Development of high vorticity structures and geometrical properties of the vortex line representation

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OUTLINE

- Motivation: Collapse and the Kolmogorov-Obukhov theory
- Vortex line representation (VLR) and new Cauchy invariant
- Compressibility of VLR and folding of vortex lines
- Numerical experiment
- Conclusion

Motivation: Collapse and the Kolmogorov-Obukhov theory

- According to the Kolmogorov-Obukhov theory (1941) velocity fluctuations at spatial scales l from the inertial range obey the power-law $\langle|\delta v|\rangle \propto \varepsilon^{1/3} l^{1/3}$, where ε is the mean energy flux from large to small scales. This formula is easily obtained from the dimensional analysis.
- Similarly, fluctuations for the vorticity field $\omega = \nabla \times \mathbf{v}$ diverge at small scales as $\langle|\delta\omega|\rangle \propto \varepsilon^{1/3} l^{-2/3}$, while the time of energy transfer from the energy-contained scale l_E to the viscous ones is finite and estimated as $T \sim l_E^{2/3} \varepsilon^{-1/3}$.
- These two relations allow to link the Kolmogorov spectrum formation with the blowup in the vorticity field (collapse).

Motivation: Collapse and the Kolmogorov-Obukhov theory

- Kolmogorov's arguments assume locality of interaction and isotropy of the turbulence in the inertial interval. This implies that the dynamics at these scales can be described by the Euler equations and the emergence of the Kolmogorov energy spectrum can be expected before the viscous scales are excited, i.e., in a fully inviscid flow.
- This conjecture was verified numerically in our previous papers (2015, 2016, 2017), where we showed that the Kolmogorov spectrum is developed through the formation of pancake-like structures of enhanced vorticity. Such pancakes can be treated as coherent structures.
- At the stage of turbulence onset turbulence is far from isotropic, its spectrum contains a few number of jets. Each jet corresponds to its own pancake.

Motivation: Collapse and the Kolmogorov-Obukhov theory

- We also established numerically the asymptotic Kolmogorov-type scaling,

$$\omega_{\max}(t) \propto \ell(t)^{-2/3},$$

between the vorticity maximum on the pancake and the pancake thickness.

- No tendency to finite-time blowup was observed for generic initial conditions, with nearly exponential growth of vorticity in time.
- In the present paper we develop a new concept of folding for continuously distributed vortex lines.

Motivation: Collapse and the Kolmogorov-Obukhov theory

- The underlying idea that enables the folding phenomenon is that the "flow" of continuously distributed vortex lines is compressible, despite the incompressibility of the fluid: the vortex lines representation (VLR), E.K.& V. Ruban, 1998. Our new theory based on the VLR explains the $2/3$-law as a result of the classical fold catastrophe.
- The discussed approach is applicable for a larger class of "frozen-in-fluid" fields advected by incompressible fluid, for instance, the magnetic field in MHD or the di-vorticity field for 2D Euler.
- By means of a new adaptive numerical scheme based on the VLR we observed numerically the compressible character of continuously distributed vortex lines and verified the details of the folding phenomenon.

VLR and new Cauchy invariant

It is well known that the Euler equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\nabla p, \qquad \text{div } \mathbf{v} = 0,$$

has infinite (continuous) number of integrals of motion. These are the so called Cauchy (1815) invariants. They can be obtained from the Kelvin (1869) theorem (Hankel in 1861)

$$\Gamma = \oint_{C[t]} (\mathbf{v} \cdot d\mathbf{l}) = \text{inv}$$

with the movable together with fluid contour $C[t]$. Passing in this integral to the Lagrangian variables,

$$\mathbf{r} = \mathbf{r}(\mathbf{a}, t), \ \frac{d\mathbf{r}}{dt} = \mathbf{v}(\mathbf{r}, t), \ \mathbf{r}|_{t=0} = \mathbf{a}$$

we arrive at

$$\Gamma = \oint_{C[a]} \dot{x}_i \cdot \frac{\partial x_i}{\partial a_k} \, da_k \ , \ \text{with fixed } C[a].$$

VLR and new Cauchy invariant

Hence we get the Cauchy invariants

$$\mathbf{I} = \text{curl}_a \left(\dot{x}_i \frac{\partial x_i}{\partial \mathbf{a}} \right) \equiv \omega_{\mathbf{0}}(\mathbf{a})$$

which are constraints in Euler. They characterize the frozenness of the vorticity into fluid. The latter means that fluid (Lagrangian) particles can not leave its own vortex line where they were initially. Thus, the particles have one independent degree of freedom – motion along vortex line. But such a motion does not change the vorticity:

$$\frac{\partial \omega}{\partial t} = \text{rot}\, [\mathbf{v} \times \omega].$$

REMARK: VLR and new Cauchy invariant

Thus, the Helmholtz equation contains only one velocity component normal to the vortex line, $\mathbf{v_n}$. The tangent velocity $\mathbf{v}_\tau$ plays a passive role providing incompressibility. Decomposing, $\mathbf{v} = \mathbf{v_n} + \mathbf{v}_\tau$, in the Euler *incompressible* equations leads to the equation of motion of charged *compressible* fluid moving in an electromagnetic field:

$$\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n \nabla)\mathbf{v}_n = \mathbf{E} + [\mathbf{v}_n \times \mathbf{H}],$$

where

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{H} = \text{rot } \mathbf{A}$$

with $\varphi = p + v_\tau^2/2, \ \mathbf{A} = \mathbf{v}_\tau$. Thus, two Maxwell equations are satisfied with the gauge: $\text{div}\,\mathbf{A} = -\text{div}\,\mathbf{v}_n \neq 0.$

VLR and new Cauchy invariant

Now perform transform in a new charged *compressible* hydrodynamics to the Lagrangian description:

$$\dot{\mathbf{r}} = \mathbf{v}_n(\mathbf{r}, t) \text{ with } \mathbf{r}|_{t=0} = \mathbf{a}.$$

Under this transform the new hydrodynamics become the Hamilton equations:

$$\dot{\mathbf{P}} = -\partial h/\partial \mathbf{r}, \qquad \dot{\mathbf{r}} = \partial h/\partial \mathbf{P},$$

$\mathbf{P} = \mathbf{v}_n + \mathbf{A} \equiv \mathbf{v}$ is the generalized momentum, and the Hamiltonian $h = (\mathbf{P} - \mathbf{A})^2/2 + \varphi \equiv p + \mathbf{v}^2/2$ ($\equiv$ the Bernoulli "invariant").

The Kelvin (Liouville) theorem says that $\Gamma = \oint (\mathbf{P} \cdot d\mathbf{R}) = \text{inv}$.

Transform in Γ to new Lagrangian coordinates leads to a new Cauchy invariant :

$$\mathbf{I} = \text{rot}_a \left(P_i \frac{\partial x_i}{\partial \mathbf{a}} \right) \equiv \omega_\mathbf{0}(\mathbf{a}).$$

VLR and new Cauchy invariant

Hence, one can see that the only one velocity component normal to the vortex line, $\mathbf{v_n}$, can change ω. To define ω it is enough to know all trajectories of the equation

$$\dot{\mathbf{r}} = \mathbf{v}_n(\mathbf{r}, t), \ \mathbf{r}|_{t=0} = \mathbf{a}$$

or, by another words, mapping $\mathbf{r} = \mathbf{r}(\mathbf{a}, t)$. In terms of this mapping the Helmholtz Eq. can be integrated:

$$\omega(\mathbf{r}, t) = \frac{(\omega_0(\mathbf{a}) \cdot \nabla_a)\mathbf{r}(\mathbf{a}, t)}{J(\mathbf{a}, t)}$$

where $J(\mathbf{a}, t) = \partial(\mathbf{r})/\partial(\mathbf{a})$ is the Jacobian of the mapping. In this Eq. $\omega_0(\mathbf{a})$ is a new Cauchy invariant. Due to the vorticity frozenness, $\mathbf{v_n}$ is the velocity of vortex lines.

VLR and new Cauchy invariant

These equations together with

$$\omega(\mathbf{r},t) = \nabla_r \times \mathbf{v}(\mathbf{r},t) \text{ and } \mathrm{div}_r\, \mathbf{v}(\mathbf{r},t) = 0$$

form the complete system of equations in the vortex line representation (Kuznetsov, Ruban (1998), Kuznetsov (2002, 2006)).

In the general case, $\mathrm{div}_r \mathbf{v}_n \neq 0$ and therefore $\mathbf{r} = \mathbf{r}(\mathbf{a},t)$ is the compressible mapping: the Jacobian is not fixed and can take arbitrary values! This means that continuously distributed vortex lines can be compressed.

The quantity $n = J^{-1}$ plays the role of vortex line density:

$$n_t + \mathrm{div}_r(n\mathbf{v}_n) = 0, \;\; \mathrm{div}_r \mathbf{v}_n \neq 0.$$

VLR and new Cauchy invariant

- The maximal value of $\max|\omega| \equiv \omega_{max}$ satisfies

$$\frac{d\omega_{max}}{dt} = \omega_{max}(\tau(\nabla\tau)\mathbf{v}).$$

  Hence for quasi one-dimensional (pancake) structures we approximately have $\omega_{max}J \approx$ const.

