Magnetic Laplacians, generalized Bergman kernels and Berezin-Toeplitz quantization on symplectic manifolds

Yuri A. Kordyukov

Russian Academy of Sciences, Ufa, Russia

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Classical phase space

- (X, ω) — a symplectic manifold, $\dim X = 2n$.
- $f, g \in C^\infty(X) \mapsto \{f, g\} \in C^\infty(X)$ the Poisson bracket

$$\{f, g\} = \omega(X_f, X_g),$$

X_f is the Hamiltonian vector field of $f \in C^\infty(X)$.

A quantization of (X, ω) is a family of maps

$$f \in C^\infty(X) \mapsto T^h(f) \in \mathcal{L}(\mathcal{H}^h),$$

$\mathcal{L}(\mathcal{H}^h)$ is the algebra of bounded linear operators in a Hilbert space $\mathcal{H}^h$, satisfying the conditions:

- $\|T^h(f)\| = \|f\| + \mathcal{O}(h)$, $h \to 0$.
- $[T^h(f), T^h(g)] = T^h(\{f, g\}) + \mathcal{O}(h)$, $h \to 0$.

- Geometric quantization.
  Kostant, Souriau (1970)
- Berezin-Toeplitz quantization.
  F. A. Berezin (1974), L. Boutet de Monvel, V. Guillemin (1981)
  For compact Kähler manifolds:
  M. Bordemann, E. Meinrenken, M. Schlichenmaier (1994)
- Spin^c Dirac quantization.
  M. Vergne, V. Guillemin (1994)
  For compact symplectic manifolds
  X. Ma, G. Marinescu (2008)
- Bochner Laplacian quantization.
  V. Guillemin - A. Uribe (1988)
  For compact almost-Kähler manifolds
  Borthwick- A. Uribe (1996)
  For compact symplectic manifolds
  Yu. K. (2017), L. Ioos - W. Lu - X. Ma - G. Marinescu, 2017

Hermitian line bundle

(L, h^L, ∇^L) a Hermitian line bundle on a smooth manifold X:

- $L \to X$ a complex line bundle on X:
  locally, over some open $\Omega \subset X$, $L|_\Omega \cong \Omega \times \mathbb{C}$.
- h^L a Hermitian structure in the fibers of L:

$$s, s' \in L \to (s, s')_{h^L} \in \mathbb{C},$$

- ∇^L a connection (covariant derivative): for $U \in C^\infty(X, TX)$

$$\nabla^L_U : C^\infty(X, L) \to C^\infty(X, L),$$

  which is Hermitian:

$$\nabla^L_U (s, s')_{h^L} = (\nabla^L_U s, s')_{h^L} + (s, \nabla^L_U s')_{h^L}, \quad s, s' \in C^\infty(X, L).$$

Example

- $X = \mathbb{R}^{2n}$, $L = X \times \mathbb{C} \to X$ the trivial line bundle, $C^\infty(X, L) \cong C^\infty(X)$.
- The Hermitian structure is given by $h \in C^\infty(X)$: for $z \in \mathbb{R}^{2n}$

$$|s|_h^2 = h(z)|s|^2, \quad s \in L_z = \{z\} \times \mathbb{C};$$

- The connection

$$\nabla_U^L = \frac{\partial}{\partial U} + \Gamma(U), \quad U \in TX,$$

$\Gamma = \sum_{j=1}^{2n} \Gamma_j(z) dz^j \in \Omega^1(X)$ the connection form;
- ∇^L is Hermitian $\Leftrightarrow \Gamma + \bar{\Gamma} = -h^{-1} dh$.

Prequantum bundle

Let (L, h^L, ∇^L) be a Hermitian line bundle on X.
The curvature of ∇^L is the differential two-form R^L on X:

$$R^L(U, V) = \nabla^L_U \nabla^L_V - \nabla^L_V \nabla^L_U - \nabla^L_{[U,V]}, \quad U, V \in TX.$$

For the connection $\nabla^L_U = \frac{\partial}{\partial U} + \Gamma(U)$, its curvature is given by

$$R^L = d\Gamma.$$

A prequantum bundle is a Hermitian line bundle (L, h^L, ∇^L), satisfying:

$$\frac{i}{2\pi} R^L = \omega.$$

(X, ω) is called quantizable $\Leftrightarrow$ a prequantum bundle exists
$(\Leftrightarrow [\omega] \in H^2(X, \mathbb{Z}))$.

