Quantum chaos, Hurwitz numbers. integrable systems

Aleksandr Orlov

Novosibirsk, March 2018
devoted to the memory of Boris Valerianovich Chirikov

$\mathbb{C}P^1$ case

A.Okounkov, R.Pandparihande
T.Ekehdal,S.Lando,M.Shapiro,A.Wainstein(ELSW)
I.Goulden, D.Jackson
A.Mironov, A.Morozov, S.Natanzon, A.Aleksandrov
V.Kazarian, S.Lando [KP hierarchy]
S.Shadrin, P.Dunin-Barkovskij, M.Mulase
S.Rangoolam, P.Zograf, L.Chekhov [a matrix model]
A.O., J.Harnad, S.Natanzon

$\mathbb{R}P^2$ case
[not yet properly studied]:
A.O. and S.Natanzon - we found the relation
to integrable systems and matrix integrals

A model of quantum chaos: Wigner-Dysson unitary enssemble

Probability measure

$$d\mu(M) = c\prod_{i\geq j}^{N} dX_{ij}e^{-X_{ij}^2}\prod_{i>j}^{N} dY_{ij}e^{-Y_{ij}^2}$$

where $M = X + iY$ is $N \times N$ Hermitian matrix and c is the normalization constant defined via $\int d\mu(M) = 1$.

The expectation

$$< f >= \int f(M)d\mu(M)$$

Spectral correlation functions

Given set $\lambda = (\lambda_1, \dots, \lambda_k)$, $\lambda_1 \geq \dots \geq \lambda_k > 0$
Theorem. For $\lambda_1 + \dots + \lambda_k =: |\lambda| \leq N$:

$$< \mathrm{tr}\, M^{\lambda_1} \cdots \mathrm{tr}\, M^{\lambda_k} > =$$

$$z_\lambda N^{|\lambda|} \sum_\chi N^\chi \sum_{\text{all} * \text{allowed by Riem}-\text{Hurwitz}} \mathrm{Hur}_{\mathrm{S}^2}^{\Sigma_\chi}(2^{\mathrm{L}}, \lambda, *)$$

Here $\mathrm{Hur}_{\mathrm{S}^2}^{\Sigma_\chi}(2^{\mathrm{L}}, \lambda, *)$ counts branched d-fold ($d = |\lambda| = 2L$) non-equivalent covers of the Riemann sphere by Riemann surfaces Σ_χ of genus g ($\chi = 2 - 2g$) with 3 critical points with profiles λ, 2^L and any profile * whose length $\ell(*)$ is defined by Riemman-Hurwitz formula $\chi = \ell(*) - L + k$. For $\lambda = (1^{m_1} 2^{m_2} \cdots)$, $z_\lambda := \prod_{i=1}^\infty i^{m_i} m_i!$

Hurwitz numbers (orientable case). Definition

Let partitions $\Delta^{(1)},\dots,\Delta^{(\mathrm{F})}$ be of the same weight d
Def. $\mathrm{Hur}_{\Sigma_{\chi^{\mathrm{base}}}}^{\Sigma_\chi^{\mathrm{cov}}}(\Delta^{(1)},\dots,\Delta^{(\mathrm{F})})$ is the number of the solutions to

$$X_1\cdots X_F\prod_{i=1}^{g^{\mathrm{base}}} a_i b_i a_i^{-1} b_i^{-1} = 1$$

devided by $d!$. Here $X_1,\dots,X_F,a_1,b_1,\dots,a_g,b_g\in S_d$ and each X_i belongs to the cyclic class $C_{\Delta^{(i)}}$, $\chi^{\mathrm{base}} = 2-2g^{\mathrm{base}}$.
(Topol. interpretation: The fundamental group of $\Sigma_{\chi^{\mathrm{base}}}$ without F points is generated by $x_1,\dots,x_F,A_1,B_1,\dots,A_g,B_g$ with the relation $x_1\cdots x_F\prod_{i=1}^{g^{\mathrm{base}}} A_iB_iA_i^{-1}B_i^{-1}=1$. Hurw number enumerates such hom ρ from fund group to S_d that $\rho(x_i)\in C_i$)

