Dynamics in Siberia 26.02-4.03.2018

Connections on a principal G-bundle and related symplectic structures

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joint work with Grzegorz Jakimowicz and Aneta Sliżewska
„Symmetries of the space of connections on a principal G-bundle and related symplectic structures", arXiv:1712.08420

- G-principal bundle over a manifold M

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \Big\downarrow \mu \\ & & M \cong P/G \end{array}$$

where the free action of G we denote by

$$\kappa : P \times G \to P, \quad \kappa(p,g) := pg$$

and

$$\kappa_g : P \to P \qquad \kappa_g(p) := pg$$
$$\kappa_p : G \to P \qquad \kappa_p(g) := pg$$

• TG is a Lie group with the product and the inverse:

$$X_g \bullet Y_h := TL_g(h)Y_h + TR_h(g)X_g, \tag{1}$$

$$X_g^{-1} := -TL_{g^{-1}}(e) \circ TR_{g^{-1}}(g)X_g \tag{2}$$

where $X_g \in T_gG$, $Y_h \in T_hG$ and $L_g(h) := gh$, $R_g(h) := hg$.
For $e \in G$ - unit element of G and $\mathbf{0} : G \to TG$ - zero section of the tangent bundle TG one has

$$X_e \bullet Y_e = X_e + Y_e, \qquad \mathbf{0}_g \bullet \mathbf{0}_h = \mathbf{0}_{gh}, \tag{3}$$

$$X_g \bullet Y_e \bullet X_g^{-1} = (TR_{g^{-1}}(e) \circ TL_g(e))Y_e =: Ad_g Y_e \tag{4}$$

So, the Lie algebra T_eG could be considered as an abelian normal subgroup of TG and the zero section $\mathbf{0} : G \to TG$ is a group monomorphism.

The diffeomorphism

$$I : G \times T_eG \ni (g, X_e) \mapsto TR_g(e)X_e =: X_g \in TG \tag{5}$$

allows us to consider TG as the semidirect product $G \ltimes_{Ad_G} T_eG$ of G by the T_eG, where the group product of $(g, X_e), (h, Y_e) \in G \ltimes_{Ad_G} T_eG$ is given by

$$(g, X_e) \bullet (h, Y_e) = I^{-1}(I(g, X_e) \cdot I(h, Y_e)) = \tag{6}$$

$$= (gh, X_e + T(R_{g^{-1}} \circ L_g)(e)Y_e) = (gh, X_e + Ad_gY_e).$$

Using the above isomorphisms and the equality

$$\kappa_g \circ \kappa_p = \kappa_p \circ R_g, \tag{7}$$

we obtain the action

$$\Phi_{(g,X_e)}(v_p) = T\kappa_g(p)(v_p + T\kappa_p(e)X_e) \tag{8}$$

of $G \ltimes_{Ad_G} T_eG$ on the tangent bundle TP.
Applying the above action we obtain the following isomorphisms

$$TP/T^vP \cong TP/T_eG, \tag{9}$$
$$TP/TG \cong (TP/T_eG)/G \cong (TP/G)/T_eG, \tag{10}$$
$$TM = T(P/G) \cong TP/TG, \tag{11}$$

of vector bundles, where we write $T^vP := KerT\mu$ for the vertical subbundle of TP.

• We consider the group $Aut_0(TP)$ of smooth automorphisms $A : TP \to TP$ of the tangent bundle covering the identity map of P, i.e. for any $p \in P$ one has the map $A(p) : T_pP \to T_pP$ which is an isomorphism of the tangent space T_pP and $A(p)$ depends smoothly on p.

• $Aut_0(TP)$ is a normal subgroup of the group $Aut(TP)$ of all automorphisms of TP.

• The subgroup $Aut_{TG}(TP) \subset Aut_0(TP)$ consisting of those elements of $Aut_0(TP)$ whose action on TP commutes with the action (8) of $TG \cong G \ltimes_{Ad_g} T_eG$ on TP, i.e.

$$A(pg) \circ \Phi_{(g,X_e)} = \Phi_{(g,X_e)} \circ A(p). \tag{12}$$

• The group $Aut_{TG}(TP)$ acts also on vector bundles $TP/G \to M$ and $TM \to M$.