Example: the 2-sphere

- X the two-dimensional sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\},$$

  equipped with the Riemannian metric induced by the standard Euclidean metric in $\mathbb{R}^3$.

- ω is a scalar multiple of the volume 2-form dx_g:

$$\omega = s\, dx_g, \quad s \in \mathbb{R}\ .$$

- (X, ω) is quantizable $\Leftrightarrow$ the area $4\pi s = n \in \mathbb{Z}$.
- The corresponding prequantum line bundle (L_n, ∇_n) is a well-known Wu-Yang magnetic monopole, which provides a natural topological interpretation of Dirac's monopole of magnetic charge $g = nh/2e$.

The Bochner-Laplacian

g a Riemannian metric on X.
∇^{TX} the Levi-Civita connection of g.

The Bochner-Laplacian Δ^L associated with a Hermitian line bunlde (L, h^L, ∇^L) acts on $C^\infty(X, L)$:

$$\Delta^L = (\nabla^L)^* \nabla^L.$$

If $\{e_j\}_{j=1,\dots,2n}$ is an orthonormal frame of TX, then Δ^L is given by

$$\Delta^L = -\sum_j \left[(\nabla^L_{e_j})^2 - \nabla^L_{\nabla^{TX}_{e_j} e_j} \right].$$

Example: magnetic Laplacian

- $X = \mathbb{R}^{2n}$, $L = X \times \mathbb{C}$ the trivial line bundle.
- The Hermitian structure ($h(z) = 1$):

$$|s(z)|_h^2 = |s(z)|^2.$$

- The connection form

$$\Gamma = -i\mathbf{A},$$

  where

$$\mathbf{A} = \sum_{j=1}^{2n} A_j(X) dX_j$$

  is a real-valued one form (a magnetic potential).

Example: magnetic Laplacian

- The Bochner-Laplacian is the magnetic Schrödinger operator:

$$\Delta^L = -\sum_{j=1}^{2n} \left(\frac{\partial}{\partial X_j} - iA_j(X) \right)^2 .$$

- The curvature $R^L (= d\Gamma) = -i\mathbf{B}$.
  B is a real-valued two form (the magnetic field):

$$\mathbf{B} = \sum_{j,k=1}^{2n} B_{jk}(X) dX_j \wedge dX_k, \quad B_{jk} = \frac{\partial A_k}{\partial X_j} - \frac{\partial A_j}{\partial X_k};$$

- $\omega (= \frac{i}{2\pi} R^L) = \frac{1}{2\pi}\mathbf{B}$.
- So **B** is non-degenerate (of full rank).

The key observation in $n=2$:
$\mathbf{A}=A_1(X)dX_1+A_2(X)dX_2$, $\mathbf{B}=B_{12}(X)dX_1\wedge dX_2$.

$$\left(\left(\frac{\partial}{\partial X_1}-iA_1(X)\right)+i\left(\frac{\partial}{\partial X_2}-iA_2(X)\right)\right)^*\times \\ \times\left(\left(\frac{\partial}{\partial X_1}-iA_1(X)\right)-i\left(\frac{\partial}{\partial X_2}+iA_2(X)\right)\right).$$

$$=-\left(\left(\frac{\partial}{\partial X_1}-iA_1(X)\right)-i\left(\frac{\partial}{\partial X_2}-iA_2(X)\right)\right)\times \\ \times\left(\left(\frac{\partial}{\partial X_1}-iA_1(X)\right)+i\left(\frac{\partial}{\partial X_2}-iA_2(X)\right)\right).$$

$$=-\left(\frac{\partial}{\partial X_1}-iA_1(X)\right)^2-\left(\frac{\partial}{\partial X_2}-iA_2(X)\right)^2-\frac{\partial A_2}{\partial X_1}+\frac{\partial A_1}{\partial X_2}$$