- The VLR equations are written in mixed Eulerian ($\mathbf{x}$-space) and Lagrangian ($\mathbf{a}$-space) variables. For numerical study, the VLR Eqs can be rewritten in terms of the inverse mapping $\mathbf{a} = \mathbf{a}(\mathbf{x}, t)$ which obeys the equation

$$\frac{\partial\mathbf{a}}{\partial t} + (\mathbf{v}_n \cdot \nabla)\mathbf{a} = 0.$$

VLR and new Cauchy invariant

Eq. for the vorticity ω can be rewritten in the form

$$\omega_i(\mathbf{x},t) = \frac{1}{2}\varepsilon_{ijk}\,\varepsilon_{\alpha\beta\gamma}\,\omega_{0\alpha}(\mathbf{a})\,\frac{\partial a_\beta}{\partial x_j}\,\frac{\partial a_\gamma}{\partial x_k}.$$

Here $\omega_0(\mathbf{a})$ is the initial vorticity at $t=0$. The two equations together with the relations

$$\mathbf{v} = \mathrm{rot}^{-1}\omega = -\Delta^{-1}\,(\nabla\times\omega), \quad \mathbf{v}_n = \mathbf{v} - \frac{(\mathbf{v}\cdot\omega)}{\omega^2}\omega$$

for the velocity and the normal velocity represent complete VLR system of equations written in the Eulerian coordinates $(\mathbf{x},t)$.

Folding of vortex lines

REMARK 1: Wave breaking, as blowup, is well known for compressible flows resulting in appearance of shocks, which can be considered as the formation of folds. Breaking in gasdynamics is possible due to compressible character of the mapping.

REMARK 2: Breaking/folding of vortex lines is impossible in 2D and for cylindrically symmetric flows without swirl (Majda, 1990) because $\omega \perp \mathbf{v}$ and div $\mathbf{v}_n = 0$, and consequently $J = 1$.

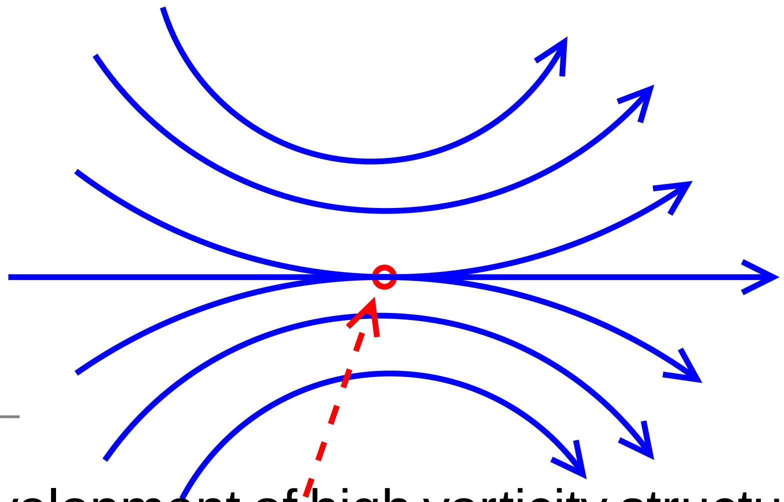
Thus, breaking/folding of vortex lines is 3D phenomenon.

Up to now it has not been known whether this process happens in a finite or infinite time.

Folding of vortex lines

In our numerics exponential increasing of the vorticity maximum and formation around this maximum a structure of the pancake type with exponential decreasing of its width were observed, instead of blow-up. Such structures appear around each vorticity maximum and are shown to have self-similar behavior. (First numerics by M. Brachet, et. el. (1992).)

Geometrically breaking results in touching of vortex lines (in a finite or infinite time).



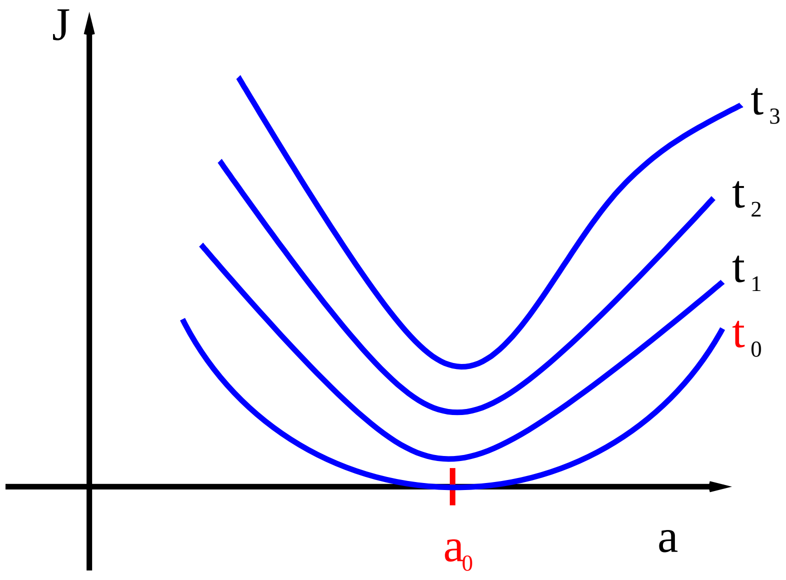
Development of high vorticity structures and geometrical properties of the vortex line representa

Folding of vortex lines

Let us assume that breaking/folding takes place. Consider the equation $J(\mathbf{a},t)=0$ and find its positive roots $t=\tilde{t}(\mathbf{a})>0$. Then the collapse (or touching) time will be

$$t_0 = \min_a \tilde{t}(\mathbf{a}).$$

Near the minimal point $\mathbf{a}=\mathbf{a}_0$ as the expansion of J takes the form:



$t_0 > t_1 > t_2 > t_3$

$$J(a,t) = \alpha\tau(t) + \gamma_{ij}\Delta a_i \Delta a_j$$

- concavity condition

$\alpha > 0$, $\tau(t)\to 0$ as $t\to t_0$,

γ_{ij} is positive definite (non-degenerate) time independent matrix,

$\Delta\mathbf{a} = \mathbf{a} - \mathbf{a}_0$.

Self-similar asymptotics

REMARK: The assumption about linear dependence of J_{min} on $\tau(t)$ is familiar to the Landau assumption in his theory of the second-order phase transitions.
This expansion results in the self-similar asymptotics for vorticity:

$$\omega(\mathbf{r},t) = \frac{(\omega_{\mathbf{0}}(\mathbf{a}) \cdot \nabla_a)\mathbf{r}|_{a_0}}{\tau(\alpha + \gamma_{ij}\eta_i\eta_j)}, \ \ \eta = \frac{\Delta a}{\tau^{1/2}}.$$

Now the main problem is
to transform from the auxiliary a-space to the physical $\mathbf{r}$-space.

Self-similar asymptotics

Consider first the 1D case when

$$J = \frac{\partial x}{\partial a} = \alpha\tau + \gamma a^2 \quad \rightarrow x = \alpha\tau a + \frac{1}{3}\gamma a^3.$$

Thus, $a \sim \tau^{1/2}, \;\; x \sim \tau^{3/2}$, i.e. in the physical space compression happens more rapidly than in the space of Lagrangian markers !! At distances $\gamma a^2 \gg \alpha\tau$ we have the time-independent asymptotics,

$$J \sim x^{2/3}.$$

Thus, any changes happen at the region $\gamma a^2 \leq \alpha\tau$.

Self-similar asymptotics

3D case

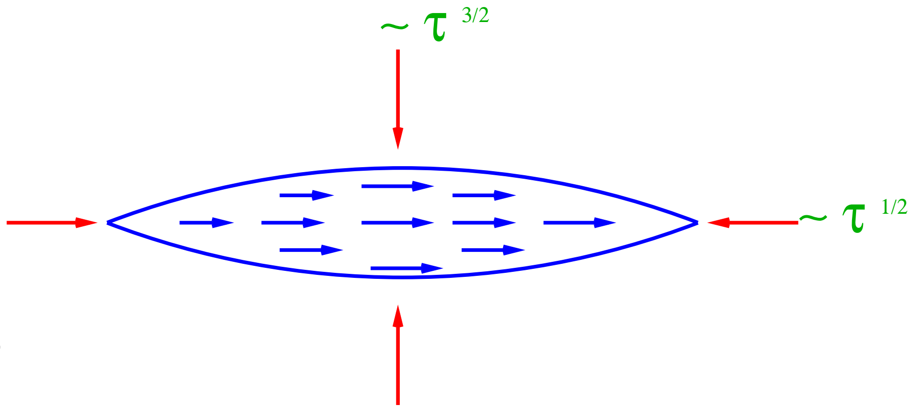
The Jacobian $J = \lambda_1\lambda_2\lambda_3 \to 0$ means that one eigenvalue, say, $\lambda_1 \to 0$ and $\lambda_2, \lambda_3 \to$ const as $t \to t_0$ and $a \to a_0$. Hence it follows that near singular point there are two different self similarities:

along "soft" (λ_1) direction $x_1 \sim \tau^{3/2}$ (like in 1D);

along "hard" (λ_2, λ_3) directions $x_{2,3} \sim \tau^{1/2}$,

so that

$$\omega = \frac{1}{\tau}\mathbf{g}\left(\frac{x_1}{\tau^{3/2}}, \frac{x_\perp}{\tau^{1/2}}\right).$$