Proposition

$A \in Aut_{TG}(TP)$ if and only if

$$A(p) \circ T\kappa_p(e) = T\kappa_p(e) \qquad (13)$$

$$A(pg) \circ T\kappa_g(p) = T\kappa_g(p) \circ A(p) \qquad (14)$$

for any $g \in G$ and $p \in P$.

• We define the subgroup $Aut_N TP \subset Aut_{TG} TP$ consisting of $A \in Aut_{TG} TP$ such that $A(p) = id_p + B(p)$, where $B(p) : T_p P \to T_p^v P$. The conditions (13) and (14) on $A(p)$ written in terms of $B(p)$ assume the form

$$B(p) \circ T\kappa_p(e) = 0 \tag{15}$$

$$B(pg) \circ T\kappa_g(p) = T\kappa_g(p) \circ B(p). \tag{16}$$

From the definition of $B(p)$ and (15) one has $ImB(p) \subset T_p^v P \subset KerB(p)$. Thus it follows that $B_1(p)B_2(p) = 0$ for any $id + B_1,\ id + B_2 \in Aut_N P$. So, one has

$$A_1(p) \circ A_2(p) = (id_p + B_1(p))(id_p + B_2(p)) = id_p + B_1(p) + B_2(p) \tag{17}$$

for $A_1(p), A_2(p) \in Aut_N TP$. This shows that $Aut_N TP$ is a commutative subgroup of $Aut_{TG} TP$.
We will identify $Aut_N TP$ also with the vector subspace $End_N TP$ of the endomorphism $B : TP \to TP$, such that $ImB(p) \subset T_p^v P \subset KerB(p)$ for any $p \in P$.

• A connection form on P is a T_eG-valued differential one-form α satisfying the conditions

$$\alpha_p \circ T\kappa_p(e) = id_{T_eG} \tag{18}$$

$$\alpha_{pg} \circ T\kappa_g(p) = Ad_{g^{-1}} \circ \alpha_p \tag{19}$$

valid for value α_p of α at $p \in P$ and $g \in G$. Using α one defines the decomposition

$$T_pP = T_p^vP \oplus T_p^{\alpha,h}P \tag{20}$$

of T_pP on the vertical T_p^vP and the horizontal $T_p^{\alpha,h}P := Ker\alpha_p$ subspaces which satisfy the G-equivariance properties

$$T\kappa_g(p)T_p^vP = T_{pg}^vP, \tag{21}$$

$$T\kappa_g(p)T_p^{\alpha,h}P = T_{pg}^{\alpha,h}P. \tag{22}$$

From the decomposition (20) for any $p \in P$ one obtains the vector spaces isomorphism

$$\Gamma_\alpha(p) : T_{\mu(p)}M \to T_p^{\alpha,h}P \tag{23}$$

such that

$$\Gamma_\alpha(pg) = T\kappa_g(p) \circ \Gamma_\alpha(p) \tag{24}$$

and

$$T\mu(p) \circ \Gamma_\alpha(p) = id_{\mu(p)}, \qquad \Gamma_\alpha(p) \circ T\mu(p) = \Pi_\alpha^h(p), \tag{25}$$

where $\Pi_\alpha^h(p)$ is defined by the decomposition

$$id_p = \Pi_\alpha^v(p) + \Pi_\alpha^h(p) \tag{26}$$

of the identity map of T_pP on the projections corresponding to (20).

Proposition

(i) One has the following short exact sequence

$$\{0\} \to Aut_N TP \xrightarrow{\iota} Aut_{TG} TP \xrightarrow{\lambda} Aut_0 TM \to \{id\} \tag{27}$$

of the group morphisms, where ι is the inclusion map and λ is an epimorphism covering the identity map of M defined by

$$(\lambda(A)(\mu(p))(T\mu(p))v_p) := (T\mu(p) \circ A(p))v_p \tag{28}$$

for $v_p \in T_pP$.