Hurwitz numbers (non-orientable case). Definition

Let partitions $\Delta^{(1)},\dots,\Delta^{(\mathrm{F})}$ be of the same weight d
Def. $\mathrm{Hur}_{\Sigma_{\chi^{\mathrm{base}}}}^{\Sigma_{\chi}^{\mathrm{cov}}}(\Delta^{(1)},\dots,\Delta^{(\mathrm{F})})$ is the number of the solutions to

$$X_1\cdots X_F\prod_{i=1}^{g^{\mathrm{base}}}R_i^2=1$$

devided by $d!$. Here $X_1,\dots,X_F,R_1,\dots,R_g\in S_d$ and each X_i belongs to the cyclic class $C_{\Delta^{(i)}}$, $\chi^{\mathrm{base}}=2-g^{\mathrm{base}}$.
(The fundamental group of $\Sigma_{\chi^{\mathrm{base}}}$ without F points is generated by $x_1,\dots,x_F,r_1,\dots,r_g$ with the relation $x_1\cdots x_F\prod_{i=1}^{g^{\mathrm{base}}}r_i^2=1$)

Geometrical definition

Let us consider a connected compact surface without boundary Ω and a branched covering $f : \Sigma \to \Omega$ by a connected or non-connected surface Σ. We will consider a covering f of the degree d. It means that the preimage $f^{-1}(z)$ consists of d points $z \in \Omega$ except some finite number of points. This points are called *critical values of* f. We consider only isolated critical points. Consider the preimage $f^{-1}(z) = \{p_1, \dots, p_\ell\}$ of $z \in \Omega$. Denote by δ_i the degree of f at p_i. It means that in the neighborhood of p_i the function f is homeomorphic to $x \mapsto x^{\delta_i}$. The set $\Delta = (\delta_1 \dots, \delta_\ell)$ is the partition of d, that is called *topological type of* z.

Fix now points $z_1, \dots, z_{\mathrm{F}}$ and partitions $\Delta^{(1)}, \dots, \Delta^{(\mathrm{F})}$ of d. Denote by

$$\widetilde{C}_{\Omega(z_1 \dots, z_{\mathrm{F}})}(d; \Delta^{(1)}, \dots, \Delta^{(\mathrm{F})})$$

the set of all branched covering $f : \Sigma \to \Omega$ with critical points $z_1, \dots, z_{\mathrm{F}}$ of topological types $\Delta^{(1)}, \dots, \Delta^{(\mathrm{F})}$.

Geometrical definition

Coverings $f_1 : \Sigma_1 \to \Omega$ and $f_2 : \Sigma_2 \to \Omega$ are called isomorphic if there exists an homeomorphism $\varphi : \Sigma_1 \to \Sigma_2$ such that $f_1 = f_2\varphi$. Denote by $\mathtt{Aut}(f)$ the group of automorphisms of the covering f. Isomorphic coverings have isomorphic groups of automorphisms of degree $|\mathtt{Aut}(f)|$.
Consider now the set $C_{\Omega(z_1\dots,z_{\mathrm{F}})}(d;\Delta^{(1)},\dots,\Delta^{(\mathrm{F})})$ of isomorphic classes in $\widetilde{C}_{\Omega(z_1\dots,z_{\mathrm{F}})}(d;\Delta^{(1)},\dots,\Delta^{(\mathrm{F})})$. This is a finite set. The sum

$$H^{\mathrm{E,F}}(d;\Delta^{(1)},\dots,\Delta^{(\mathrm{F})}) = \sum_{f\in C_{\Omega(z_1\dots,z_{\mathrm{F}})}(d;\Delta^{(1)},\dots,\Delta^{(\mathrm{F})})} \frac{1}{|\mathtt{Aut}(f)|} \quad ,$$

don't depend on the location of the points $z_1\dots,z_{\mathrm{F}}$ and is called *Hurwitz number*.

Geometrical definition. Examples

Example 1. Let $f : \Sigma \to \mathbb{CP}^1$ be a covering without critical points. Then, each d-fold cover is the disjoint union of d Riemann spheres: $\mathbb{CP}^1 \coprod \cdots \coprod \mathbb{CP}^1$, then $|\mathrm{Aut} f| = d!$ and $H^{2,0}(d) = \frac{1}{d!}$
Example 2. Let $f : \Sigma \to \mathbb{CP}^1$ be a d-fold covering with two critical points with the profiles $\Delta^{(1)} = \Delta^{(2)} = (d)$. (One may think of $f = x^d$). Then $H_{S^2}^{S_2}((d),(d)) = \frac{1}{d}$. Let us note that Σ is connected in this case and its Euler characteristic $\chi = 2$.