$$=\Delta^L-B_{12}.$$

The renormalized Bochner-Laplacian

- $J_0 : TX \to TX$ a skew-adjoint linear endomorphism:

$$\omega(u,v) = g(J_0 u, v), \quad u, v \in TX;$$

- τ is a smooth function on X given by

$$\tau(x) = \pi \operatorname{Tr}[(-J_0^2(x))^{1/2}], \quad x \in X.$$

- L^p the p-th tensor power of L, $p \in \mathbb{N}$;
- $\nabla^{L^p}_U : C^\infty(X, L^p) \to C^\infty(X, L^p)$ the induced connection on L^p:

$$\nabla^{L^p}_U = \frac{\partial}{\partial U} + p\Gamma^L(U), \quad U \in TX.$$

Definition (V. Guillemin - A. Uribe, 1988)

The renormalized Bochner-Laplacian Δ_p acts on $C^\infty(X, L^p)$:

$$\Delta_p = \Delta^{L^p} - p\tau.$$

Magnetic Laplacian

- $X=\mathbb{R}^{2n}$, $L=X\times\mathbb{C}$ the trivial line bundle.
- The Hermitian structure $|s(z)|_h^2=|s(z)|^2$.
- The connection form $\Gamma=-i\mathbf{A}$, where $\mathbf{A}=\sum_{j=1}^{2n}A_j(X)dX_j$ is a real-valued one form.
- The Bochner-Laplacian

$$\Delta^{L^p}=-\sum_{j=1}^{2n}\left(\frac{\partial}{\partial X_j}-ipA_j(X)\right)^2,\quad p=\frac{1}{\hbar}.$$

- $J_0=\frac{1}{2\pi}B$, where $B:TX\to TX$ be a skew-adjoint operator

$$\mathbf{B}(u,v)=g(Bu,v),\quad u,v\in TX.$$

- $\tau(x)=\frac{1}{2}\operatorname{Tr}(B^*B)^{1/2}=\operatorname{Tr}^+(B)$.

Complex manifolds

- $X = \mathbb{C}^n$, $L = X \times \mathbb{C}$ the trivial line bundle.
- The Hermitian structure is given by $h \in C^\infty(X)$: for $z = x + iy \in \mathbb{C}^n$
$$|s|_h^2 = h(z)|s|^2, \quad s \in L_z;$$
- The Hermitian connection
$$\nabla^L = d + \Gamma, \quad \Gamma + \bar{\Gamma} = -h^{-1}dh;$$

  Assume that Γ is compatible with the complex structure of $\mathbb{C}^n$ (a holomorphic Hermitian connection — the Chern connection), then, Γ is a $(1,0)$-form:
$$\Gamma = \partial \log h = \sum_{j=1}^{n} h^{-1}\frac{\partial h}{\partial z_j}dz_j;$$

Complex manifolds

- The curvature $R = d\Gamma$ is a purely imaginary 2-form: $(1,1)$-form

$$R = \bar{\partial}\partial \log h.$$

- For the symplectic form ω, we have

$$\omega = \frac{i}{2\pi}\bar{\partial}\partial \log h.$$

- ω is positive if $h = e^{-\varphi}$, $\varphi : X \to \mathbb{C}$ a smooth strictly plurisubharmonic function:

$$\omega = \frac{i}{2\pi}\sum_{j,k=1}^{n} \frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k, \quad \left(\frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k}\right)_{j,k=1,\ldots,n} > 0.$$

Kähler manifolds

- A particular case: the Hermitian structure is given by

$$|s|_h^2 = h(z)|s|^2, \quad h(z) = e^{-\frac{\pi}{2}|z|^2};$$

- The connection form

$$\Gamma = \partial \log h = -\pi \sum_{j=1}^{n} \bar{z}_j dz_j;$$

- The symplectic form ω is the canonical symplectic form:

$$\omega = \frac{i}{2\pi}\bar{\partial}\partial \log h = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\bar{z}_j = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

- J_0 is a complex structure, the standard complex structure on $\mathbb{C}^n$.
- ω is a Kähler form on $\mathbb{C}^n$ and $(\mathbb{C}^n, J_0)$ is a Kähler manifold.