Proposition

(ii) A fixed connection α defines the injection $\sigma_\alpha : Aut_0TM \to Aut_{TG}TP$ by

$$\sigma_\alpha(\tilde{A})(p) := \Pi^v_\alpha(p) + \Gamma_\alpha(p) \circ \tilde{A}(\mu(p)) \circ T\mu(p), \tag{29}$$

where $\tilde{A} \in Aut_0TM$, and the surjection $\beta_\alpha : Aut_{TG}TP \to Aut_NTP$ by $\beta_\alpha(A) := A\sigma_\alpha(\lambda(A))^{-1}$, where $A \in Aut_{TG}TP$, which are arranged into the short exact sequence

$$\{\mathrm{id}_{TM}\} \to Aut_0TM \xrightarrow{\sigma_\alpha} Aut_{TG}TP \xrightarrow{\beta_\alpha} Aut_NTP \to \{\mathrm{id}_{TP}\}, \tag{30}$$

inverse to the sequence (27). The map σ_α is a monomorphism

$$\sigma_\alpha(\tilde{A}_1\tilde{A}_2) = \sigma_\alpha(\tilde{A}_1)\sigma_\alpha(\tilde{A}_2)$$

of the groups and β_α satisfies

$$\beta_\alpha(A_1A_2) = \beta_\alpha(A_1)\sigma_\alpha(\lambda(A_1))\beta_\alpha(A_2)\sigma_\alpha(\lambda(A_1))^{-1}.$$

Proposition

(iii) The decomposition

$$A(p) = (\mathrm{id}_p + B(p))\sigma_\alpha(\tilde{A})(p) \tag{31}$$

of $A \in Aut_{TG}TP$, where $\mathrm{id}_p + B(p) \in Aut_N TP$ and $\tilde{A} \in Aut_0 TM$, defines an isomorphism of $Aut_{TG}TP$ with the semidirect product group $Aut_0 TM \ltimes_\alpha End_N TP$, where the product of $(\tilde{A}_1, B_1), (\tilde{A}_2, B_2) \in Aut_0 TM \ltimes_\alpha End_N TP$ is given by

$$[(\tilde{A}_1, B_1) \cdot (\tilde{A}_2, B_2)](p) := \tag{32}$$

$$= (\tilde{A}_1(\mu(p))\tilde{A}_2(\mu(p)), B_1(p) + B_2(p) \circ \Gamma_\alpha(p) \circ \tilde{A}_1^{-1}(\mu(p)) \circ T\mu(p)).$$

• Let $ConnP(M,G)$ be the space of all connections on $P(M,G)$. We define

$$\phi_A(\alpha)_p := \alpha_p \circ A(p)^{-1} \tag{33}$$

the left action $\phi_A : ConnP(M,G) \to ConnP(M,G)$ of $Aut_{TG}TP$ on $ConnP(M,G)$, i.e. ϕ satisfies $\phi_{A_1 A_2} = \phi_{A_1} \circ \phi_{A_1}$ for $A_1, A_2 \in Aut_{TG}TP$.

Proposition

There are the following properties:

1. The action of $Aut_{TG}TP$ defined in (33) is transitive.
2. The horizontal lift Γ_α defined by $\alpha \in ConnP(M,G)$, see (23), satisfies

$$A(p) \circ \Gamma_\alpha(p) = \Gamma_{\phi_A}(\alpha)(p) \circ \lambda(A)(\mu(p)) \qquad (34)$$

   for $A \in Aut_{TG}TP$.
3. The action (33) restricted to the subgroup $Aut_N TP$ is free and transitive.
4. The subgroup $\sigma_\alpha(Aut_0 TM)$ is the stabilizer of α with respect to the action (33).

The following proposition shows that one can define the group $Aut_{TG}TP$ in terms of connections space $ConnP(M,G)$.

Proposition

If $A \in Aut_0(TP)$ and $\phi_A(ConnP(M,G)) \subset ConnP(M,G)$ then $A \in Aut_{TG}(TP)$.

• We recall that the standard symplectic form on T^*P is $\omega_0 = d\gamma_0$, where $\gamma_0 \in C^\infty T^*(T^*P)$ is the canonical one-form on T^*P defined at $\varphi \in T^*P$ by

$$\langle \gamma_{0\varphi}, \xi_\varphi \rangle := \langle \varphi, T\pi^*(\varphi)\xi_\varphi \rangle,$$

where $\pi^* : T^*P \to P$ is the projection of T^*P on the base and $\xi_\varphi \in T_\varphi(T^*P)$.