Geometrical definition. Examples

Example 3. Let $f:\Sigma\to\mathbb{RP}^2$ be a covering without critical points. Then, if Σ is connected, then $\Sigma=\mathbb{RP}^2$, $\deg f=1$ or $\Sigma=S^2$, $\deg f=2$. Next, if $d=3$, then $\Sigma=\mathbb{RP}^2\coprod\mathbb{RP}^2\coprod\mathbb{RP}^2$ or $\Sigma=\mathbb{RP}^2\coprod S^2$. Thus $H^{1,0}(3)=\frac{1}{3!}+\frac{1}{2!}=\frac{2}{3}$.
Example 4. Let $f:\Sigma\to\mathbb{RP}^2$ be a covering with a single critical point with profile Δ, and Σ is connected. Due to RH the $\chi=\ell(\Delta)$. (One may think of $f=z^d$ defined in the unit disc where we identify z and $-z$ if $|z|=1$). In case we cover the Riemann sphere by the Riemann sphere $z\to z^m$ we get two critical points with the same profiles. However we cover $\mathbb{RP}^2$ by the Riemann sphere, then we have the composition of the mapping $z\to z^m$ on the Riemann sphere and the factorization by antipodal involution $z\to-\frac{1}{\bar z}$. Thus we have the ramification profile (m,m) at the single critical point 0 of $\mathbb{RP}^2$. The automorphism group consists of rotations on $\frac{2\pi}{m}$ and antipodal involution $z\to-\frac{1}{\bar z}$.

Geometrical definition. Examples

Thus we get that

$$H_{RP^2}\left(2m;(m,m)\right)=\frac{1}{2m}$$

From RiemHurw we see that $1=\ell(\Delta)$ in this case. Now let us cover $\mathbb{RP}^2$ by $\mathbb{RP}^2$ via $z\to z^d$. We see that $\ell(\Delta)=1$. For even d we have the critical point 0, in addition each point of the unit circle $|z|=1$ is critical (a folding), while from the beginning we restrict our consideration only on isolated critical points. For odd $d=2m-1$ there is the single critical point 0, the automorphism group consists of rotations on the angle $\frac{2\pi}{2m-1}$. Thus in this case

$$H^{1,1}_{RP^2}\left(2m-1;(2m-1)\right)=\frac{1}{2m-1}$$

Mednykh-Pozdnyakova formula for Hurwitz numbers (both for orientable and non-or. cases)

Theorem (Mednykh-Pozdnyakova)

$$\mathrm{Hur}_{\Sigma_{\chi^{\mathrm{base}}}}^{\Sigma_\chi^{\mathrm{cov}}}(\Delta^{(1)},\dots,\Delta^{(\mathrm{F})}) = \sum_\lambda \left(\frac{\dim\lambda}{\mathrm{d}!}\right)^{\chi^{\mathrm{base}}} \mathrm{f}_\lambda(\Delta^{(1)})\cdots \mathrm{f}_\lambda(\Delta^{(\mathrm{F})})$$

where $\mathrm{f}_\lambda(\Delta)$ can be defined from the characteristic map relation

$$s_\lambda(\mathbf{p}) = s_\lambda(\mathbf{p}_\infty)\left(\mathrm{p}_1^{\mathrm{d}} + \sum_{\Delta\neq 1^{\mathrm{d}}} \mathrm{f}_\lambda(\Delta)\mathbf{p}_\Delta\right)$$

where s_λ is the Schur function and
$s_\lambda(\mathbf{p}_\infty) = s_\lambda(1,0,0,\dots) = \frac{\dim\lambda}{d!}$, and for a partition
$\Delta = (\delta_1,\delta_2,\dots)$, we denote $\mathbf{p}_\Delta = \mathrm{p}_{\delta_1}\mathrm{p}_{\delta_2}\cdots$