Kähler manifolds

- $\tau(z) = \pi \operatorname{Tr}[(-J_0^2(z))^{1/2}] = 2\pi n, z \in X.$
- The renormalized Bochner-Laplacian:

$$\Delta_p = -\sum_j \left[(\nabla^{L^p}_{\partial/\partial x_j})^2 + (\nabla^{L^p}_{\partial/\partial y_j})^2 \right] - 2\pi np$$

$$= -\sum_j \left[\left(\frac{\partial}{\partial x_j} - \pi p \bar{z}_j \right)^2 + \left(\frac{\partial}{\partial y_j} - \pi i p \bar{z}_j \right)^2 \right] - 2\pi np$$

$$= -\sum_j \left[\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} - \pi p \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right]$$

- For the Kodaira-Laplacian $\square^{L^p} = \bar{\partial}^{L^p *} \bar{\partial}^{L^p}$, we have

$$\square^{L^p} = -\frac{1}{2} \sum_j \left(\frac{\partial}{\partial z_j} - \pi p \bar{z}_j \right) \frac{\partial}{\partial \bar{z}_j}.$$

The almost complex structure

- $J_0 : TX \to TX$ a skew-adjoint linear endomorphism such that

$$\omega(u, v) = g(J_0 u, v), \quad u, v \in TX;$$

- $J : TX \to TX$ the linear endomorphism given by

$$J = J_0(-J_0^2)^{-1/2}.$$

- J is an almost complex structure on X, $J^2 = -Id_{TX}$, compatible with ω and g:

$$\omega(Ju, Jv) = \omega(u, v), \quad g(Ju, Jv) = g(u, v), \quad u, v \in TX.$$

- ω is positive: for $u \in TX \setminus 0$

$$\omega(u, Ju) = -g(JJ_0 u, u) = g((-J_0^2)^{1/2} u, u) > 0.$$

- If $J_0 = J$ and J is integrable, then (X, J) is a Kähler manifold.

Spectral gap

Theorem (Guillemin-Uribe, 1988; Ma-Marinescu, 2002)

There exists $C_L > 0$ such that for any p

$$\sigma(\Delta_p) \subset [-C_L, C_L] \cup [2p\mu_0 - C_L, +\infty),$$

where the constant μ_0 is given by

$$\mu_0 = \inf_{u \in T_xX, x \in X} \frac{iR^L_x(u, J(x)u)}{|u|^2_g}.$$

Example

$$\mu_0 = \inf_{u \in TX} \frac{|(B^*B)^{1/4}u|^2_g}{|u|^2_g} = \inf_{x \in X} \inf (B^*B(x))^{1/2}.$$

Generalized Bergman projection

- $\mathcal{H}_p$ the linear subspace of $L^2(X, L^p)$ spanned by the eigensections of Δ_p corresponding to eigenvalues in $[-C_L, C_L]$.
- $P_{\mathcal{H}_p}$ the orthogonal projection in $L^2(X, L^p)$ onto $\mathcal{H}_p$ (generalized Bergman projection).

Example

(X, ω) a compact Kaehler manifold, L a holomorphic line bundle:

- $\Delta_p = 2\square^{L^p}$, where $\square^{L^p}$ is the Kodaira Laplacian on L^p:

$$\sigma(\Delta_p) \subset \{0\} \cup [2p\mu_0 - C_L, +\infty),$$

- $\mathcal{H}_p$ is the space $H^0(X, L^p)$ of holomorphic sections of L^p.
- $P_{\mathcal{H}_p}$ the usual Bergman projection.

