# On cascades with surface dynamics on $3$-manifolds

V. Grines, Nizhny Novgorod, Russia

## Goal of the lecture

The main goal is to present some results on interrelations between topology of $M^3$ and dynamics of cascades on it.

## Example given by R. Thom 1960

Represent $2$-torus $\mathbb{T}^2$ as the factor space $\mathbb{R}^2/\sim$, where $\sim$ means:
$(x,y) \sim (x',y')$ if there is a pair $m,n \in \mathbb{Z}$ such that
$(x' = x+m, y' = y+n$.
Consider automorphism $f_A : \mathbb{T}^2 \to \mathbb{T}^2$ given by formulas:

$$\begin{cases} \bar{x} = ax + by \mod 1 \\ \bar{y} = cx + dy \mod 1 \end{cases},$$

where matrice $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is integer, unimodular $(\det A = \pm 1)$ and hyperbolic (that is modulus of eigenvalues are not equal to one).
$f_A$ is R. Thom example which he explained to S. Smale.

## Hyperbolic invariant set

Let $f : M^n \to M^n$ be diffeomorphism given on closed manifold.

### Definition

*An invariant set* $\Omega(f)$ *is hyperbolic if there is continuous* $df$*-invariant splitting*

$$T_{\Omega(f)}M^n = E^s_{\Omega(f)} \oplus E^u_{\Omega(f)}$$

*of tangent subbundle* $T_{\Omega(f)}M^n$ *in sum of stable and unstable subbundels such that the following estimates hold:*

$$\|df^k(v)\| \leq C\lambda^k\|v\|, \quad \|df^{-k}(w)\| \leq C\lambda^k\|w\|$$

*for some real numbers* $C > 0$ *and* $0 < \lambda < 1$*,*
*and for any* $v \in E^s_{\Omega(f)}, w \in E^u_{\Omega(f)}, k \in \mathbb{N}$.

## The concept of topological conjugacy

**Definition**

*Two dynamical system* $\bar{x} = f(x)$, $\bar{x} = g(x)$, $x \in X$ *are called topologically conjugated if there is homeomorphism* $h : X \to X$ *such that* $h(f(x)) = g(h(x))$ *for any* $x \in X$.

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow h & & \downarrow h \\ X & \xrightarrow{g} & X \end{array}$$

# Anosov diffeomorphisms

## Definition

*Diffeomorphism $f : M^n \to M^n$ is called Anosov diffeomorphism if manifold $M^n$ is hyperbolic set.*

## Theorem (n=2 - Sinai, n=3 - Franks, Newhous, n>3 - Franks, Newhous, Manning.)

*Any anosov diffeomorphism $f : \mathbb{T}^n \to \mathbb{T}^n$ topologically conjugated to hyperbolic algebraic automorphism $\bar{x} = A_f x \mod 1$, where matrice $A_f$ is integer, unimodular $(\det\, A = \pm 1)$ and hyperbolic (that is modulus of eigenvalues are not equal to one).*

Moreover if $n = 3$ and $f : M^3 \to M^3$ is Anosov diffeomorphism, then $M^3$ is torus and $f$ is topologically conjugated to hyperbolic algebraic automorphism.

# A-diffeomorphisms

## Definition

*Diffeomorphism $f : M^n \to M^n$ is called $A$-diffeomorphism if $f$ satisfies to S. Smale axiom $A$, that is*

1. *nonwandering set $NW(f)$ is hyperbolic;*
2. *set of periodic points is dense in $NW(f)$.*

## Definition

*A diffeomorphism $f \in Diff(M^n)$ is called structural stable if there is a neighborhood $U(f)$ of $f$ in $Diff(M^n)$ such that if $f' \in U(f)$ then $f'$ and f are topologically conjugated.*

Axiom $A$ and the strong condition of transversality are necessary and sufficient condition for the structural stability of a diffeomorphism $f : M^n \to M^n$.

