Metric in the space of measures: from fortification to cosmology

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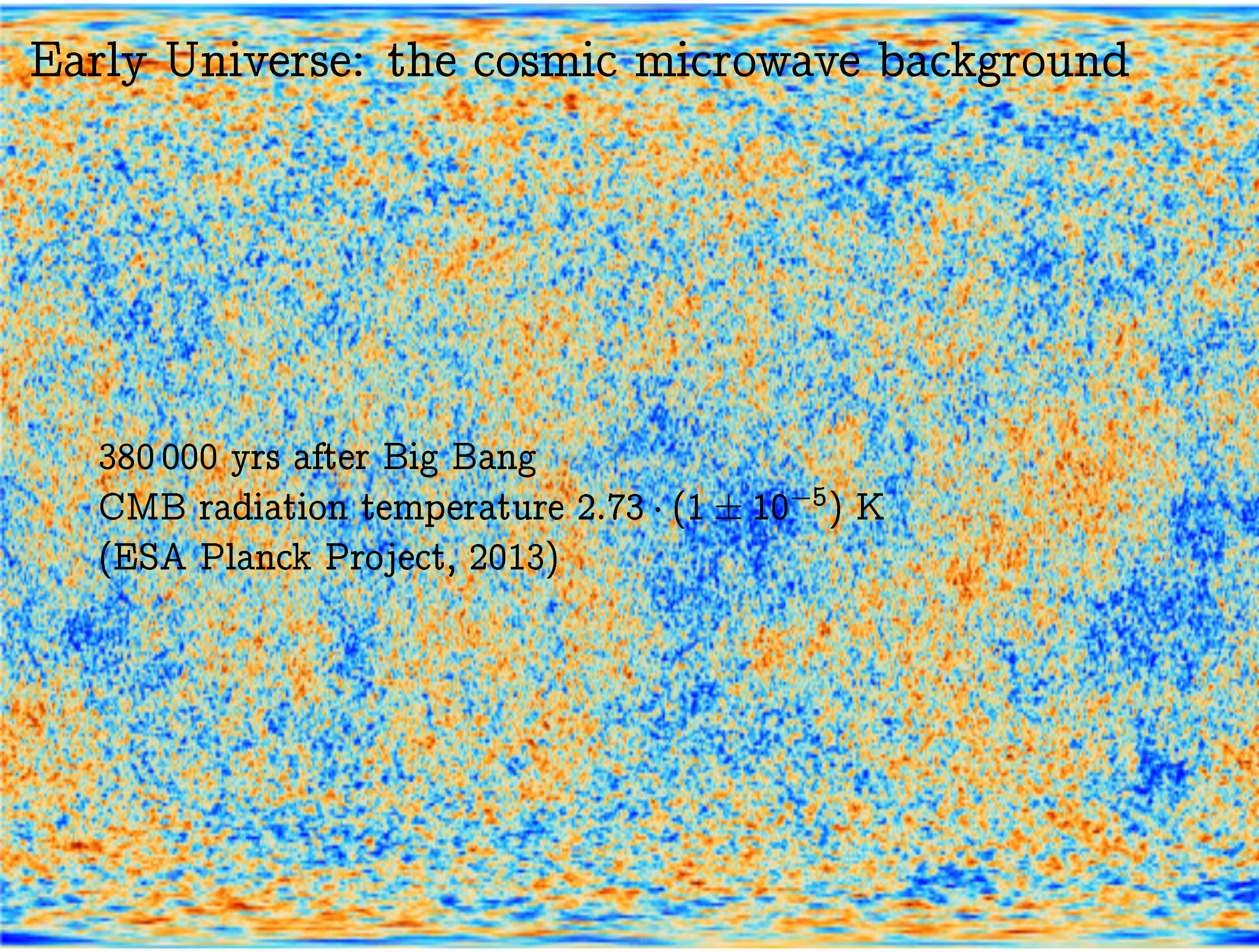