Toeplitz operators

A Toeplitz operator is a sequence of bounded linear operators $T_p : L^2(X, L^p) \to L^2(X, L^p), p \in \mathbb{N}$:

- For any $p \in \mathbb{N}$, we have

$$T_p = P_{\mathcal{H}_p} T_p P_{\mathcal{H}_p}.$$

- There exists a sequence $g_l \in C^\infty(X)$ such that

$$T_p = P_{\mathcal{H}_p} \left(\sum_{l=0}^{\infty} p^{-l} g_l \right) P_{\mathcal{H}_p} + \mathcal{O}(p^{-\infty}),$$

  i.e. for any natural k there exists $C_k > 0$ such that

$$\left\| T_p - P_{\mathcal{H}_p} \left(\sum_{l=0}^{k} p^{-l} g_l \right) P_{\mathcal{H}_p} \right\| \leqslant C_k p^{-k-1}.$$

Algebra of Toeplitz operators

Theorem (Yu.K., 2017, Ioos-Lu-Ma-Marinescu, 2017)

The product $T_{f,p}T_{g,p}$ of the Toeplitz operators

$$T_{f,p}=P_{\mathcal{H}_p}fP_{\mathcal{H}_p},\quad T_{g,p}=P_{\mathcal{H}_p}gP_{\mathcal{H}_p},\quad f,g\in C^\infty(X),$$

is a Toeplitz operator. It admits the asymptotic expansion

$$T_{f,p}T_{g,p}=\sum_{r=0}^{\infty}p^{-r}T_{C_r(f,g),p}+\mathcal{O}(p^{-\infty}),$$

with some $C_r(f,g)\in C^\infty(X)$, where C_r are bidifferential operators:

$$C_0(f,g)=fg,\quad C_1(f,g)-C_1(f,g)=i\{f,g\},$$

where $\{f,g\}$ is the Poisson bracket on $(X,2\pi\omega)$.

Generalized Bergman kernels

- $\mathcal{H}_p$ the linear subspace of $L^2(X, L^p)$ spanned by the eigensections of Δ_p corresponding to eigenvalues in $[-C_L, C_L]$.
- $P_{\mathcal{H}_p}$ the orthogonal projection in $L^2(X, L^p)$ onto $\mathcal{H}_p$ (generalized Bergman projection).

Definition

The generalized Bergman kernel of Δ_p is the smooth kernel P_p of the operator $P_{\mathcal{H}_p}$ with respect to the Riemannian volume form dv_X.

Example

(X, ω) a compact Kaehler manifold, L a holomorphic line bundle:

- $\Delta_p = 2\square^{L^p}$, where $\square^{L^p}$ is the Kodaira Laplacian on L^p.
- $\mathcal{H}_p$ is the space $H^0(X, L^p)$ of holomorphic sections of L^p.
- P_p the usual Bergman kernel.

Off-diagonal estimates

Theorem (Yu. K. 2017)

There exists $c > 0$ such that for any $k \in \mathbb{N}$, there exists $C_k > 0$ such that for any $p \in \mathbb{N}$, $x, x' \in X$, we have

$$|P_p(x,x')|_{C^k} \leqslant C_k p^{n+\frac{k}{2}} e^{-c\sqrt{p}\rho(x,x')}.$$

- *ρ is the geodesic distance;*
- *$|P_p(x,x')|_{C^k}$ denotes the pointwise C^k-seminorm of the section $P_p \in C^\infty(X \times X, (L^p \otimes E) \boxtimes (L^p \otimes E)^*)$ at a point $(x,x') \in X \times X$.*

Remark

The estimate holds for noncompact Riemannian manifolds of bounded geometry (Yu. K., X. Ma, G. Marinescu, in preparation).

Normal coordinates

Fix $x_0 \in X$.

- Identification via the exponential map:

$$\exp_{x_0}^X : B^{T_{x_0}X}(0, a^X) \subset T_{x_0}X \xrightarrow{\cong} B^X(x_0, a^X) \subset X;$$

  where a^X is the injectivity radius of (X, g), $B^{T_{x_0}X}(0, a^X)$ and $B^X(x_0, a^X)$ are the open balls.
- A trivialization of the line bundle L over $B^X(x_0, a^X)$, identifying the fiber L_Z of L at $Z \in B^{T_{x_0}X}(0, a^X) \cong B^X(x_0, a^X)$ with L_{x_0} by parallel transport with respect to the connection ∇^L along the curve $\gamma_Z : [0, 1] \ni u \to \exp_{x_0}^X(uZ)$.
- κ a smooth positive function on $B^{T_{x_0}X}(0, a^X) \cong B^X(x_0, a^X)$:

$$dv_X(Z) = \kappa(Z) dv_{TX}(Z), \quad Z \in B^{T_{x_0}X}(0, a^X).$$

  where dv_X is the Riemannian volume form of (X, g) and dv_{TX} the Riemannian volume form $(T_{x_0}X, g_{x_0})$.