## Basic sets

According to S. Smale spectral theorem nonwandering set $NW(f)$ of any $A$-diffeomorphism is the union of pair disjoint closed invariant sets each of which contains dense orbit under action of diffeomorphism $f$:

$$NW(f) = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_l,$$

where $l \geq 1$.

Important example.

Let $f : M^n \to M^n$ be Anosov diffeomorphism ($n \geq 3$). If all leaves of unstable (stable) invariant foliation have dimension $n-1$ then $M^n$ is the torus $\mathbb{T}^n$ and nonwandering set $NW(f)$ consists of unique basic set which coincides with $\mathbb{T}^n$ and has topological dimension $n$. In partiucar It is true for $n = 2$ or $n = 3$.

## Two-dimesional basic sets

The future aim of my talk to describe situation then nonwandering set of $A$-diffeomorphism $f : M^3 \to M^3$ contains two-dimensional basic sets.

## Attrators and repellers

### Definition

*A basic set $\mathcal{B}$ of diffeomorphism $f : M^n \to M^n$ is called attractor if there is a closed neighborhood $U$ of the set $\mathcal{B}$ such that $f(U) \subset int\ U$, $\bigcap\limits_{j \geq 0} f^j(U) = \mathcal{B}$. An invariant set is called repeller if it is attractor for $f^{-1}$.*

According to R. Plykin any basic set $\mathcal{B}$ of A-diffeomorphisms $f : M^n \to M^n$ such that $dim\ \mathcal{B} = n - 1$ is attractor or repeller. In particular if $n = 2$ then any two-dimensionsl basic set is attractor or repeller.

# Examples of $A$-diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ with two-dimensional attractor and repller

$A$-diffeomorphisms $f$ given on $\mathbb{T}^3 = \mathbb{T}^2 \times S^1$ whose nonwandering set consists of exactly $2$-dimensional attractor and $2$-dimensional repeller being 2-dimension tori. Restrictions of diffeomorphism $f$ to each basic set is topologically conjugated with Anosov diffeomorphism.


$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

# Surface basic set

## Definition

*A basic set of diffeomorphism $f : M^3 \to M^3$ is called surface basic set if it belongs to a $f$-invariant closed 2-dimensional manifold $M^2$.*

## Theorem (Grines, Medvedev, Zhuzhoma. Mathematical Notes, 2005, 78:6.)

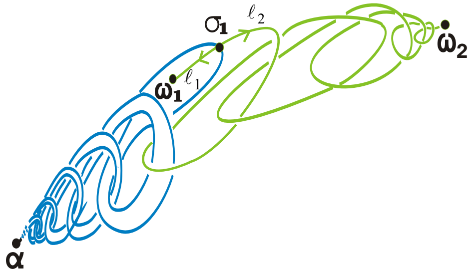
*Let $f : M^3 \to M^3$ diffeomorphism, nonwandering set of which contains a connected two-dimensional surface attractor $\mathcal{B}$. Then $\mathcal{B} = M^2$, $M^2$ is cylindrically embedded torus, and the restriction $f$ to $M^2$ is conjugated with an Anosov automorphism of the torus.*

## Remark

*The two-dimensional torus may be no smooth at any point (Kaplan J, Mallet-Parret J, Yorke J, 1984).*

## Widly embbeded invariant manifold

In 1977 D. Pixton constructed an example of Morse-Smale diffeomorphism whose nonwandering set consists of fixed source $\alpha$, fixed saddle point $\sigma$ and fixed sinks $\omega_1$, $\omega_2$ such that closer of two dimensional unstable manifold of $\sigma$ is wildly embedded sphere.



## The structure of the ambient manifold $M^3$

Denote by $M_\tau$ quotient space obtained from $\mathbb{T}^2 \times [0,1]$ by identifying the points $(z,1)$ and $(\tau(z),0)$, where $\tau : \mathbb{T}^2 \to \mathbb{T}^2$ be a homeomorphism.