• By definition a *linear vector field* on T^*P is a pair (ξ, χ) of vector fields $\xi \in C^\infty T(T^*P)$ and $\chi \in C^\infty TP$ such that

$$\begin{array}{ccc} T^*P & \xrightarrow{\quad \xi \quad} & T(T^*P) \\ \pi^* \big\downarrow & & \big\downarrow T\pi^* \\ P & \xrightarrow{\quad \chi \quad} & TP \end{array}$$

defines a morphism of vector bundles. Note here that $T\pi^*(\varphi)\xi_\varphi = \chi_{\pi^*(\varphi)}$.

• We will denote by $LinC^\infty T(T^*P)$ the Lie algebra of linear vector fields over the vector bundle $\pi^* : T^*P \to P$. The Lie bracket of (ξ_1, χ_1), $(\xi_2, \chi_2) \in LinC^\infty T(T^*P)$ is defined by

$$[(\xi_1, \chi_1), (\xi_2, \chi_2)] := ([\xi_1, \xi_2], [\chi_1, \chi_2])$$

and the vector space structure on $LinC^\infty T(T^*P)$ by

$$c_1(\xi_1, \chi_1) + c_2(\xi_2, \chi_2) := (c_1\xi_1 + c_2\xi_2, c_1\chi_1 + c_2\chi_2).$$

Let $LinC^\infty(T^*P)$ denote the vector space of smooth fibre-wise linear functions on T^*P. Spaces $LinC^\infty(T(T^*P))$ and $LinC^\infty(T^*P)$ have structures of $C^\infty(P)$-modules defined by $f(\xi, \chi) := ((f \circ \pi^*)\xi, f\chi)$ and by $fl := (f \circ \pi^*)l$, respectively, where $f \in C^\infty(P)$ and $l \in LinC^\infty(T^*P)$.

Definition

• A differential one-form $\gamma \in C^\infty T^*(T^*P)$ is called a *generalized canonical form* on T^*P if:

(i) $\gamma_\varphi \neq 0$ for any $\varphi \in T^*P$,

(ii) $kerT\pi^*(\varphi) \subset ker\ \gamma_\varphi := \{\xi_\varphi \in T_\varphi(T^*P) :\ \langle\gamma_\varphi, \xi_\varphi\rangle = 0\}$,

(iii) $\langle\gamma, \xi\rangle \in LinC^\infty(T^*P)$ for any $\xi \in LinC^\infty T(T^*P)$.

The space of generalized canonical forms on T^*P will be denoted by $CanT^*P$. Let us note here that $\gamma_0 \in CanT^*P$.

Proposition

(i) The map $\Theta : Aut_0TP \to CanT^*P$ defined by

$$\langle \Theta(A)_\varphi, \xi_\varphi \rangle := \langle \varphi, A(\pi^*(\varphi))T\pi^*(\varphi))\xi_\varphi \rangle, \quad (35)$$

where $\xi_\varphi \in T_\varphi(T^*P)$, is bijective.

Proposition

(ii) The natural left action $L^* : Aut_0TP \times CanT^*P \to CanT^*P$ of Aut_0TP on $CanT^*P$ defined by

$$\langle (L_A^*(\gamma))_\varphi, \xi_\varphi \rangle := \langle \gamma_{A^*(\varphi)}, TA^*(\varphi)\xi_\varphi \rangle, \tag{36}$$

where $A^* : T^*P \to T^*P$ is the dual of $A \in Aut_0TP$, is a transitive and free action. Furthermore,

$$L_A^* \circ \Theta = \Theta \circ L_A, \tag{37}$$

where $L_A A' := AA'$, i.e. $L_A^* \Theta(A') = \Theta(AA')$.

• From the above proposition we conclude that $\gamma \in CanT^*P$ is the pull-back $\gamma = \Theta(A) = L_A^*\gamma_0$ of the canonical form γ_0. So, $\omega_A := d\Theta(A)$ is a symplectic form.

• It is reasonable to define the space

$$Can_{TG}T^*P := \Theta(Aut_{TG}TP)$$

which is an $Aut_{TG}TP$-invariant subspace of the space $CanT^*P$.