The asymptotic expansion (Ma-Marinescu, 2008)

For any $x_0 \in X$ and for $Z, Z' \in T_{x_0}X$

$$\frac{1}{p^n} P_p(Z, Z') \sim \sum_{r=0}^{+\infty} F_r(\sqrt{p}Z, \sqrt{p}Z')\kappa^{-\frac{1}{2}}(Z)\kappa^{-\frac{1}{2}}(Z')p^{-\frac{r}{2}},$$

where $|Z|, |Z'| < c/\sqrt{p}$, $c > 0$ - near off-diagonal asymptotics

$$F_r(Z, Z') = J_r(Z, Z')\mathcal{P}_{x_0}(Z, Z'),$$

$J_r(Z, Z')$ are polynomials in Z, Z' with the same parity as r and $\deg J_r \leqslant 3r$;
$\mathcal{P}_{x_0}(Z, Z')$ the Bergman kernel in the complex space $T_{x_0}X$.

The proof is based on local index theory, in particular, analytic localization technique due to J.-M. Bismut.

The Bergman kernel $\mathcal{P}_{x_0}$

The function $\mathcal{P}_{x_0}(Z, Z')$ is the Bergman kernel of the operator

$$\mathcal{L}_0 = -\sum_{j=1}^{2n}\left(\frac{\partial}{\partial e_j} + \frac{1}{2}R^L_{x_0}(Z, e_j)\right)^2 - \tau(x_0),$$

- $\{e_j\}_{j=1,\dots,2n}$ is an orthonormal base in $T_{x_0}X$.
- $\frac{\partial}{\partial U}$ the ordinary differentiation operator on $T_{x_0}X$ in the direction $U \in T_{x_0}X$.

Thus, $\mathcal{P}_{x_0}(Z, Z')$ is the smooth Schwartz kernel (with respect to $dv_{TX}(Z)$) of the orthogonal projection in $L^2(T_{x_0}X)$ to the kernel of $\mathcal{L}_0$.

The Bergman kernel $\mathcal{P}_{x_0}$

The almost complex structure J_{x_0} induces a splitting

$$T_{x_0}X \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}_{x_0}X \oplus T^{(0,1)}_{x_0}X,$$

where $T^{(1,0)}_{x_0}X$ and $T^{(0,1)}_{x_0}X$ are the eigenspaces of J_{x_0} corresponding to eigenvalues i and $-i$ respectively.

$$\mathcal{J}_{x_0} = -2\pi i J_0.$$

Then $\mathcal{J}_{x_0} : T^{(1,0)}_{x_0}X \to T^{(1,0)}_{x_0}X$ is positive, and $\mathcal{J}_{x_0} : T_{x_0}X \to T_{x_0}X$ is skew-adjoint. Denote by $\det_{\mathbb{C}}$ the determinant function of the complex space $T^{(1,0)}_{x_0}X$.

$$\mathcal{P}_{x_0}(Z, Z') = \frac{\det_{\mathbb{C}} \mathcal{J}_{x_0}}{(2\pi)^n} \exp\left(-\frac{1}{4}\langle(\mathcal{J}^2_{x_0})^{1/2}(Z - Z'), (Z - Z')\rangle + \frac{1}{2}\langle\mathcal{J}_{x_0}Z, Z'\rangle\right).$$

Off-diagonal estimates of the remainder (Yu.K., 2017)

There exists $\varepsilon \in (0, a^X)$ such that, for any $j, m, m' \in \mathbb{N}$, there exist positive constants C, c and M such that for any $p \geqslant 1$ and $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon$, we have

$$\sup_{|\alpha|+|\alpha'|\leqslant m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}} \left(\frac{1}{p^n} P_p(Z, Z') \right.\right.$$
$$\left.\left. - \sum_{r=0}^{j} F_r(\sqrt{p}Z, \sqrt{p}Z')\kappa^{-\frac{1}{2}}(Z)\kappa^{-\frac{1}{2}}(Z')p^{-\frac{r}{2}} \right) \right|_{\mathcal{C}^{m'}(X)}$$
$$\leqslant Cp^{-\frac{j-m+1}{2}}(1+\sqrt{p}|Z|+\sqrt{p}|Z'|)^M \exp(-c\sqrt{\mu_0 p}|Z-Z'|) + \mathcal{O}(p^{-\infty}),$$