This results in formation of pancake structure (compare with Zeldovich)

Self-similar asymptotics

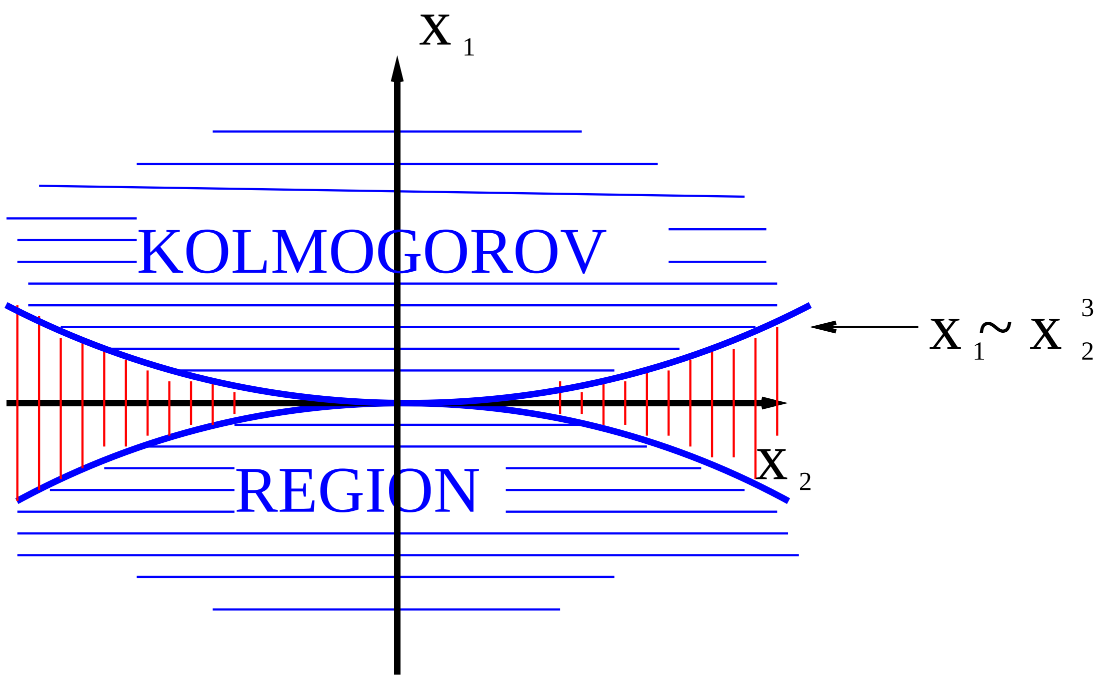
As $\tau \to 0$ when $\gamma_{ij}\Delta a_i \Delta a_j \gg \alpha\tau$ the vorticity has a time-independent, very anisotropic distribution. The main dependence of ω is connected with x_1-direction:

$$\omega \approx \frac{\mathbf{b}}{x_1^{2/3}}$$

with $\mathbf{b} =$ const and KOLMOGOROV index $2/3$!.
This dependence is realized everywhere except regions between two cubic paraboloids $-cx_\perp^3 < x_1 < cx_\perp^3$ In this narrow region vorticity at $\tau = 0$ behaves like

$$\omega \approx \frac{\mathbf{b_1}}{x_\perp^2}.$$

Self-similar asymptotics



In Kolmogorov region the vorticity can be estimated as

$$\omega \sim \frac{\epsilon^{1/3}}{x_1^{2/3}}$$

where $\epsilon \sim \omega_0^3 L^2,\ L \sim \gamma^{-1/2}$.

VLR for exact solution

As it was shown by us (JFM, **813**, R1 (1-10) (2017)) 3D Euler has exact solution which in Cartesian coordinates has the form

$$\mathbf{v}(\mathbf{x},t) = -\omega_{\max}(t)\,\ell_1(t)\,f\left(\frac{x_1}{\ell_1(t)}\right)\mathbf{n}_3 + \begin{pmatrix} -\beta_1(t)\,x_1 \\ \beta_2(t)\,x_2 \\ \beta_3(t)\,x_3 \end{pmatrix},$$

$$\omega(\mathbf{x},t) = \omega_{\max}(t) f'\left(\frac{x_1}{\ell_1(t)}\right)\mathbf{n}_2.$$

Here $\omega_{\max}(t)$ and $\ell_1(t)$ are the vorticity maximum and the pancake thickness, $f(x_1)$ is arbitrary smooth function.

VLR for exact solution

- $\beta_1(t)$, $\beta_2(t)$ and $\beta_3(t)$ are given by

$$\beta_1 = -\dot{\ell}_1/\ell_1, \quad \beta_2 = \dot{\omega}_{\max}/\omega_{\max}, \quad -\beta_1 + \beta_2 + \beta_3 = 0.$$

- There exists the analog of this solution by Lundgren (1982) which describes axi-symmetric flow. But nobody before us has found the 1D (pancake) solution.
- This solution has infinite energy in $\mathbb{R}^3$ and allows for an arbitrary time-dependency of $\omega(t)$ and $\ell_1(t)$, in particular, the one leading to a finite-time blowup.

VLR for exact solution

- It can be extended for the Navier–Stokes equations with kinematic viscosity ν, if the function $f(\xi, t)$ changes with time as $f_t - \frac{\nu}{\ell_1^2} f_{\xi\xi} = 0$.
- Comparison of this solution for 3D Euler with the simulations gives a good agreement at the pancake region for $\omega_{\max}(t) \propto e^{t/T_\omega}$ and $\ell_1(t) \propto e^{-t/T_\ell}$.
- The velocity component normal to vorticity:

$$\mathbf{v}_n(\mathbf{x}, t) = -\omega_{\max}(t)\, \ell_1(t)\, f\left(\frac{x_1}{\ell_1(t)}\right) \mathbf{n}_3 + \begin{pmatrix} -\beta_1 x_1 \\ 0 \\ \beta_3 x_3 \end{pmatrix}.$$

VLR for exact solution

- For exponential pancake development the VLR mapping is written as

$$x_1 = a_1\, e^{-\beta_1 t}, \quad x_2 = a_2, \quad x_3 = a_3\, e^{\beta_3 t} - f(a_1)\,\frac{\sinh(\beta_3 t)}{\beta_3},$$

with the corresponding Jacobi matrix,

$$\hat{J}(\mathbf{a},t) = \begin{pmatrix} e^{-\beta_1 t} & 0 & 0 \\ 0 & 1 & 0 \\ -f'(a_1)\,\frac{\sinh(\beta_3 t)}{\beta_3} & 0 & e^{\beta_3 t} \end{pmatrix}, \quad J(\mathbf{a},t) = \det \hat{J} = e^{-\beta_2 t}.$$