**Theorem (V. Grines, Yu. Levchenko, V. Medvedev, O. Pochinka. Nonlinearity. 2015. Vol. 28. P. 4081-4102)**

*Let a closed oriented 3-manifold $M^3$ admits A-diffeomorphism $f$ such that nonwandering set $NW(f)$ consists of 2-dimensional surface basic sets.*
*Then $M^3$ is diffeomorphic to $M_{\widehat{J}}$, where $\widehat{J}$ algebraic automorphism of the torus given by the matrix $J$, which is either hyperbolic or coincides with the matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or with the matrix $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.*

## Topological classification of structurally stable diffeomorphisms on $M^3$ whose nonwandering set consists of 2-dimensional surface basic sets

Let us denote by $\Phi$ the class of model diffeomorphisms given on mapping tori $M_{\widehat{J}}$.

**Theorem (V. Grines,Yu. Levchenko, V. Medvedev, O. Pochinka, Nonlinearity. 2015. Vol. 28. P. 4081-4102)**

*Any sructurally stable diffeomorphism on $M^3$ whose nonwandering set consists of 2-dimensional surface attractors and repellers is topologically conjugated with some model diffeomorphism from class $\Phi$.*

## Expanding attrators and attracting repellers

If $\mathcal{B}$ is an attractor then for any point $x \in \mathcal{B}$ unstable manifold $W^u(x)$ belongs to $\mathcal{B}$.

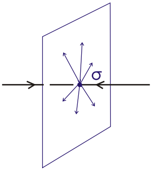
### Definition

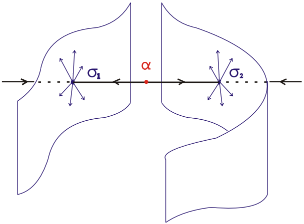
*An attractor $\mathcal{B}$ of $f$ is called expanding attractor of $f$ if it's topological dimension is equal to dimension of $W^u(x)$ for any point $x \in \mathcal{B}$. A repeller $\mathcal{B}$ is called attracting repeller if it is expanding attractor of $f^{-1}$.*

According to R. Plykin any expanding attractor of codimension one of $f : M^n \to M^n$ is locally homeomorphic to product of $(n-1)$-disk and Cantor set.

# S.Smale surgery operation. Examples of $2$-dimensional expanding attractor

Let $\bar{x} = Ax \mod 1$ ($x = (x_1, x_2, x_3)$) automorphism of torus $\mathbb{T}^3$, where matrice $A$ is integer, unimodular ($\det A = 1$) and hyperbolic, eigenvalues: $0 < \lambda_1 < 1, \lambda_2 > 1, \lambda_3 > 1$).
Applying to this automorphism S.Smale surgery operation we get DA-diffeomorphisms with 2-dimensional expanding attractor. Topological dimension of such attractor is equal to $2$ - dimension of unstable manifold of any point belonging to attractor.

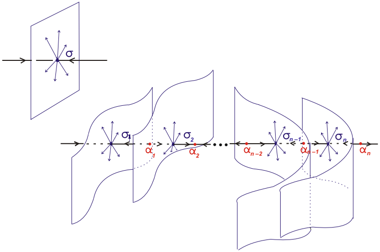


## Classification structurally stable diffeomorphisms with two-dimensional expanding attractors on $M^3$

**Theorem (Grines, Zhuzhoma. Trans. Amer. Math. Soc., 357 (2005).)**

*Let $f : M^3 \to M^3$ is structurally stable diffeomorphism, nonwandering set of which contains a two-dimensional expanding attractor. Then the manifold $M^3$ is diffeomorphic to the torus $\mathbb{T}^3$ and $f$ is topologically conjugated with the diffeomorphism obtained from Anosov diffeomorphism by the generalized surgery operation.*



# Ch. Bonatti problem. Topological structure of basic set of dimension 2

Solenoid

### Theorem (A. Brown, 2010)

*Any connected two-dimensional basic sets of diffeomorphisms of three-dimensional manifold is exactly one of the following:*

1. *expanding attractor*
2. *attracting repeller*
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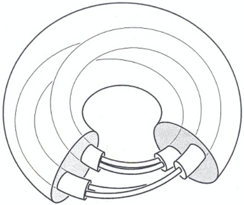