Proposition

1. The generalized canonical form $\Theta(A)$ belongs to $Can_{TG}T^*P$ if and only if $(\phi_g^*)^*\Theta(A) = \Theta(A)$ and $J_A = J_0$.
2. One can consider $Can_{TG}T^*P$ as the orbit of the subgroup $Aut_{TG}TP \subset Aut_0TP$ taken through γ_0 with respect to the free action L^* defined in (36).
3. If $A \in Aut_0TP$ and $L_A^*(Can_{TG}T^*P) \subset Can_{TG}T^*P$ then $A \in Aut_{TG}TP$.

Corrolary

Fixing a connection α one obtains an embedding $\iota_\alpha : ConnP(M,G) \hookrightarrow Can_{TG}T^*P$ of the connection space into the space of generalized canonical forms defined as follows

$$\iota_\alpha(\alpha') := \varphi \circ T\pi^*(\varphi) + \varphi \circ T\kappa_{\pi^*(\varphi)}(e) \circ (\alpha'_{\pi^*(\varphi)} - \alpha_{\pi^*(\varphi)}) \circ T\pi^*(\varphi). \tag{38}$$

The symplectic form $d\iota_\alpha(\alpha')$ is the pullback $L^*_{\mathrm{id}_{TP}+B}\omega_0$ of the standard symplectic form ω_0 by the bundle morphism $(\mathrm{id}_{TP} + B)^* : T^*P \to T^*P$, where $\mathrm{id}_{TP} + B \in Aut_N TP$.

• A G-equivariant diffeomorphism $I_\alpha : T^*P \xrightarrow{\sim} \overline{P} \times T^*_e G$ dependent on a fixed connection α

$$I_\alpha(\varphi) := (\Gamma^*_\alpha(\pi^*(\varphi))(\varphi), \pi^*(\varphi), J_0(\varphi)),$$

where

$$\overline{P} := \{(\tilde{\varphi}, p) \in T^*M \times P : \ \tilde{\pi}^*(\tilde{\varphi}) = \mu(p)\}$$

is the total space of the principal bundle $\overline{P}(T^*M, G)$ being the pullback of the principal bundle $P(M, G)$ to T^*M by the projection $\tilde{\pi}^* : T^*M \to M$ of T^*M on the base M.

The correctness of the above definition follows from $\tilde{\pi}^* \circ \Gamma_\alpha^* = \mu \circ \pi^*$. The map $I_\alpha^{-1} : \overline{P} \times T_e^* G \to T^* P$ given by

$$I_\alpha^{-1}(\tilde{\varphi}, p, \chi) = \tilde{\varphi} \circ T\mu(p) + \chi \circ \alpha_p \tag{39}$$

is the inverse to I_α.

The natural right action of $Aut_{TG}TP$ on T^*P, defined for $A \in Aut_{TG}TP$ by $(A^*\varphi)(\pi^*(\varphi)) := \varphi \circ A(\pi^*(\varphi))$, and the action of G on T^*P defined in (??) transported by I_α to $\overline{P} \times T_e^*G$ are given by

$$\Lambda_\alpha(A)(\tilde{\varphi}, p, \chi) := (I_\alpha \circ A^* \circ I_\alpha^{-1})(\tilde{\varphi}, p, \chi) = \tag{40}$$

$$= ((\tilde{\varphi} \circ T\mu(p) + \chi \circ \alpha_p) \circ A(p) \circ \Gamma_\alpha(p), p, \chi)$$

and by

$$\psi_g^*(\tilde{\varphi}, p, \chi) := (I_\alpha \circ \phi_g^* \circ I_\alpha^{-1})(\tilde{\varphi}, p, \chi) = (\tilde{\varphi}, pg, Ad_{g^{-1}}^*\chi), \tag{41}$$

respectively.

Setting $A = \mathrm{id}_{TP} + B$ or $A = \sigma_\alpha(\tilde{A})$ in (40) we obtain

$$\Lambda_\alpha(\mathrm{id}_{TP} + B)(\tilde{\varphi}, p, \chi) = (\tilde{\varphi} + \chi \circ \alpha_p \circ B(p) \circ \Gamma_\alpha(p), p, \chi) \quad (42)$$

or

$$\Lambda_\alpha(\sigma_\alpha(\tilde{A}))(\tilde{\varphi}, p, \chi) = (\tilde{\varphi} \circ \tilde{A}, p, \chi), \quad (43)$$

respectively.