VLR for exact solution

- Respectively, for vorticity we have

$$\omega(\mathbf{x},t) = \frac{\widehat{J}\,\omega_0(\mathbf{a})}{J},$$

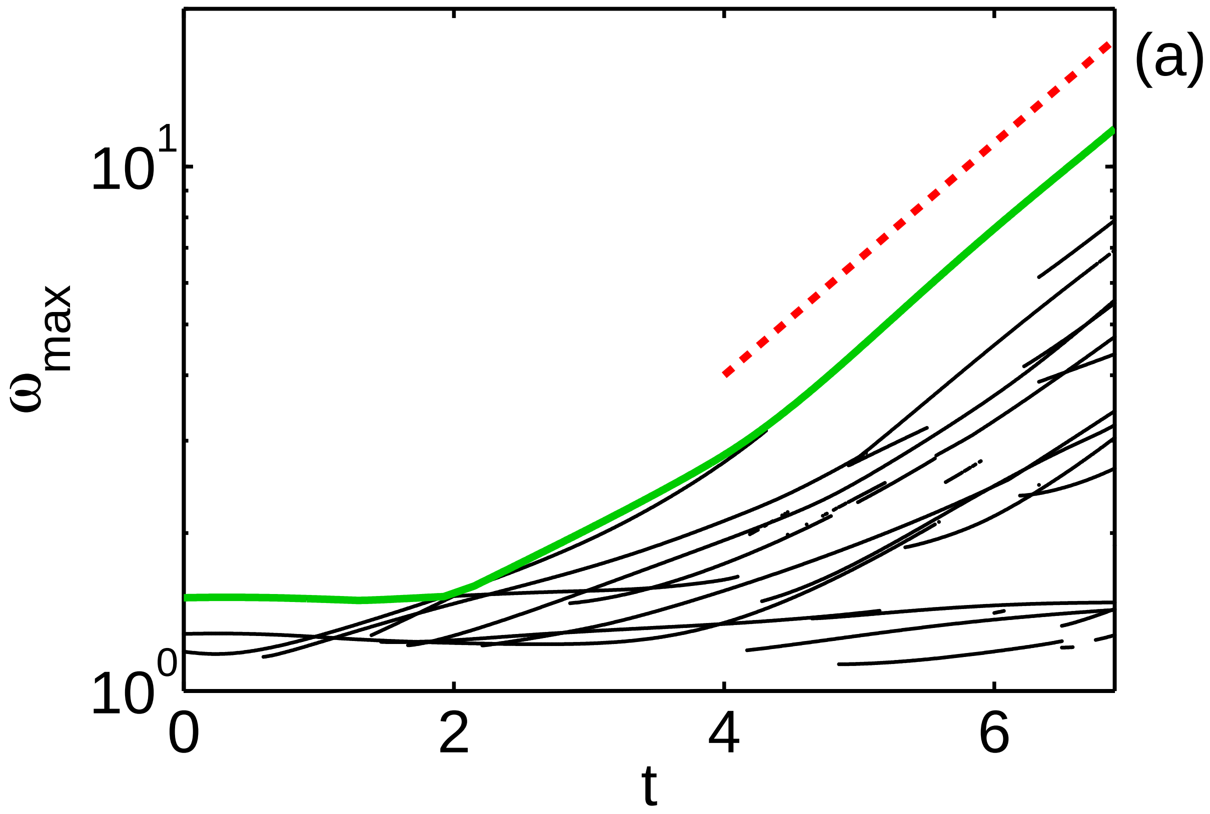
that coincides with our solution.
Hence the Jacobian is inverse-proportional to the vorticity $J(t) \propto 1/\omega_{\max}(t)$, and does not depend on spatial coordinates.

Numerical experiment

We use two numerical schemes based on direct integration of the Euler equations for ω and the VLR formulation in the periodic box $\mathbf{r} = (x, y, z) \in [-\pi, \pi]^3$ using the pseudo-spectral method with high-order Fourier filtering. During simulations, the number of nodes is adapted independently along each coordinate providing an optimal anisotropic rectangular grid. We tested several large-scale initial conditions in the form of random truncated (up to second harmonics) Fourier series considered as a perturbation of the shear flow $\omega_x = \sin z,\ \omega_y = \cos z,\ \omega_z = 0$. This paper is based on one selected simulation with the final grid $486 \times 1024 \times 2048$.

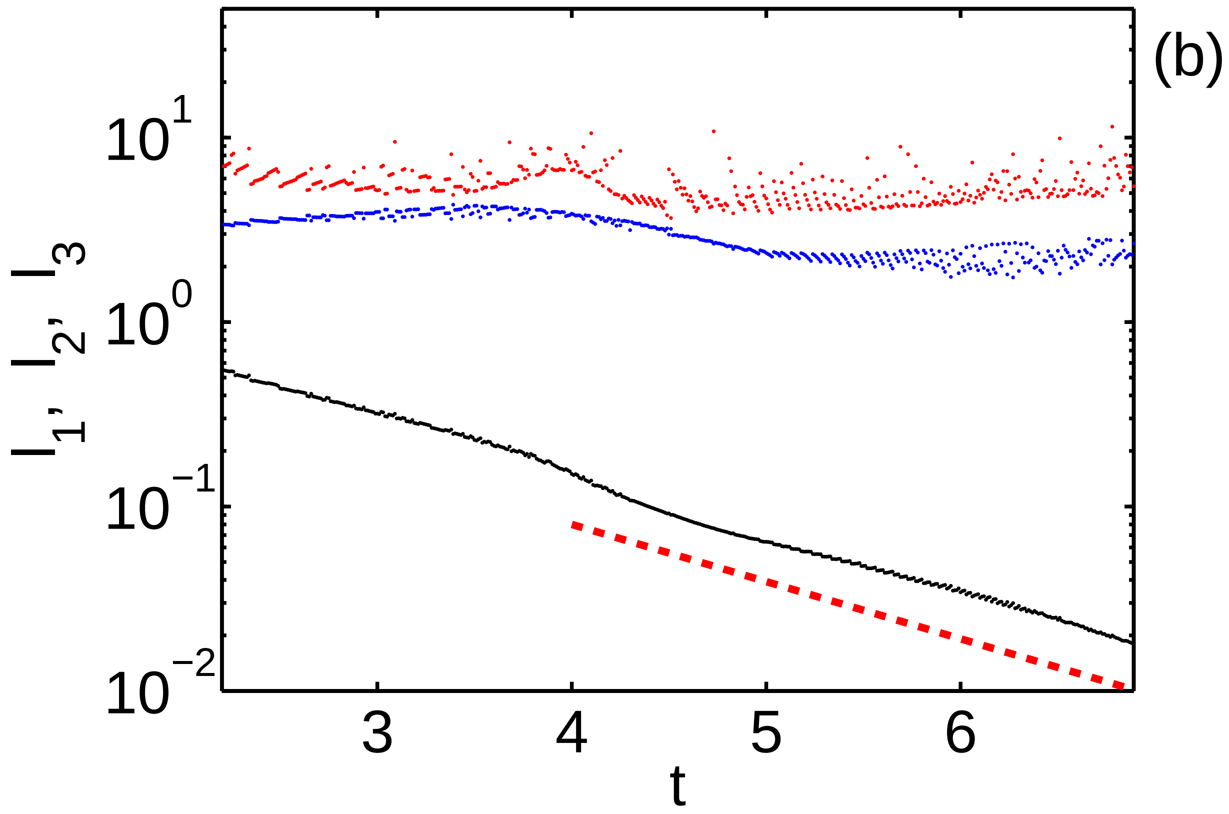
Numerical experiment: direct integration

Evolution of local vorticity maximums (logarithmic vertical scale). Green line shows the global maximum, dashed red line indicates the slope $\propto e^{t/T_\omega}$ with $T_\omega = 2$.



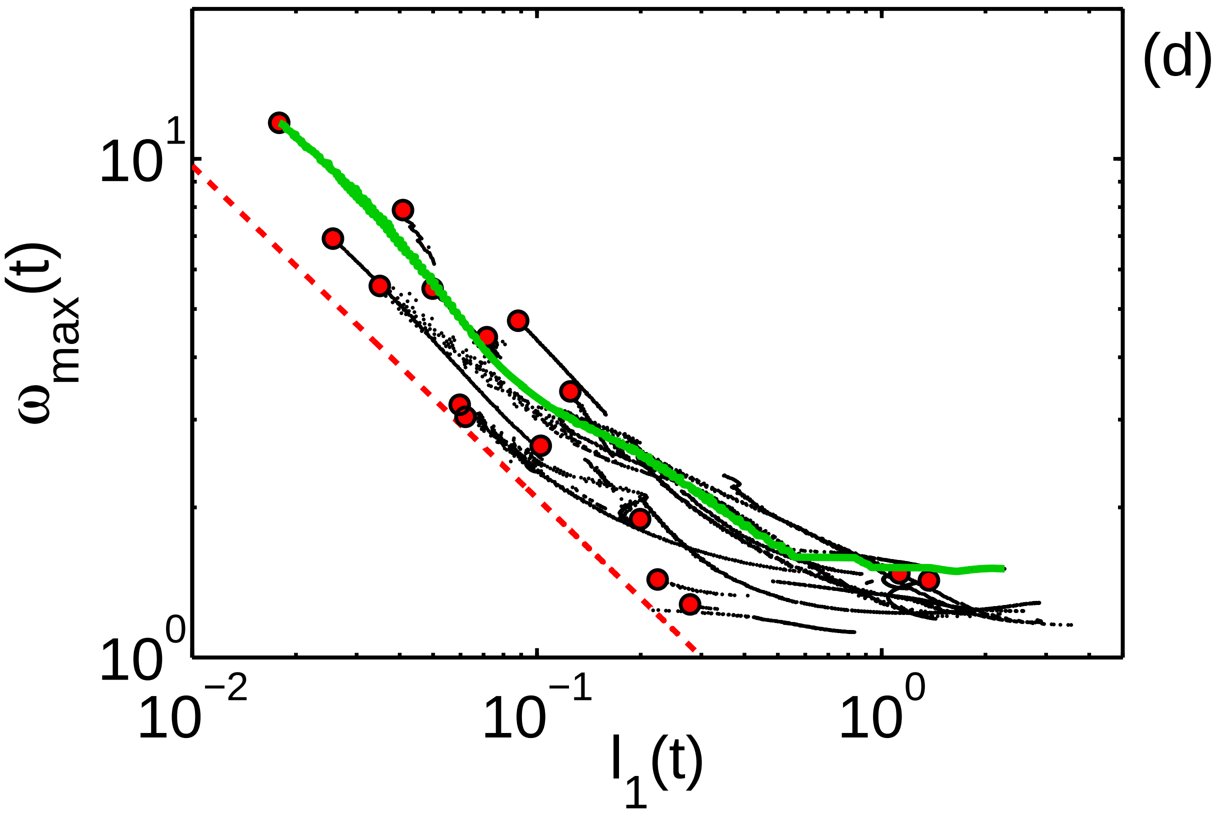
Numerical experiment: direct integration

Evolution of characteristic spatial scales ℓ_1 (black), ℓ_2 (blue) and ℓ_3 (red) for the global vorticity maximum. Dashed red line indicates the slope $\propto e^{-t/T_\ell}$ with $T_\ell = 1.4$.