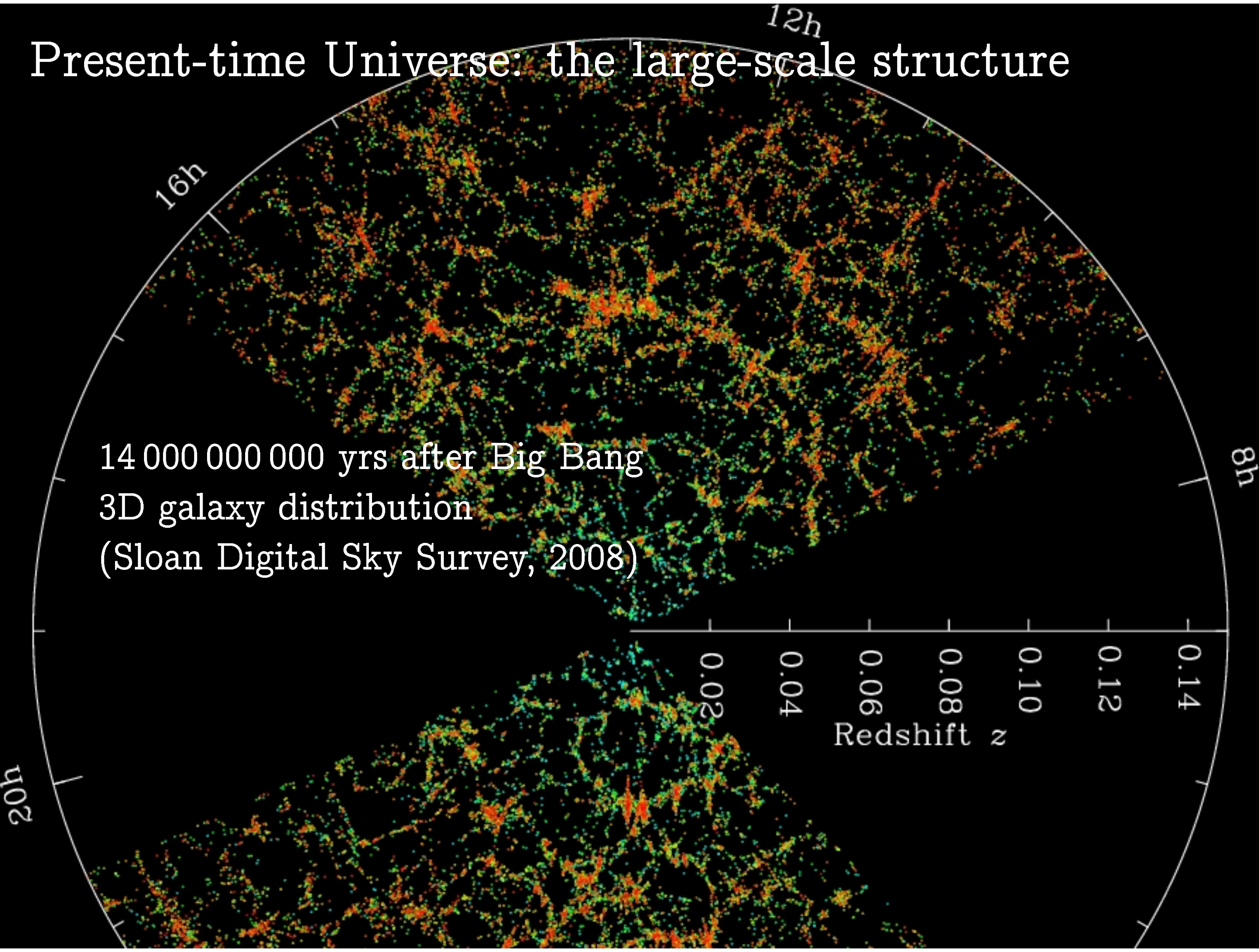
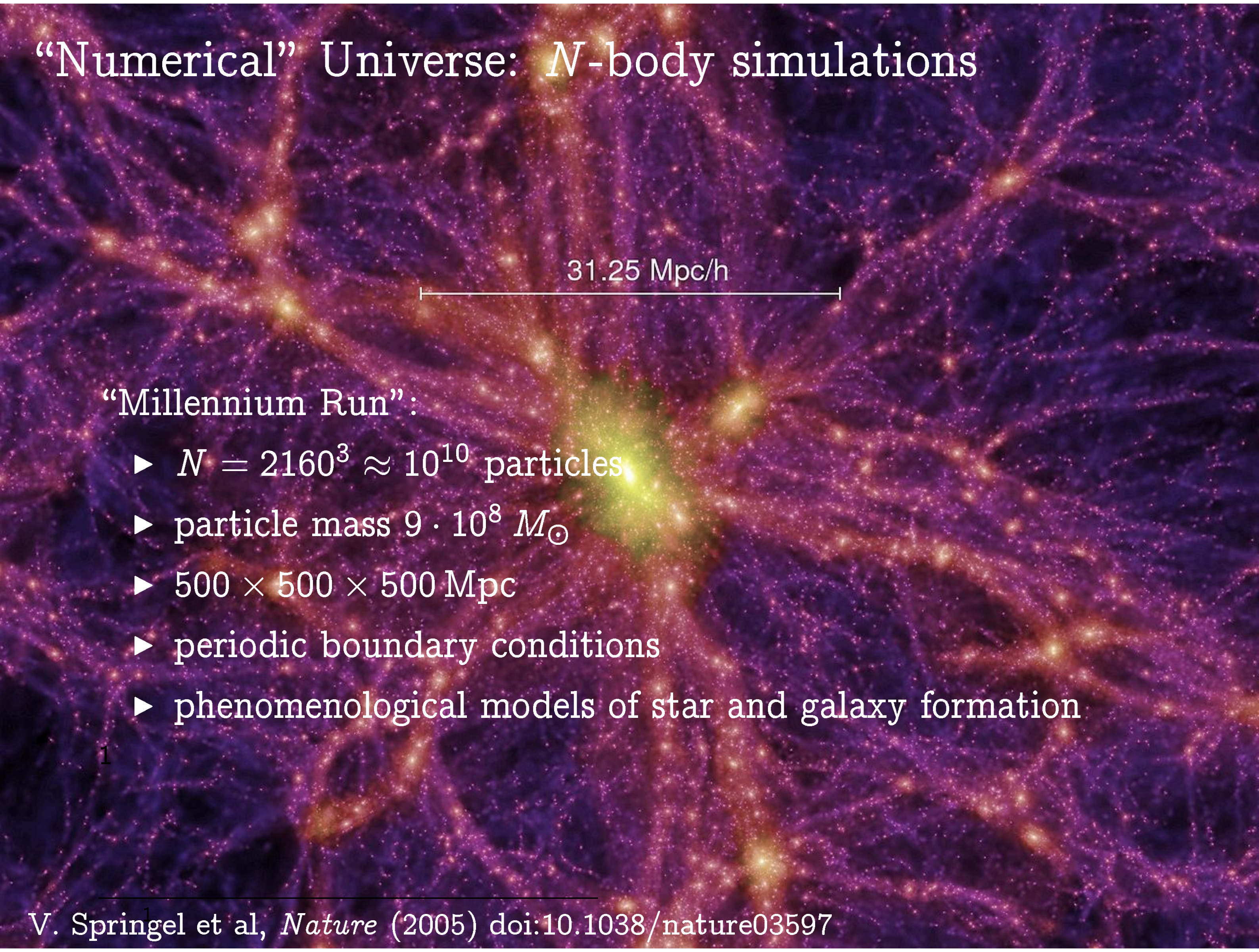