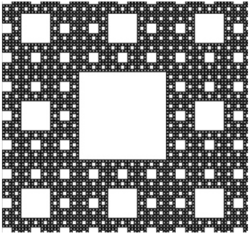

# Topological classification of structurally stable diffeomorphisms on $M^3$ whose nonwandering set consists of two-dimensional basic sets

**Theorem (Corallary from V. Grines, E. Zhuzoma, A. Brown results.)**

*Let $f : M^3 \to M^3$ be structurally stable diffeomorphism whose all basic set from $NW(f)$ has topological dimension 2. Then $\Omega(f)$ consists of surface basic sets.*

Let us denote by $\Phi$ the class of model diffeomorphisms given on mapping tori $M_{\widehat{J}}$.

**Theorem (V. Grines,Yu. Levchenko, V. Medvedev, O. Pochinka, Nonlinearity, 2015)**

*Any sructurally stable diffeomorphism on $M^3$ whose all basic sets from $NW(f)$ has topological dimension 2 is topologically conjugated with some model diffeomorphism from class $\Phi$.*

Developments in Mathematics 46

Viacheslav Z. Grines · Timur V. Medvedev · Olga V. Pochinka

## Dynamical Systems on 2- and 3-Manifolds

This book provides an introduction to the topological classification of smooth structurally stable diffeomorphisms on closed orientable 2- and 3-manifolds. The topological classification is one of the main problems of the theory of dynamical systems and the results presented in this book are mostly for dynamical systems satisfying Smale's Axiom A. The main results on the topological classification of discrete dynamical systems are widely scattered among many papers and surveys. This book presents these results fluidly, systematically, and, for the first time, in one publication. Additionally, this book discusses the recent results on the topological classification of Axiom A diffeomorphisms focusing on the nontrivial effects of the dynamical systems on 2- and 3-manifolds. The classical methods and approaches that are considered to be promising for further research are also discussed.

The reader needs to be familiar with the basic concepts of the qualitative theory of dynamical systems, which are presented in Part 1 for convenience. The book is accessible to ambitious undergraduates, graduates, and researchers in dynamical systems and low dimensional topology. This volume consists of 10 chapters; each chapter contains its own set of references and a section on further reading. Proofs are presented with the exact statements of the results. In Chapter 10 the authors briefly state the necessary definitions and results from algebra, geometry, and topology. When stating ancillary results at the beginning of each part, the authors refer to other readily available sources.

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Grines · Medvedev · Pochinka

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Springer

THANK YOU!

## Expanding attrators and attracting repellers

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According to R. Plykin any expanding attractor of codimension one of $f : M^n \to M^n$ is locally homeomorphic to product of $(n-1)$-disk and Cantor set.

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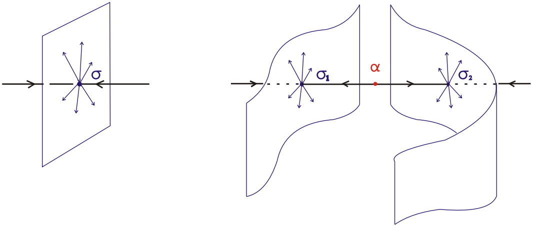
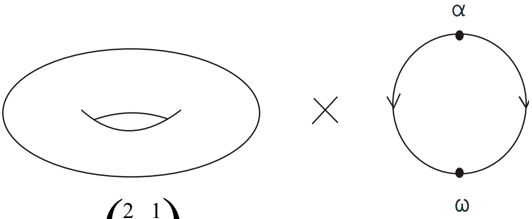
## $n = 3$, $M^3$– closed orientable 3-manifold. Examples of two-dimensional expanding attractor

DA-diffeomorphisms with 2-dimensional expanding attractors It means that topological dimension of such attractor is equal to dimension of unstable manifold of any point belonging to attractor.