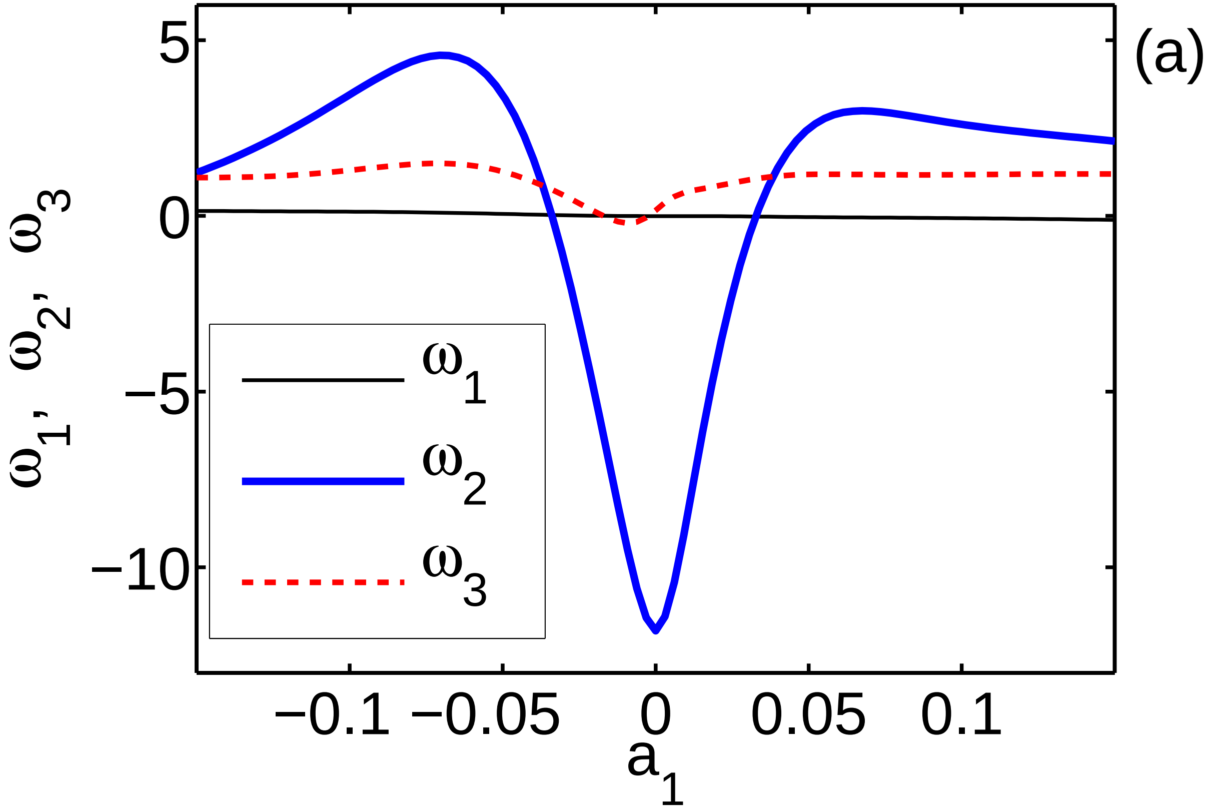
Numerical experiments: direct integration

Vorticity local maximums $\omega_{\max}(t)$ vs. lengths $\ell_1(t)$ during the evolution of the pancake structures. Green line shows the global maximum, red circles mark local maximums at the final time. Dashed red line indicates the power-law $\omega_{\max} \propto \ell_1^{-2/3}$.



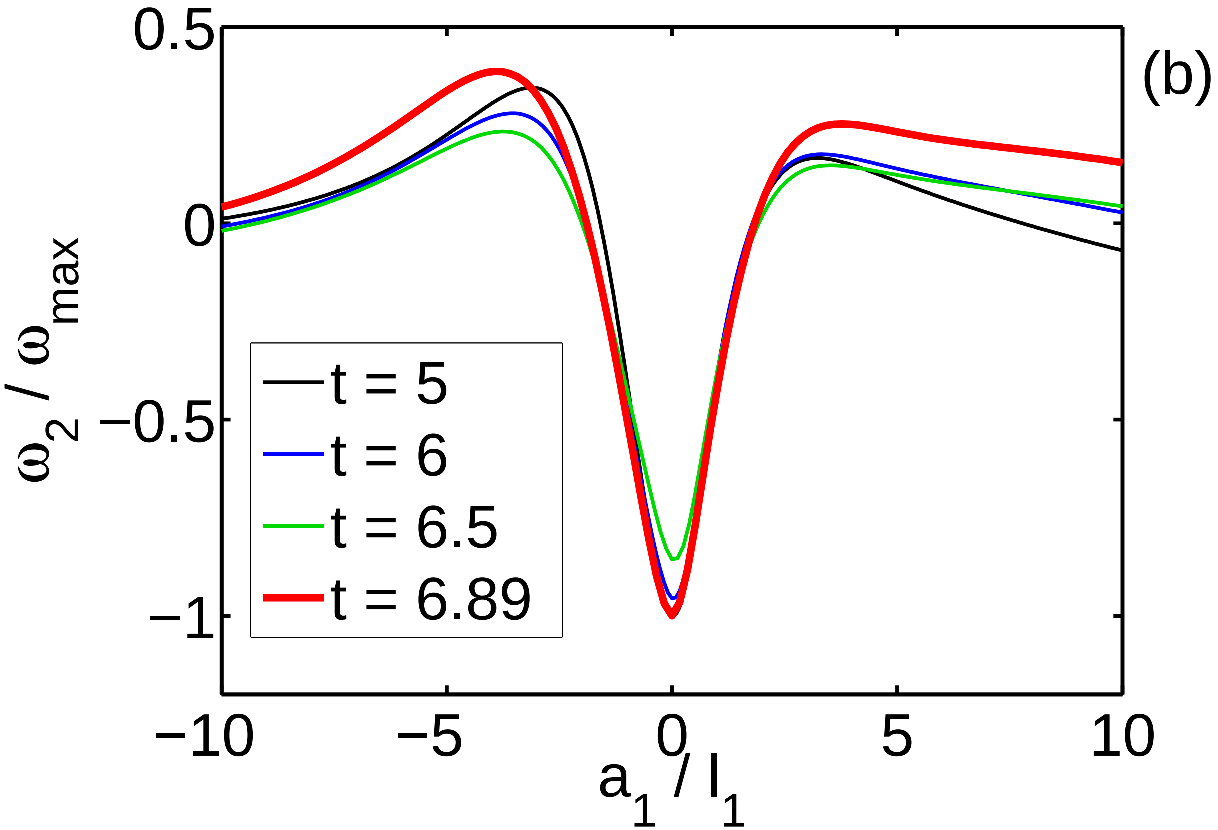
Numerical experiments: direct integration

Components of the vorticity vector $\omega = (\omega_1, \omega_2, \omega_3)$ as functions of a_1 perpendicular to the pancake, at the final time.