Early Universe: the cosmic microwave background
380 000 yrs after Big Bang
CMB radiation temperature $2.73 \cdot (1 \pm 10^{-5})$ K
(ESA Planck Project, 2013)

Present-time Universe: the large-scale structure



“Numerical” Universe: N-body simulations



“Millennium Run”:

- $N = 2160^3 \approx 10^{10}$ particles
- particle mass $9 \cdot 10^8\ M_\odot$
- $500 \times 500 \times 500\,\mathrm{Mpc}$
- periodic boundary conditions
- phenomenological models of star and galaxy formation

[1]

[1] V. Springel et al, *Nature* (2005) doi:10.1038/nature03597

The Euler–Poisson equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}) = 0$$ mass conservation

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\, \boldsymbol{V} = -\nabla \Phi$$ Euler equation

$$\nabla^2 \Phi = 4\pi G \rho$$ Poisson equation

- Newtonian approximation (no relativity)
- $\rho(t, \boldsymbol{r})$ density, $\boldsymbol{V}(t, \boldsymbol{r})$ velocity, $\Phi(t, \boldsymbol{r})$ gravitational potential

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- Newtonian approximation (no relativity)
- $\rho(t, \boldsymbol{r})$ density, $\boldsymbol{V}(t, \boldsymbol{r})$ velocity,
  $\Phi(t, \boldsymbol{r})$ gravitational potential
- Hubble expansion: $\boldsymbol{r} = a(t)\boldsymbol{x}, \quad \boldsymbol{V}(t, \boldsymbol{r}) = \dot{a}\boldsymbol{x} = \frac{\dot{a}}{a}\boldsymbol{r}$:

$$\rho = 3/(8\pi G a^3), \quad \nabla \Phi = -\frac{\ddot{a}}{a}\boldsymbol{r}, \quad -3\frac{\ddot{a}}{a} = \frac{3}{2a^3} \Rightarrow a = (3t/2)^{2/3}$$

The Euler–Poisson equations: comoving coordinates

$$\rho = \frac{3}{8\pi G a^3}(1+\delta), \quad \boldsymbol{V} = \frac{\dot{a}}{a}\boldsymbol{r} + a\boldsymbol{v}, \quad \nabla\Phi = -\frac{\ddot{a}}{a}\boldsymbol{r} + a\nabla_x\phi$$

$$\frac{\partial\delta}{\partial t} + \nabla_x \cdot ((1+\delta)\boldsymbol{v}) = 0$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla_x)\boldsymbol{v} = -2\frac{\dot{a}}{a}\boldsymbol{v} - \nabla_x\phi$$

$$\nabla_x^2\phi = \frac{3}{2a^3}\delta$$

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- Linearization: $\frac{\partial\delta}{\partial t} + \nabla_x \cdot \boldsymbol{v} = 0, \quad \frac{\partial \boldsymbol{v}}{\partial t} = -2\frac{\dot{a}}{a}\boldsymbol{v} - \nabla_x\phi$

$$\frac{\partial^2\delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial\delta}{\partial t} - \frac{3}{2a^3}\delta = 0 \quad \Rightarrow \quad \delta \sim t^{2/3} \sim a \text{ or } \delta \sim t^{-1} \sim a^{-3/2}$$

The Euler–Poisson equations: "convenient" variables

- rescale velocity $\boldsymbol{v} = \dot{a}\,\boldsymbol{u}$, potential $\phi = \dfrac{3\dot{a}^2}{2a}\psi$

$$\frac{\partial \delta}{\partial a} + \nabla_x((1+\delta)\boldsymbol{u}) = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial a} + (\boldsymbol{u} \cdot \nabla_x)\boldsymbol{u} = -\frac{3}{2a}(\boldsymbol{u} + \nabla_x \psi)$$

$$\nabla_x^2 \psi = \frac{\delta}{a}$$

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$$\frac{\partial u}{\partial a} + (u \cdot \nabla_x)u = -\frac{3}{2a}(u + \nabla_x \psi)$$

$$\nabla_x^2 \psi = \frac{\delta}{a}$$

- "slaving" as $a \to 0$:

$\psi(0, x)$ finite, $\quad \delta \to 0, \quad u \to -\nabla_x \psi(0, x)$ and stays potential

The Euler–Poisson equations: Lagrangian variables

- $\boldsymbol{x} = \boldsymbol{x}(a, \boldsymbol{q}), \quad \boldsymbol{u} = \partial \boldsymbol{x}/\partial a, \quad \partial_a + \boldsymbol{u}\nabla_x = D_a,$
  $1 + \delta = |\partial \boldsymbol{x}/\partial \boldsymbol{q}|^{-1}$

$$D_a^2 \boldsymbol{x} = -\frac{3}{2a}(D_a \boldsymbol{x} + \nabla_x \psi),$$

$$\nabla_x^2 \psi = \frac{1}{a}\left(\left|\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{q}}\right|^{-1} - 1\right)$$

The Euler–Poisson equations: Lagrangian variables

- $x = x(a, q), \quad u = \partial x/\partial a, \quad \partial_a + u\nabla_x = D_a,$
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$$\nabla_x^2 \psi = \frac{1}{a}\left(\left|\frac{\partial x}{\partial q}\right|^{-1} - 1\right)$$