Facts

(i) The action Λ_α of $Aut_{TG}TP$ on $\overline{P} \times T^*_eG$ is reduced to an action of $Aut_{TG}TP$ on T^*M which preserves the cotangent spaces T^*_mM, $m \in M$, and is realized on them by affine maps, see (40), (42) and (43).

(ii) The action (41) of G does not change $\tilde{\varphi}$ and commute with the action (40) of the group $Aut_{TG}TP$.

Using $I_\alpha^{-1} : \overline{P} \times T_e^* G \to T^* P$ we pull the generalized canonical form $\Theta(A)$ back to $\overline{P} \times T_e^* G$. For $A = (\mathrm{id}_{TP} + B)\sigma_\alpha(\tilde{A})$ we have

$$(I_\alpha^{-1})^* \Theta(A)(\tilde{\varphi}, p, \chi) = \tag{44}$$

$$= \tilde{\varphi} \circ \tilde{A}(\mu(p)) \circ T(\tilde{\pi}^* \circ pr_1)(\tilde{\varphi}, p, \chi) + \chi \circ \alpha_p \circ A(p) \circ Tpr_2(\tilde{\varphi}, p, \chi) =$$

$$= pr_1^*(\tilde{\Theta}(\tilde{A})(\tilde{\varphi}, p, \chi) + \langle pr_3(\tilde{\varphi}, p, \chi), pr_2^*(\Phi_{A^{-1}}(\alpha))(\tilde{\varphi}, p, \chi)\rangle,$$

where $pr_3(\tilde{\varphi}, p, \chi) := \chi$.

- The symplectic form corresponding to (44) is given by

$$d((I_\alpha^{-1})^* \Theta(A)) = \tag{45}$$

$$= pr_1^*(d\tilde{\Theta}(\tilde{A})) + \langle d\, pr_3 \overset{\wedge}{,} pr_2^*(\Phi_{A^{-1}}(\alpha))\rangle + \langle pr_3, pr_2^*(d\Phi_{A^{-1}}(\alpha))\rangle.$$

Considering P as the configuration space of a physical system which has a symmetry described by G one consequently assumes that its Hamiltonian $H \in C^\infty(T^*P)$ is a G-invariant function on T^*P, i.e. $H \circ \phi_g^* = H$ for $g \in G$. Hence it is natural to consider the class of Hamiltonian systems on G-symplectic manifold (T^*P, ω_A, J_0) with a G-invariant Hamiltonians H.

Using the isomorphism $(T^*P, \omega_A, J_0) \cong (\overline{P} \times T_e^*G, (I_\alpha^{-1})^*\omega_A, pr_3)$ of G-symplectic manifolds, where the symplectic form $(I_\alpha^{-1})^*\omega_A$ is presented in (45) and the momentum map is $J_0 \circ I_\alpha = pr_3$, one defines the G-invariant Hamiltonian $H \in C^\infty(\overline{P} \times T_e^*G)$ as follows

$$H(\tilde{\varphi}, p, \chi) := (\tilde{H} \circ \overline{\mu})(\tilde{\varphi}, p, \chi) + (C \circ pr_3)(\tilde{\varphi}, p, \chi),$$

where $\overline{\mu} : \overline{P} \to T^*M$ is the projection of the total space $\overline{P}$ of the principal G-bundle $\overline{P}(T^*M, G)$ on the base T^*M and $\tilde{H} \in C^\infty(T^*M)$. Coming back to the phase space (T^*P, ω_A, J_0) one obtains the G-Hamiltonian system with the Hamiltonian

$$H_\alpha(\varphi) := (H \circ I_\alpha)(\varphi) = (\tilde{H} \circ \Gamma_\alpha^*)(\varphi) + (C \circ J_0)(\varphi)$$

where $C \in C^\infty(T_e^*G)$ is Casimir with respect to the standard Lie-Poisson structure

The G-invariance of the Hamiltonian system $(T^*P, \omega_A, J_0, H_\alpha)$ allows ones to apply the Marsden-Weinstein reduction procedure. For this reason we consider the dual pair of Poisson manifolds

$$\begin{array}{ccccc} & & (T^*P, \omega_A) & & \\ & \swarrow^{\pi_G^*} & & \searrow^{J_0} & \\ (T^*P/G, \{\cdot,\cdot\}_{A/_G}) & & & & (T_e^*G, \{\cdot,\cdot\}_{L-P}) \end{array} \tag{46}$$

The symplectic form ω_A is a G-invariant two-form.