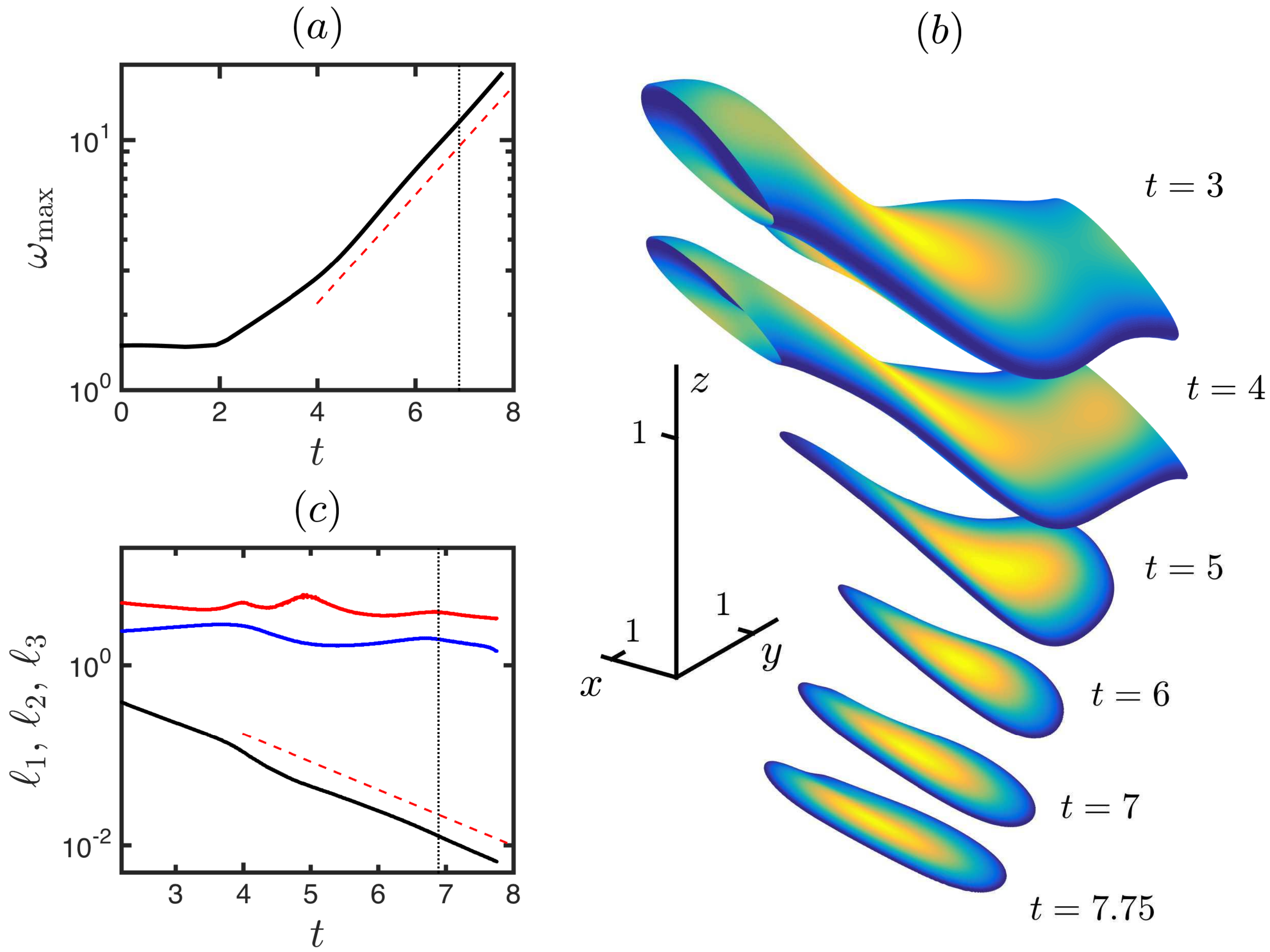


Numerical experiment: direct integration

Vorticity component $\omega_2/\omega_{\max}$ vs. coordinate a_1/ℓ_1 at different times, demonstrating the self-similarity from $\ell_1(5) = 0.064$ to $\ell_1(6.89) = 0.018$.



Numerical experiment: direct integration



Evolution of pancake (right), dependences of ω_{max} and $\ell_{1,2,3}$ (left).

Development of high vorticity structures and geometrical properties of the vortex line representa

Numerical experiment

- By use of the direct integration we found that at the maximal vorticity point

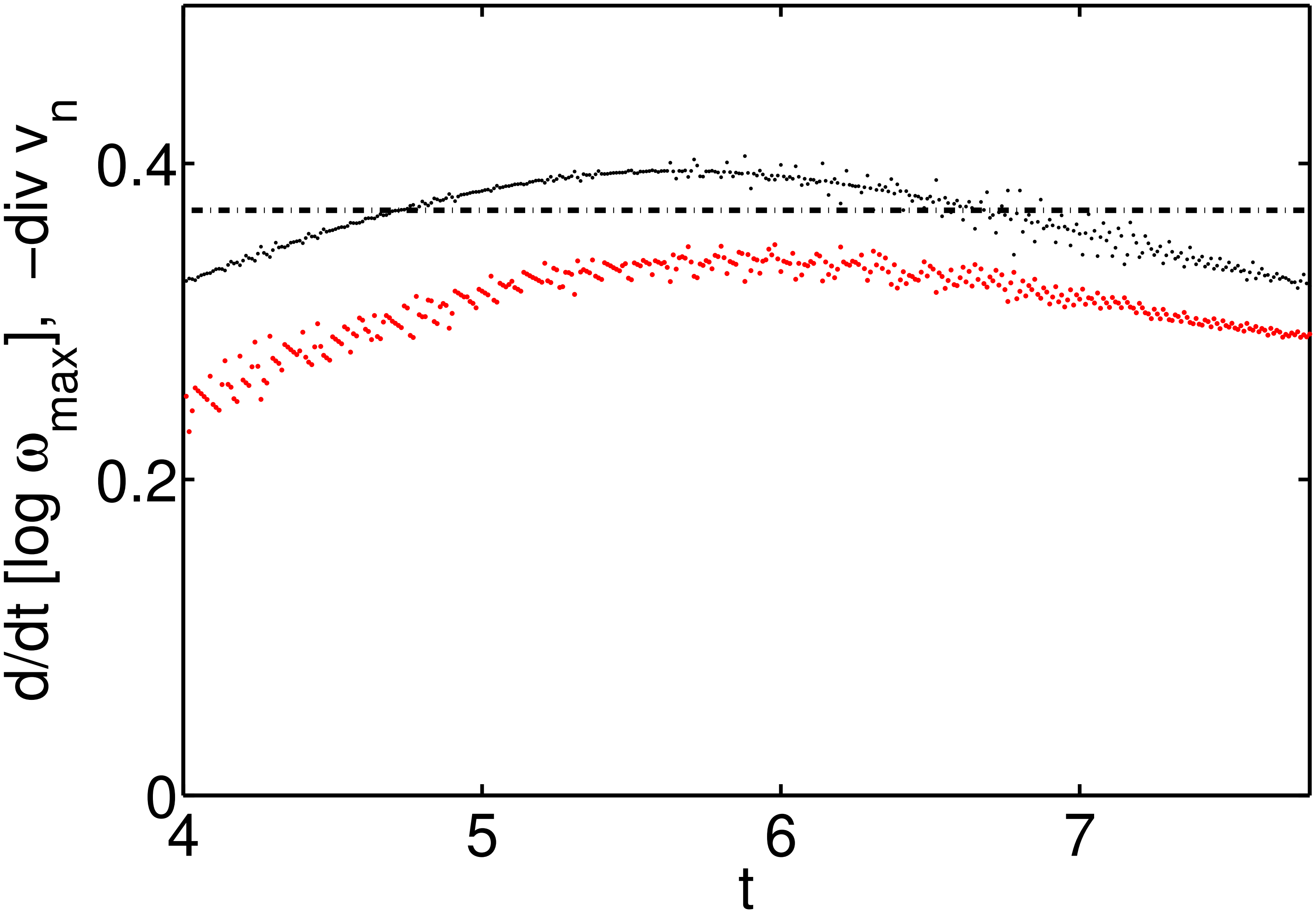
$$\frac{1}{\omega_{max}}\frac{d\omega_{max}}{dt} \approx -\mathrm{div}\,\mathbf{v_n}.$$

This means that the main contribution into the vorticity maximum comes from the denominator,

$$\omega(\mathbf{r},t) = \frac{(\omega_0(\mathbf{a})\cdot\nabla_a)\mathbf{r}(\mathbf{a},t)}{J(\mathbf{a},t)}.$$