## $n = 3$, $M^3$– closed orientable 3-manifold. Examples of surface two-dimensional attractors and repllers

Diffeomorphisms given on $\mathbb{T}^3$ with 2-dimensional attractor and repeller being 2-dimension tori. Restrictions of diffeomorphism to such basic set topologically conjugated with Anosov diffeomorphism.
It is clear that such basic set is not expanding attractor or attracting repeller.

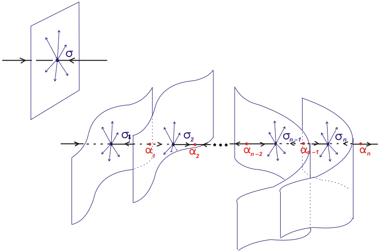



$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

## Two-dimensional expanding attractors and topology of an ambient manifold $M^3$

**Theorem (Grines, Zhuzhoma. Trans. Amer. Math. Soc., 357 (2005).)**

*Let $f : M^3 \to M^3$ is structurally stable diffeomorphism, nonwandering set of which contains a two-dimensional expanding attractor. Then the manifold $M^3$ is diffeomorphic to the torus $\mathbb{T}^3$ and $f$ is topologically conjugated with the diffeomorphism obtained from Anosov diffeomorphism by the generalized surgery operation.*



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### Remark

*The two-dimensional torus may be no smooth at any point (Kaplan J, Mallet-Parret J, Yorke J, 1984).*

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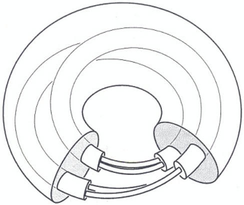

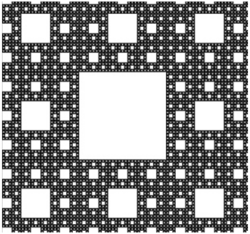

## The structure of the ambient manifold $M^3$

Denote by $M_\tau$ quotient space obtained from $\mathbb{T}^2 \times [0, 1]$ by identifying the points $(z, 1)$ and $(\tau(z), 0)$, where $\tau : \mathbb{T}^2 \to \mathbb{T}^2$ be a homeomorphism.

**Theorem (V.Z. Grines, V.S. Medvedev, Ya. A. Levchenko (2010))**

*Let nonwandering set of $f : M^3 \to M^3$ consists of a two-dimensional surface basic sets. Then there is a homeomorphism $\tau : \mathbb{T}^2 \to \mathbb{T}^2$ such that $M^3$ is diffeomorphic to $M_\tau$ .*

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## The structure of the ambient manifold $M^3$, specification

**Theorem (V. Grines, Yu. Levchenko, O. Pochinka 2012-2014)**

*Let a closed oriented 3-manifold $M^3$ admits A-diffeomorphism $f$ such that nonwandering set $NW(f)$ consists of 2-dimensional surface attractors and repellers.*
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**Theorem (Corallary from V. Grines, E. Zhuzoma, A. Brown results.)**

*Let $f : M^3 \to M^3$ be structurally stable diffeomorphism whose nonwandering set $\Omega(f)$ has topological dimension 2. Then $\Omega(f)$ consists of surface basic sets.*

Let us denote by $\Phi$ the class of model diffeomorphisms given on mapping tori $M_{\widehat{J}}$.

**Theorem (V. Grines,Yu. Levchenko, V. Medvedev, O. Pochinka, Nonlinearity, 2015)**

*Any sructurally stable diffeomorphism on $M^3$ with two dimensional nonwandering set is topologically conjugated with some model diffeomorphism from class $\Phi$.*

THANK YOU!