The Poisson bracket $\{f, g\}_{A/G}$ of $f, g \in C^\infty(T^*P/G)$ is defined by $\{f \circ \pi_G^*, g \circ \pi_G^*\}_A$, where we identify $C^\infty(T^*P/G)$ with the Poisson subalgebra $C_G^\infty(T^*P) \subset C^\infty(T^*P)$ of G-invariant functions and $\{\cdot, \cdot\}_A$ is the Poisson bracket on $C^\infty(T^*P)$ defined by ω_A. By $\{\cdot, \cdot\}_{L-P}$ we denoted Lie-Poisson bracket on the dual T_e^*G of the Lie algebra T_eG.

Note that surjective submersions in (46) are Poisson maps and the Poisson subalgebras $(\pi_G^*)^*(C^\infty(T^*P/G))$ and $J_0^*(C^\infty(T_e^*G))$ are mutually polar. As a consequence of the above one obtains the one-to-one correspondence between the coadjoint orbits $\mathcal{O} \subset T_e^*G$ of G and the symplectic leaves $\mathcal{S} \subset T^*P/G$ of the Poisson manifold $(T^*P/G, \{\cdot,\cdot\}_{A/G})$ which is defined as follows

$$\mathcal{S} = \pi_G^*(J_0^{-1}(\mathcal{O})) \quad \text{and} \quad \mathcal{O} = J_0(\pi_G^{*\,-1}(\mathcal{S})).$$

Note that surjective submersions in (46) are Poisson maps and the Poisson subalgebras $(\pi_G^*)^*(C^\infty(T^*P/G))$ and $J_0^*(C^\infty(T_e^*G))$ are mutually polar. As a consequence of the above one obtains the one-to-one correspondence between the coadjoint orbits $\mathcal{O} \subset T_e^*G$ of G and the symplectic leaves $\mathcal{S} \subset T^*P/G$ of the Poisson manifold $(T^*P/G, \{\cdot,\cdot\}_{A/G})$ which is defined as follows

$$\mathcal{S} = \pi_G^*(J_0^{-1}(\mathcal{O})) \quad \text{and} \quad \mathcal{O} = J_0({\pi_G^*}^{-1}(\mathcal{S})).$$

Since $I_\alpha : T^*P \to \overline{P} \times T_e^*G$ is a G-equivariant map it defines a diffeomorphism

$$[I_\alpha] : T^*P/G \to \overline{P} \times_{Ad_G^*} T_e^*G$$

of the quotient manifolds which transports the Poisson structure $\{\cdot,\cdot\}_{A/G}$ of T^*P/G on the total space $\overline{P} \times_{Ad_G^*} T_e^*G$ of the vector bundle $\overline{P} \times_{Ad_G^*} T_e^*G \to T^*M$ over the symplectic manifold $(T^*M, d\tilde{\Theta}(\tilde{A}))$. Using (38) one obtains the isomorphisms $[I_{\alpha,\mathcal{O}}] = \pi_G^*(J_0^{-1}(\mathcal{O})) \xrightarrow{\sim} \overline{P} \times_{Ad_G^*} \mathcal{O}$ of symplectic leaves. If $A = \sigma_\alpha(\tilde{\mathrm{id}}_{TM}) = \mathrm{id}_{TP}$ one obtains the diffeomorphisms of symplectic leaves where the coadjoint orbit $\mathcal{O}$ is the phase space for inner degrees of freedom. In this case the symplectic manifold $(T^*M, d\tilde{\gamma}_0)$ is the phase space for external degrees of freedom and $\overline{P} \times_{Ad_G^*} \mathcal{O}$ is the total phase space of a classical particle interacted with Yang-Mills field described by α which was constructed by Sternberg.

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THANK YOU FOR ATTENTION