Two-matrix model $\longrightarrow$ 1-matrix model

Let M_1 and $-iM_2$ be $N \times N$ Hermitian matrices, and $\mathbf{p} = (\mathrm{p}_1, \mathrm{p}_2, \dots)$ and $\mathbf{p}^* = (\mathrm{p}_1^*, \mathrm{p}_2^*, \dots)$ be parameters (the coupling constants)

$$c \int dM_1 dM_2 e^{\mathrm{tr}\, M_1 M_2} e^{\sum_{m>0} \mathrm{tr}\left(M_1^m \mathrm{p_m} + \mathrm{M_2^m p_m^*}\right)}$$

$$= (N)_\lambda s_\lambda(\mathbf{p}) s_\lambda(\mathbf{p}^*)$$

(where $dM = c \prod_{i \geq j}^N dX_{ij} \prod_{i>j}^N dY_{ij}$ for $M = X + iY$). Let us take $\mathrm{p_m^*} = \frac{1}{2}\delta_{\mathrm{m},2}$, then taking the integral over M_2 we get one-matrix model

$$c \int d\mu(M_1) e^{\sum_{m>0} \mathrm{tr}\, M_1^m \mathrm{p_m}} = (N)_\lambda s_\lambda(\mathbf{p}) s_\lambda(0, \frac{1}{2}, 0, 0, \dots)$$

A model of quantum chaos with decay: complex Ginibre enssemble. Special Toda lattice tau function

$$d\mu(Z) = ce^{-\mathrm{tr}\, ZZ^\dagger} \prod_{i,j=1}^{N} d\Re Z_{ij} d\Im Z_{ij}$$

c is the normalization constant defined via $\int d\mu(Z) = 1$.

$$c\int d\mu(Z) e^{N\sum_{m>0} \mathrm{tr}\left(Z^m N^{-\frac{1}{m}}\mathrm{P_m}+(\mathrm{Z}^\dagger)^{\mathrm{m}}\mathrm{N}^{-\frac{1}{\mathrm{m}}}\mathrm{P}^*_{\mathrm{m}}\right)}$$

$$= (N)_\lambda s_\lambda(\mathbf{p}) s_\lambda(\mathbf{p}^*)$$

$$= \sum_{\Delta^1,\Delta^2,\Delta^3}^{*} N^\chi \mathrm{Hur}_{\mathrm{S}^2}^{\Sigma_\chi}(\Delta^1,\Delta^2,\Delta^3)\mathrm{P}_{\Delta^1}\mathrm{P}^*_{\Delta^2}$$

where $\mathrm{p_m} = \mathrm{N}^{1-\frac{1}{2\mathrm{m}}}\mathrm{P_m}$, $\mathrm{p}^*_{\mathrm{m}} = \mathrm{N}^{1-\frac{1}{2\mathrm{m}}}\mathrm{P}^*_{\mathrm{m}}$, conditioned by Riemann-Hurwitz reltation $\ell(\Delta^3) = \chi + d - \ell(\Delta^1) - \ell(\Delta^2)$

RP^2 Hurwitz numbers from the complex Ginibre enssemble. Special BKP tau function

Hurwitz numbers for RP^2 base surface are genereted by

$$< e^{N\sum_{m>0} \text{tr } Z^m N^{-\frac{1}{m}} \mathrm{P_m}} \tau^B(Z^\dagger) >_{\text{Ginibre ensemble}}$$

$$= (N)_\lambda s_\lambda(\mathbf{p})$$

$$= \sum_{\Delta,\Delta'}^{*} N^\chi \mathrm{Hur}_{\mathrm{RP}^2}^{\Sigma_\chi}(\Delta,\Delta')\mathrm{P}_\Delta$$

where $\mathrm{p_m} = \mathrm{N}^{1-\frac{1}{2\mathrm{m}}}\mathrm{P_m},$, conditioned by Riemann-Hurwitz reltation $\ell(\Delta') = \chi + d - \ell(\Delta)$ and $\tau^B(Z) = \prod_{i\neq}(1-z_i z_j)^{-1}(1-z_i)^{-1}$.
As usual $\mathrm{P}_\Delta = \mathrm{P}_{\delta_1}\mathrm{P}_{\delta_2}\cdots$

Product of random matrices - another Toda lattice tau function

Example 1. Consider $Z = Z_1 \cdots Z_n$ where $Z_1, \dots, Z_n$ belong n independent Ginibre ensembles

$$\int \cdots \int e^{\mathrm{Tr} V(Z,\mathbf{p}) + \mathrm{Tr} V(Z^\dagger, \bar{\mathbf{p}})} \prod_{\alpha=1}^{n} \det \left(Z_\alpha^\dagger Z_\alpha \right)^{a_\alpha} d\mu(Z_\alpha) =$$

$$= \sum_{\substack{\lambda \\ \ell(\lambda) \le N}} s_\lambda(\mathbf{p}) s_\lambda(\bar{\mathbf{p}}) \prod_{\alpha=1}^{n} \frac{s_\lambda(\mathbf{p}(a_\alpha + N))}{s_\lambda(1,0,0,\dots)}$$

where $a_1, \dots, a_n$ and $\mathbf{p}$, $\mathbf{p}^*$ are parameters and

$$V(Z, \mathbf{p}) = \sum_{m>0} \frac{1}{m} Z^m p_m$$

generates Hurwitz numbers: two arbitrary profiles and n additional points with fixed profile lengths.

