \; \cos mx^2,$$

Then

$$\mathcal{J} = 1 + \varepsilon^2 \Psi_b'(0)(n^2 + m^2)^2 + \varepsilon^2 \Psi_m'(1)(m^2 + n^2) - \frac{\varepsilon^2}{2} \frac{n^2}{\sqrt{m^2 + n^2}} + \frac{\varepsilon^2 \lambda}{2} + O(\varepsilon^4). \quad (16)$$

Regularity. Hole-filling method

The hole-filling method is applied for the proof of the regularity of solutions to variational problems. (Bensoussan & Frehse, 2013, Simon, 1993).

Example. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Let $g \in C^\infty(\partial\Omega)$. Consider the standard variational problem

$$\min_{\partial\Omega: u=g} \int_\Omega |\nabla u|^2 \, dx$$

Choose an arbitrary $a \in \Omega$ and set $B_R = \{|x - a| \leq R\}$. Consider the ball $B_{R/2}$ and the annulus $B_R \setminus B_{R/3}$

Regularity. Hole-filling method

Obviously there is $\overline{u} \in W^{1,2}(B_{R/2})$ such that

$$\int_{B_{R/2}} |\nabla \overline{u}|^2 \, dx \leq c \int_{B_R \setminus B_{R/3}} |\nabla u|^2 \, dx, \quad u = \overline{u} \text{ on } \partial B_{R/2}.$$

Hence

$$\int_{B_{R/2}} |\nabla u|^2 \, dx \leq \int_{B_{R/2}} |\nabla \overline{u}|^2 \, dx \leq c \int_{B_R \setminus B_{R/3}} |\nabla u|^2 \, dx,$$

and

$$\frac{1}{c} \int_{B_{R/3}} |\nabla u|^2 \, dx \leq \int_{B_R \setminus B_{R/3}} |\nabla u|^2 \, dx.$$

Regularity. Hole-filling method

$$(1+\frac{1}{c})\int_{B_{R/3}}|\nabla u|^2\,dx \le \int_{B_R}|\nabla u|^2\,dx$$

$$\int_{B_{R/3}}|\nabla u|^2\,dx \le \lambda\int_{B_R}|\nabla u|^2\,dx, \quad \lambda=\frac{c}{c+1},$$

$$\int_{B_r}|\nabla u|^2\,dx \le cr^{\lambda} \Rightarrow |u(x)-u(a)| \le c|x-a|^{\alpha}.$$

Structural Lemma

Let us consider C^∞-conformal *embedding* $x = \mathbf{u}(X)$ of the unit disk $D = \{|X| < 1\}$ into $\mathbb{R}^3$ with

$$\partial_i \mathbf{u}(X) = e^f(X)\mathbf{e}_i(X), \quad \mathbf{e}_3(X) = \mathbf{n}(X), \quad \text{and} \quad \mathbf{u}(0) = 0, \mathbf{e}_3(0) = \mathbf{k}. \tag{17}$$

The smooth vector fields $\mathbf{e}_i$, $1 \le i \le 3$ form a moving orthogonal frame on the surface

$$M = \{\mathbf{u}(X),\ X \in D\}.$$

Structural Lemma

Let

$$\|df\|_{L^2(D)} + \|d\mathbf{e}_i\|_{L^2(D)} \le \varepsilon, \quad \lambda^{-1} \le e^f \le \lambda < \infty. \tag{18}$$

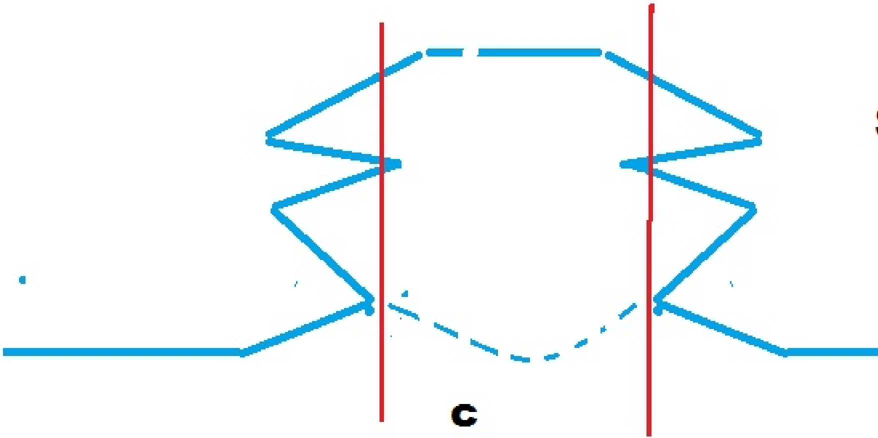
Then there exist constants ε_0 , $\varrho > 0$ c, depending only on λ, such that for all $\varepsilon \in (0, \varepsilon_0)$, there are and compact set $\mathcal{F} \subset (0, \rho)$ with the following properties: The set $\mathcal{F}$ can be covered by intervals (α_i, β_i) with total length less than $C\varepsilon$. For every $t \in (0, \rho) \setminus \mathcal{F}$ the intersection

$$M \cap \mathcal{D}_t, \quad D_t = \{x : \ (x^1)^2 + (x^2)^2 = t^2, \ x^3 \in \mathbb{R}\}.$$