- $\mathcal{C}^{m'}(X)$ is the $\mathcal{C}^{m'}$-norm for the parameter $x_0 \in X$.
- $G_p = \mathcal{O}(p^{-\infty})$ if for any $l, l_1 \in \mathbb{N}$, there exists $C_{l,l_1} > 0$ such that $\mathcal{C}^{l_1}$-norm of G_p is dominated by $C_{l,l_1} p^{-l}$.

Toeplitz operators

The Toeplitz operator

$$T_{f,p}=P_{\mathcal{H}_p}fP_{\mathcal{H}_p}:L^2(X,L^p\otimes E)\to L^2(X,L^p\otimes E).$$

The Schwartz kernel of $T_{f,p}$ is given by

$$T_{f,p}(x,x')=\int_X P_p(x,x'')f(x'')P_p(x'',x')dv_X(x'').$$

Off-diagonal estimates of kernels of Toeplitz operators

For any $\varepsilon>0$ and $k\in\mathbb{N}$, there exists $C_k>0$ such that for any $p\geqslant 1$ and $(x,x')\in X\times X$ with $d(x,x')>\varepsilon$,

$$|T_{f,p}(x,x')|_{C^k}\leqslant Ce^{-c\sqrt{p}\rho(x,x')}.$$

Full off-diagonal expansion

Let $f \in C^\infty(X, \mathrm{End}(E))$. There exist polynomials $Q_{r,x_0}(f) \in \mathrm{End}(E_{x_0})[Z, Z']$ such that, for any $k \in \mathbb{N}$:

$$p^{-n} T_{f,p,x_0}(Z, Z') \cong \sum_{r=0}^{k} (Q_{r,x_0}(f)\mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}),$$

There exist $\varepsilon' \in (0, a_X]$ and $C_0 > 0$ with the following property: for any $k, l \in \mathbb{N}$, there exist $C > 0$ and $M > 0$ such that for any $x_0 \in X$, $p \geqslant 1$ and $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon'$, we have

$$\left| p^{-n} T_{f,p,x_0}(Z, Z')\kappa^{\frac{1}{2}}(Z)\kappa^{\frac{1}{2}}(Z') - \sum_{r=0}^{k} (Q_{r,x_0}\mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z')p^{-\frac{r}{2}} \right|_{\mathcal{C}^l(X)}$$
$$\leqslant Cp^{-\frac{k+1}{2}}(1 + \sqrt{p}|Z| + \sqrt{p}|Z'|)^M \exp(-\sqrt{C_0 p}|Z - Z'|) + \mathcal{O}(p^{-\infty}).$$

The coefficients $Q_{r,x_0}(f)$

The polynomials $Q_{r,x_0}(f) \in \mathrm{End}(E_{x_0})[Z, Z']$ have the same parity as r and are given by

$$Q_{r,x_0}(f) = \sum_{r_1+r_2+|\alpha|=r} \mathcal{K}\left[F_{0,r_1,x_0}, \frac{\partial^\alpha f_{x_0}}{\partial Z^\alpha}(0)\frac{Z^\alpha}{\alpha!}F_{0,r_2,x_0}\right],$$

In particular,

$$Q_{0,x_0}(f) = f(x_0),$$
$$Q_{1,x_0}(f) = f(x_0)F_{1,x_0} + \mathcal{K}\left[F_{0,x_0}, \frac{\partial f_{x_0}}{\partial Z_j}(0)Z_j F_{0,x_0}\right].$$

Here for any polynomials $F, G \in \mathbb{C}[Z, Z']$, the polynomial $\mathcal{K}[F, G] \in \mathbb{C}[Z, Z']$ is given by the composition of the operators.