Example 1 (Continuation)

$$< \cdots >_{\mathrm{product}} = \sum_{\Delta^1,\dots,\Delta^{n+2}}^{*} \mathrm{Hur}_{\mathrm{S}^2}^{\Sigma_\chi}(\Delta^1,\dots,\Delta^{\mathrm{n}+2}) \times$$

$$\mathrm{P}_{\Delta^{\mathrm{n}+1}} \mathrm{P}^*_{\Delta^{\mathrm{n}+2}} \prod_{\alpha=1}^{\mathrm{n}} (\mathrm{N} + \mathrm{a}_\alpha)^{\ell(\Delta^\alpha)} \mathrm{P}_{\Delta^\alpha}(\mathrm{C}_\alpha)$$

where for a partition $\Delta = (\delta_1, \delta_2, \dots)$ and a matrix C we define $\mathrm{P}_\Delta(\mathrm{C}) := \mathrm{tr}\,\mathrm{C}^{\delta_1} \mathrm{tr}\,\mathrm{C}^{\delta_2} \cdots$

Specially ordered products of n random matrices

Example. Let us introduce the following products $(2g < n)$

$$X_{2g} = (Z_1 C_1) \cdots (Z_n C_n) \times$$

$$(Z_n^\dagger Z_{n-1}^\dagger \cdots Z_{2g+1}^\dagger)(Z_1^\dagger Z_2^\dagger \cdots Z_{2g}^\dagger)$$

where Z_α, C_α are complex $N \times N$ matrices and where $Z_\alpha^\dagger$ is the Hermitian conjugate of Z_α. We consider each matrix Z_α, $\alpha = 1, \ldots, n$ as the random matrix which belongs to the complex Ginibre enesemble numbered by α while the given matrices C_α are treated as sources.

Correlation functions of spectral invariants of products of random matrices $\to$ Hurwitz numbers with any base surface

Genus g base surface Σ_{2-2g}.

Proposition

$$< \mathrm{tr}\, X_{2g}^{\lambda_1} \mathrm{tr}\, X_{2g}^{\lambda_2} \cdots >_{\mathrm{product}} =$$

$$z_\lambda \sum_{\substack{\Delta^1,\dots,\Delta^{n-2g+1} \\ |\lambda|=|\Delta^j|=d,\, j\le n-2g+1}} \mathrm{Hur}_{\Sigma_{2-2\mathrm{g}}}(\lambda, \Delta^1, \dots, \Delta^{\mathrm{n-2g+1}}) \times$$

$$P_{\Delta^{n-2g+1}}(C'C'') \prod_{i=1}^{n-2g} P_{\Delta^i}(C_{2g+i})$$

where as before for a partition $\Delta = (\delta_1, \delta_2, \dots)$ and a matrix C we define $\mathrm{P}_\Delta(\mathrm{C}) := \mathrm{tr}\, \mathrm{C}^{\delta_1} \mathrm{tr}\, \mathrm{C}^{\delta_2} \cdots$ and where

$$C' = C_1 \cdots C_{2g-1}, \qquad C'' = C_2 C_4 \cdots C_{2g}$$

Hypergeometric tau functions

Relativistic Toda lattice (2-component KP) hypergeometric tau function

$$\tau^{TL}(N,n,\mathbf{p},\bar{\mathbf{p}}) = \sum_{\ell(\lambda)\le N} s_\lambda(\mathbf{p}) s_\lambda(\bar{\mathbf{p}}) \prod_{(i,j)\in\lambda} r(n+j-i)$$

generates $\mathbb{C}P^1$ Hurwitz numbers and includes all known examples

BKP (of Kac-van-de Leur) hypergeometric tau function

$$\tau^{B}(N,n,\mathbf{p}) = \sum_{\ell(\lambda)\le N} s_\lambda(\mathbf{p}) \prod_{(i,j)\in\lambda} r(n+j-i)$$

(A.O., Shiota, Takasaki) generates RP^2 Hurwitz numbers (this is completely new topic)

Thank you !