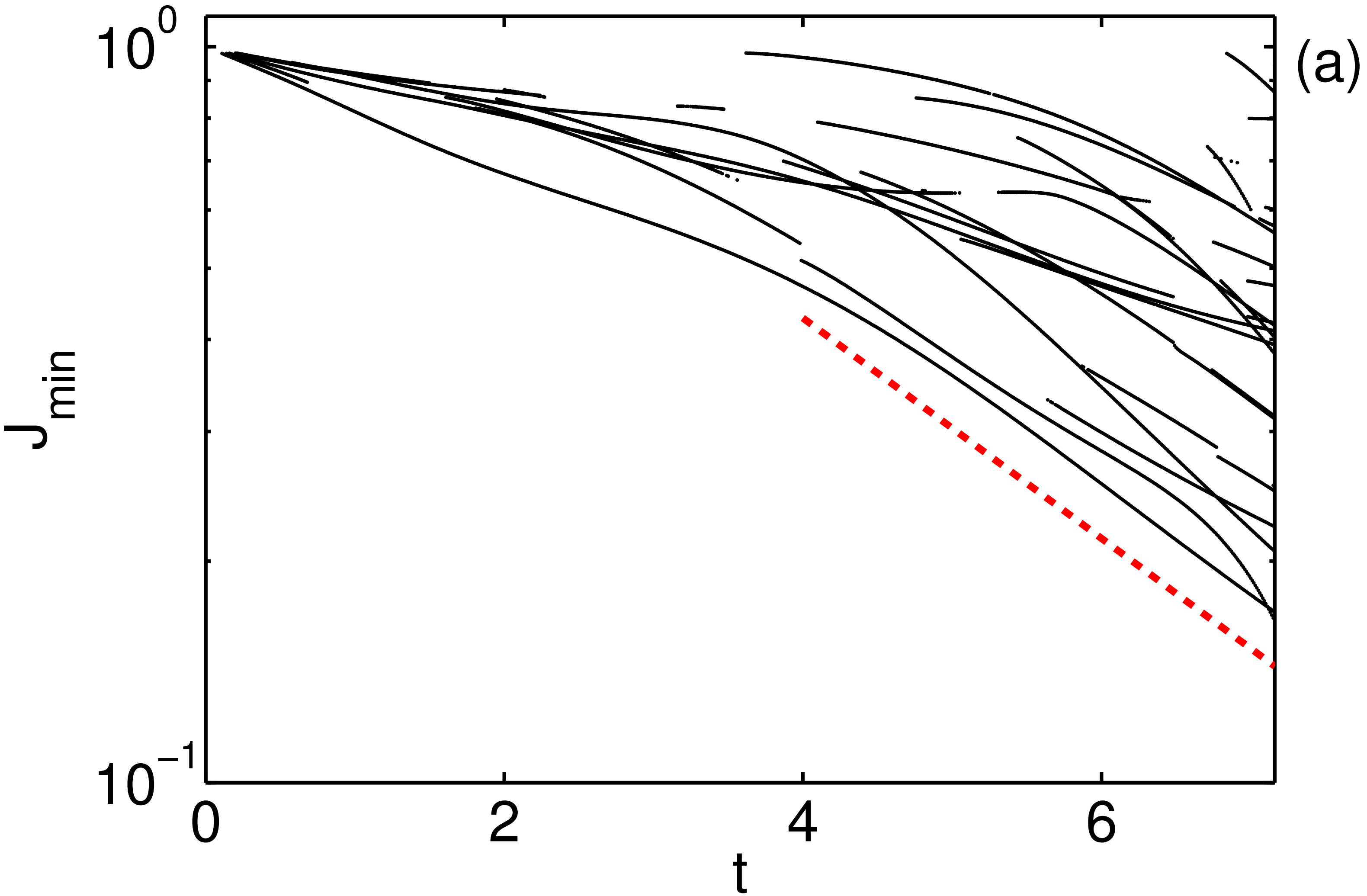
- By means of the VLR scheme it was demonstrated decreasing of the Jacobian. This means that formation of the pancake structures can be considered as folding (breaking) of the vortex lines

Direct integration: compressibility



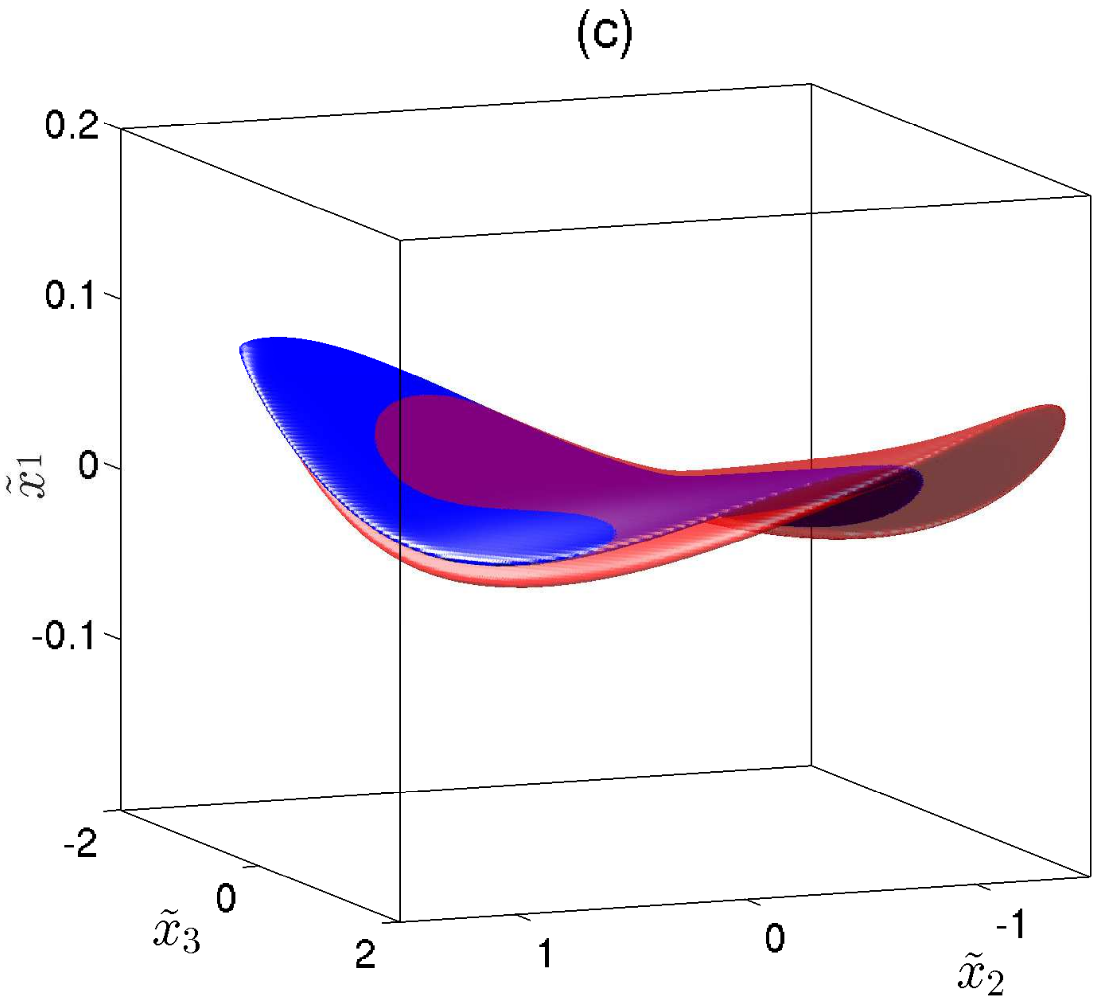
Comparison of $d \log \omega_{max}/dt$ and $-\text{div}\, \mathbf{v_n}$.

Numerical experiment: compressibility



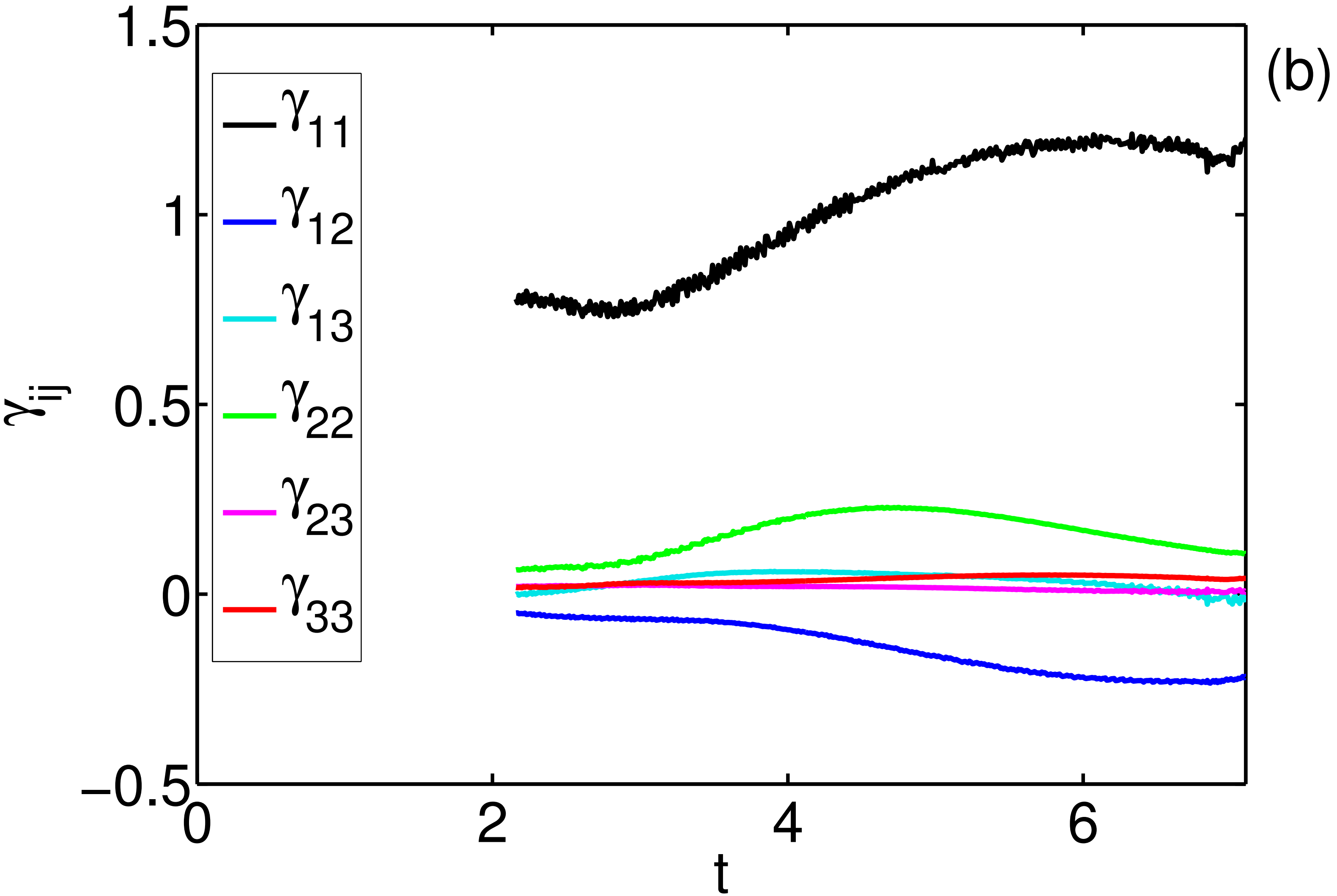
Decreasing of the Jacobian (VLR scheme)