## Structurally stable diffeomorphism on $M^{2m+1}$ does not admit non-orientable attractors

**Theorem (Zhuzhoma E., Medvedev V. (2005))**

*Let* $f : M \to M$ *be a structurally stable diffeomorphism on a closed manifold* $M^{2m+1}$, $m \geq 1$. *Then* $f$ *has no codimension one non-orientable expanding attractors.*

**Corollary (Grines, Zhuzhoma, Medvedev)**

*Let* $f : M^{2m+1} \to M^{2m+1} (m \geq 1)$ *is structurally stable diffeomorphism, nonwandering set of which contains two-dimension expanding attractor. Then the manifold* $M^{2m+1}$ *is diffeomorphic to the torus* $\mathbb{T}^3$ *and* $f$ *is topologically conjugated with the diffeomorphism obtained from 3-Anosov diffeomorphism by the generalized surgery operation.*

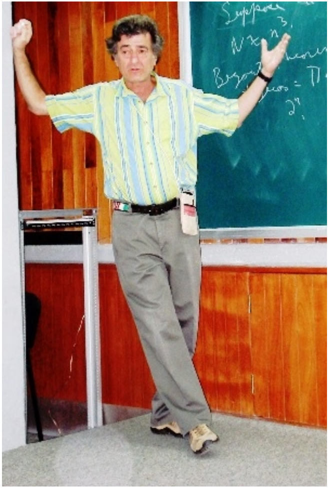

**Figure:** M. Shub

back

**Figure:** D. Sullivan

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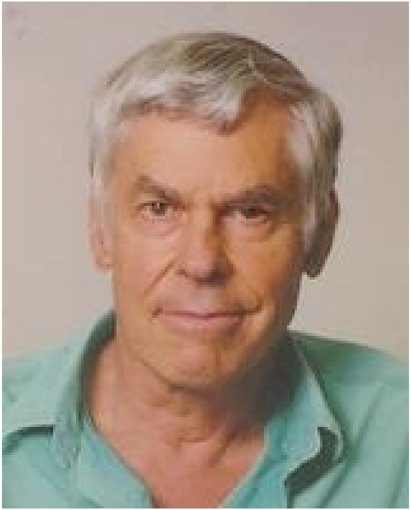

**Figure:** S. Smale

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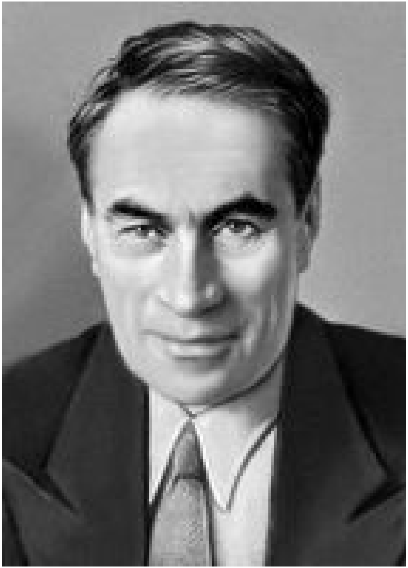

**Figure:** A. Andronov

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**Figure:** L. Pontryagin

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**Figure:** A. Lyapunov

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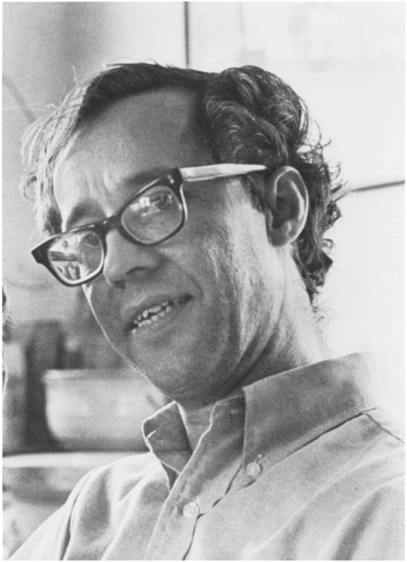

**Figure:** C. Conley

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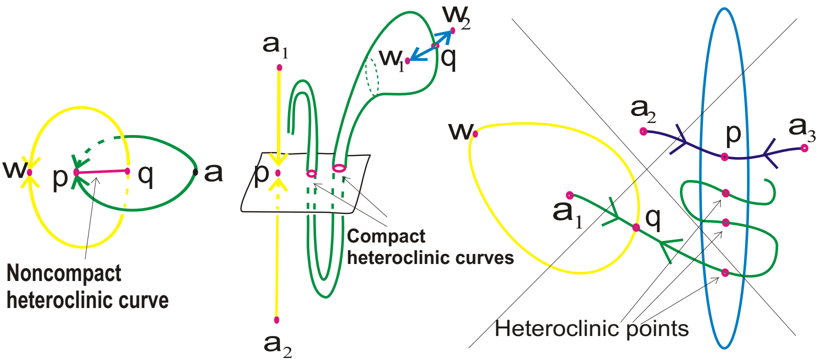