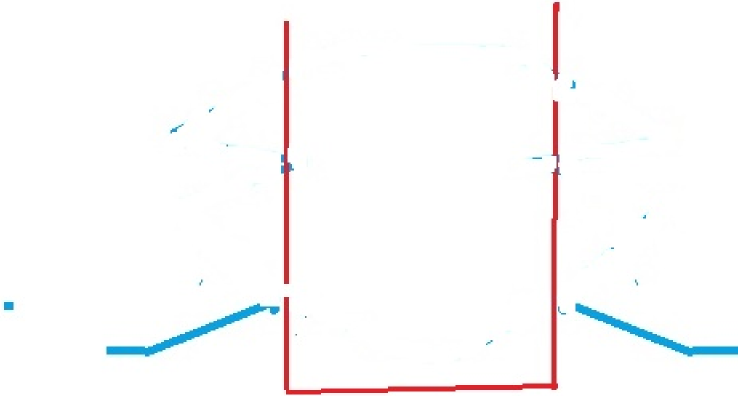
consists of finite number of closed curves $\mathcal{C}_{i,t}$ such that

$$\begin{aligned} \mathcal{C}_{i,t} : \ x^3 = \eta_{i,t}(x^1, x^2), \quad |\eta_{i,t}| + |D\eta_{i,t}| \le c\sqrt{\varepsilon}, \\ \|D^2\eta\|^2_{L^2(C_{i,t})} \le c \int_{M \setminus \mathcal{D}_t} |\mathbf{A}|^2 dM. \end{aligned} \tag{19}$$

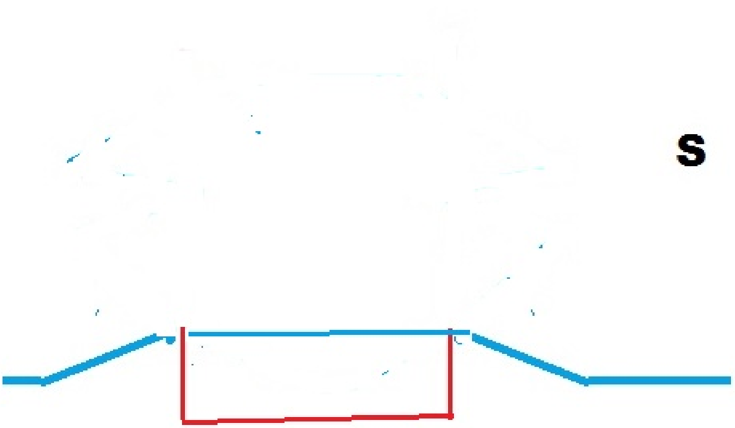
Regularity. Hole-filling method



Regularity. Hole-filling method



Regularity. Hole-filling method



Regularity

Applying hole-filling method we conclude that for every minimizer $(\mathbf{r}, S, \Phi)$ and every $X_0 \in \mathbb{R}^2$, $R > 0$,

$$\int_{|X-X_0|\leq R} |\mathbf{A}|^2 \, dX \leq cR^{\alpha}, 0 < \alpha < 1.$$

Since

$$\int_{|X-X_0|\leq R} |\mathbf{A}|^2 \, dX = \int_{|X-X_0|\leq R} |d\mathbf{n}|^2 \, dX,$$

it follows that

$$|\mathbf{n}(X) - \mathbf{n}(Y)| \leq |X - Y|^{\beta}, 0 < \beta < 1.$$

Hence S belongs to the class $C^{1+\beta}$.

Remark 1

Remark 2

Capillarity.

$$c_m \int_{\mathbb{R}^2/\Gamma} \left(|\partial_1 \mathbf{r}|^2 + |\partial_2 \mathbf{r}|^2\right) dX \quad \Rightarrow \quad c_m \int_\Sigma d\Sigma,$$

or take

$c_m = 0$.

Remark 3

Saddle critical points — ?

Remark 3

Mass. Two speeds of the wave propagation.

$$x = \mathbf{r}(X - t\mathbf{V}) + t\mathbf{c}.$$

If the density ρ of the elastic shell is not equal to zero, then the stretching energy must be replaced by the quadratic form (Toland & P., 2012)

$$c_m(|\partial_1 \mathbf{r}|^2 + |\partial_2 \mathbf{r}|^2) - \rho|(V_1\partial_1 + V_2\partial_2)\mathbf{r}|^2.$$