Numerical experiment: compressibility



Isosurfaces $|\boldsymbol{\omega}| = 0.8\,\omega_{\max}$ (red) and $J = 1.25\,J_{\min}$ (blue) at $t = 7.5$ (VLR simulation)

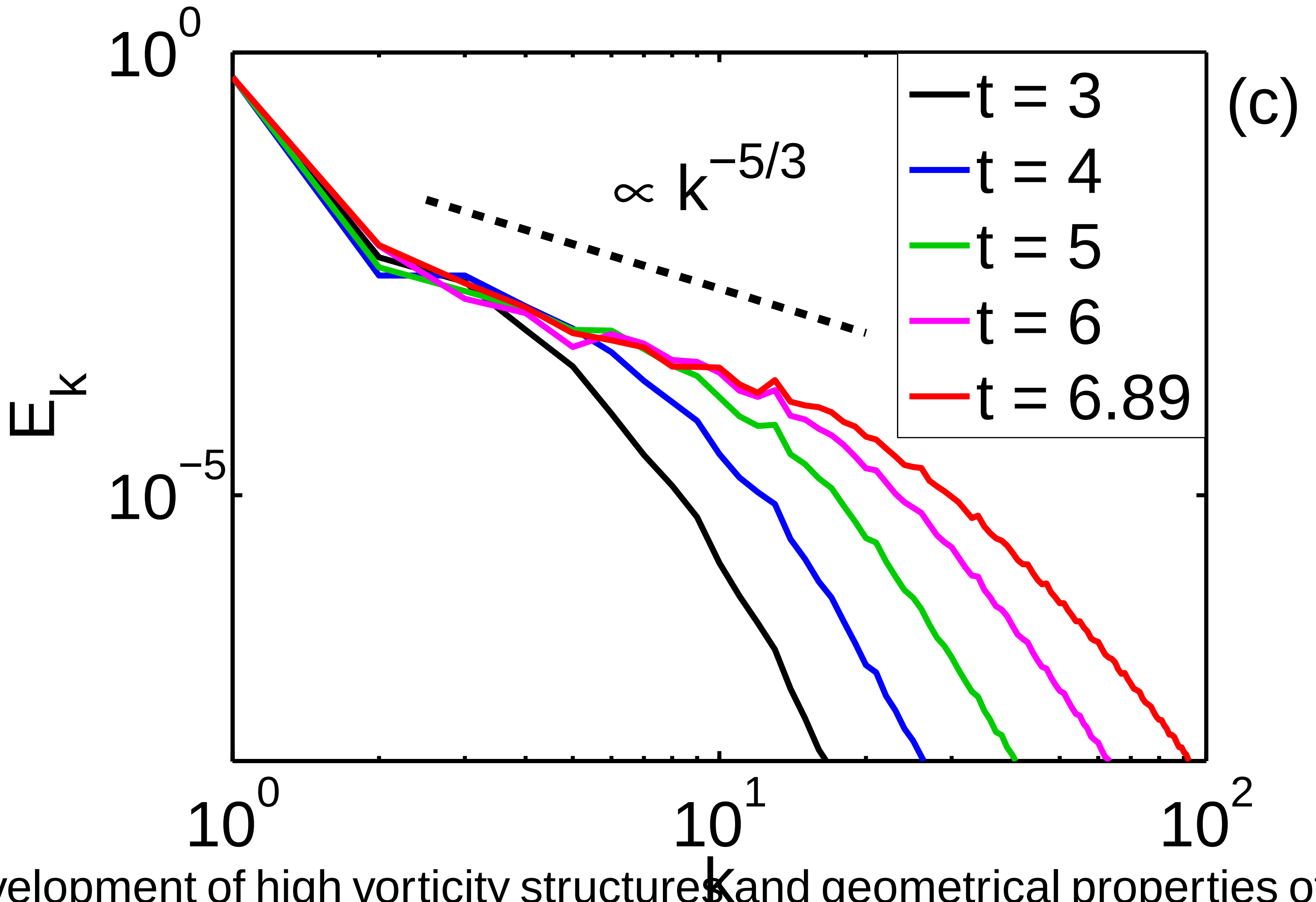
Numerical experiment: compressibility



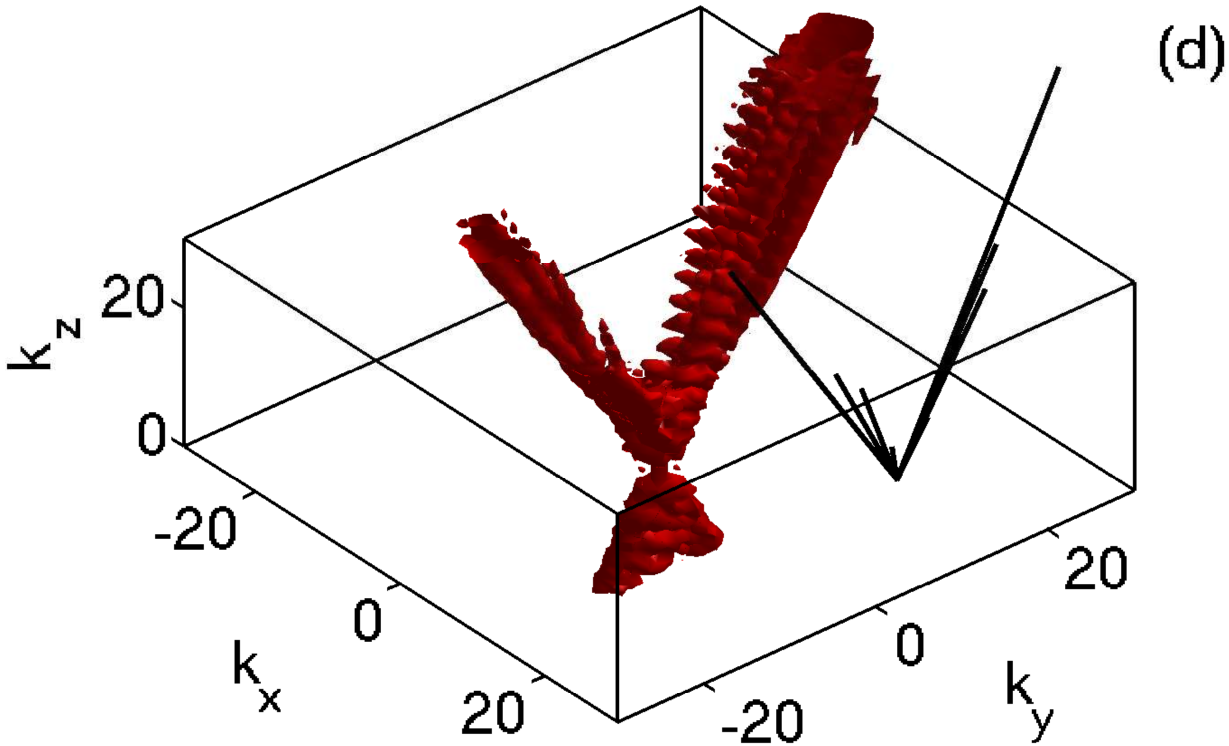
Behavior of γ_{ij} with time.

Numerical experiment

Energy spectrum at different times demonstrating the Kolmogorov power-law.



Numerical experiment: spectrum



JETS: Isosurface $|\tilde{\omega}(\mathbf{k})| = 0.2$ of the normalized vorticity field in $\mathbf{k}$- space at the final time. Solid lines show maximal $\mathbf{k}$-vectors for all jets (normalized by $1/\ell_1$).

Conclusion of the Ist part

In this talk, based on both VLR and direct numerical integration of 3D Euler, we show:

- At the stage of turbulence arising the spectrum is very far from isotropic (in the inertial interval).
- The main contribution in the spectrum in 3D is connected with appearance of coherent structures of the pancake type which in the turbulent spectrum are responsible for jets with growing in time anisotropy. (First time such structures were observed by M. Brachet, et.al. (1992).)
- The maximal pancake vorticity and its width ℓ are connected by means of the Kolmogorov type relation:

$$\omega_{max} \sim \ell^{-2/3}.$$

Conclusion of the Ist part

- Appearance of the pancake structures is a consequence of compressibility of the vorticity lines as it follows from the vortex line representation (K. & Ruban, 1998, K. 2002). These structures develop in time exponentially.
- Increasing with time number of such structures leads to formation of the Kolmogorov energy spectrum observed numerically in a fully inviscid flow, with no tendency towards finite-time blowup.

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