**Figure:** Heteroclinic intersections

**Figure:** D. Pixton

back

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**Figure:** E. Artin

**Figure:** R. Fox

back

**Figure:** Cr. Bonatti

back


**Figure:** V. Grines

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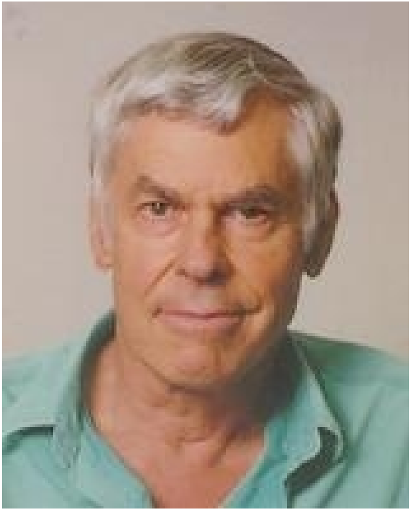

**Figure:** S. Smale

back

**Figure:** H. Debrunner

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**Figure:** R. Fox

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**Figure:** Harrold O.G.

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**Figure:** Griffith H.C.

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**Figure:** Posey E.E.

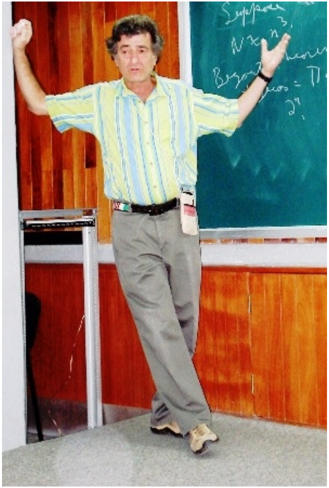

**Figure:** M. Shub

back

**Figure:** D. Sullivan

back


**Figure:** V. Grines

back

**Figure:** V. Medvedev

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**Figure:** E. Zhuzhoma

back

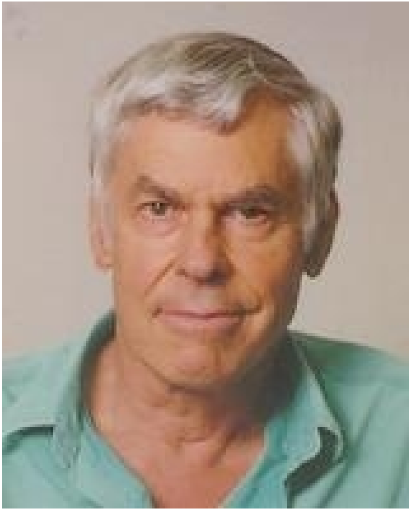


**Figure:** S. Smale

back

**Figure:** K. Meyer

**Figure:** F. Takens

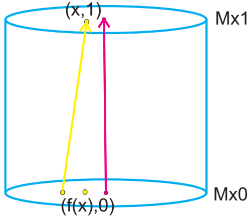
back

**Figure:** D. Pixton

back

## Suspension for a Morse-Smale diffeomorphism

Let $f : M^n \to M^n$ be Morse-Smale diffeomorphism. Denote by $W^{n+1}$ the manifold which is obtained from Cartesian product $M^n \times [0,1]$ by identification of pairs of points $(x,1)$ and $(f(x),0)$ for $x \in M^n$. Define suspension $\xi_f$ as "vertical" vector field $\frac{\partial}{\partial t}$ on $W^{n+1}$. By the construction $\xi_f$ is a Morse-Smale vector field.




back