Composition of kernels

For any polynomial $F \in \mathbb{C}[Z, Z']$, consider the operator $F\mathcal{P}$ in $L^2(T_{x_0}X) \cong L^2(\mathbb{R}^{2n})$ with the kernel $(F\mathcal{P})(Z, Z')$ with respect to dZ:

$$F\mathcal{P}u(Z) = \int_{T_{x_0}X} F(Z, Z')\mathcal{P}_{x_0}(Z, Z')u(Z')\, dZ', \quad Z \in T_{x_0}X.$$

For any polynomials $F, G \in \mathbb{C}[Z, Z']$, define the polynomial $\mathcal{K}[F, G] \in \mathbb{C}[Z, Z']$ by the condition

$$((F\mathcal{P}) \circ (G\mathcal{P}))(Z, Z') = (\mathcal{K}[F, G]\mathcal{P})(Z, Z'),$$

where $(F\mathcal{P}) \circ (G\mathcal{P})$ is the composition of the operators $F\mathcal{P}$ and $G\mathcal{P}$ in $L^2(T_{x_0}X)$.

Characterization of Toeplitz operators

A sequence $\{T_p : L^2(X, L^p) \to L^2(X, L^p)\}$ of bounded linear operators is a Toeplitz operator if and only if:

- For any $p \in \mathbb{N}$, we have
$$T_p = P_{\mathcal{H}_p} T_p P_{\mathcal{H}_p}.$$
- There exists $c > 0$ such that, for any $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists $C_k > 0$ such that for any $p \geqslant 1$ and $(x, x') \in X \times X$ with $d(x, x') > \varepsilon$,
$$|T_{f,p}(x, x')|_{C^k} \leqslant Ce^{-c\sqrt{p}\rho(x,x')}.$$
- There exist a family of polynomials $\mathcal{Q}_{r,x_0} \in \mathrm{End}(E_{x_0})[Z, Z']$, depending smoothly on x_0, of the same parity as r and $\varepsilon' \in (0, a_X/4)$ such that, for any $k \in \mathbb{N}$, $x_0 \in X$, $Z, Z' \in T_{x_0}X$, $|Z|, |Z'| < \varepsilon'$, we have
$$p^{-n} T_{p,x_0}(Z, Z') \cong \sum_{r=0}^{k} (\mathcal{Q}_{r,x_0} \mathcal{P}_{x_0})(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{-\frac{k+1}{2}}).$$

Perspectives

The algebra of Toeplitz operators is an analogue of the algebra of semiclassical pseudodifferential operators in $\mathbb{R}^n$:

$$A = a\left(x, \frac{h}{i}\frac{d}{dx}\right), \quad a \in S(1).$$

The magnetic pseudodifferential calculus in $\mathbb{R}^n$:

Iftimie, V.; Mantoiu, M.; Purice, R.
Magnetic pseudodifferential operators. Publ. Res. Inst. Math. Sci. 43 (2007), no. 3, 585–623

or with deformation quantization:

Karasev, M. V.; Osborn, T. A.
Cotangent bundle quantization: entangling of metric and magnetic field. J. Phys. A 38 (2005), no. 40, 8549–8578.

Perspectives

Semiclassical eigenvalue asymptotics (Kähler manifolds):

- Deleporte, A., Low-energy spectrum of Toeplitz operators: the case of wells, arXiv:1609.05680, to appear in J. Spectral Theory.
- Deleporte, A., Low-energy spectrum of Toeplitz operators with a miniwell, arXiv:1610.05902.

Quasimodes constructions:

- Ioos, L., Quantization and isotropic submanifolds, arXiv:1802.09930.

Perspectives

Applications to the Bochner Laplacian:

The Bochner-Laplacian

$$\Delta^{L^p} := (\nabla^{L^p})^* \nabla^{L^p} = \Delta_p + p\tau,$$

where the renormalized Bochner-Laplacian Δ_p satisfies the gap property: for any p

$$\sigma(\Delta_p) \subset [-C_L, C_L] \cup [2p\mu_0 - C_L, +\infty),$$

$\mathcal{H}_p$ corresponds to the lowest Landau levels.