- Zeldovich approximation $D_a^2 x = 0$, or in Eulerian variables

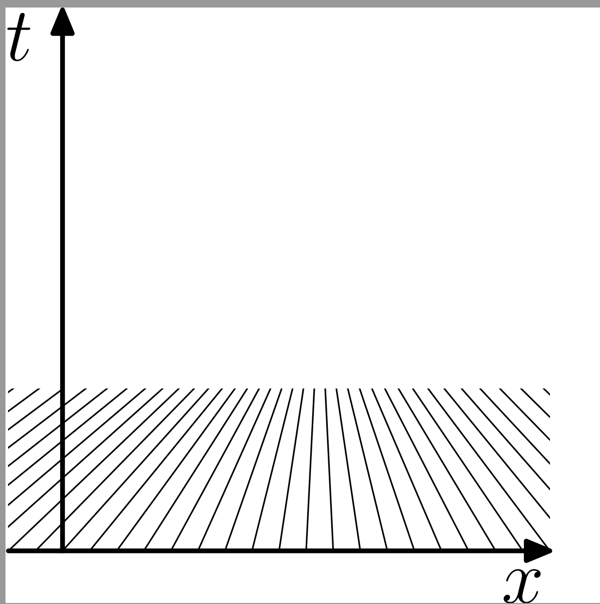
$$\frac{\partial \delta}{\partial a} + \nabla_x \cdot ((1+\delta)u) = 0 \qquad \frac{\partial u}{\partial a} + (u \cdot \nabla_x)u = 0$$

  - $x(a, q) = q + au(q)$
  - exact in dimension one

Crossing of trajectories

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = 0$$

or $\boldsymbol{u} = \mathrm{const}$ (ballistic motion)



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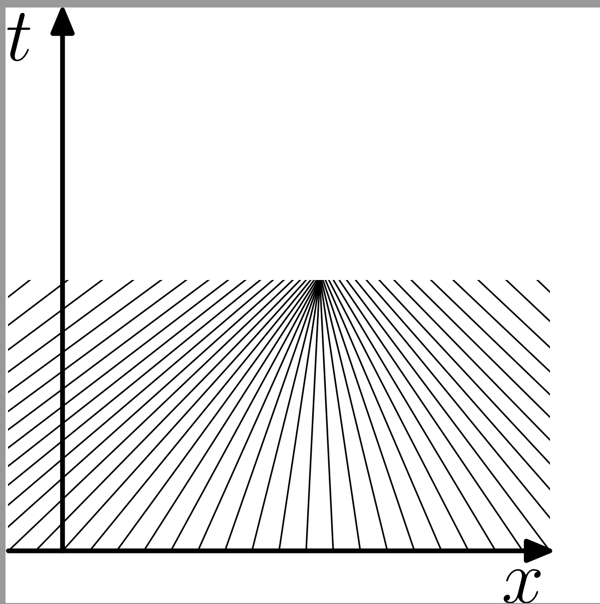

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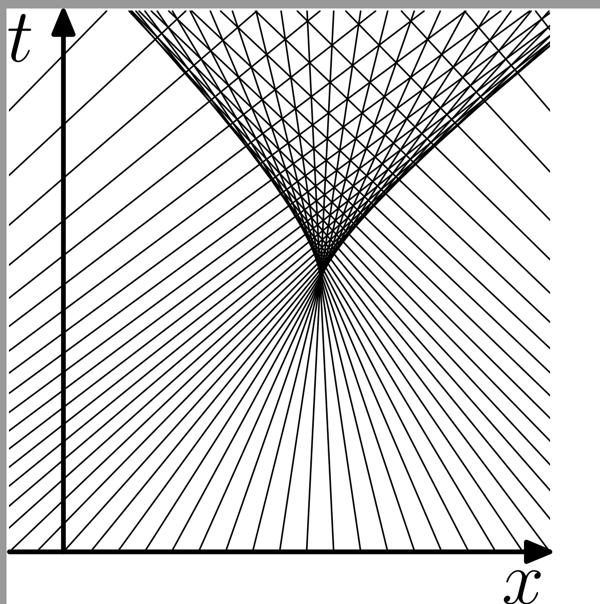
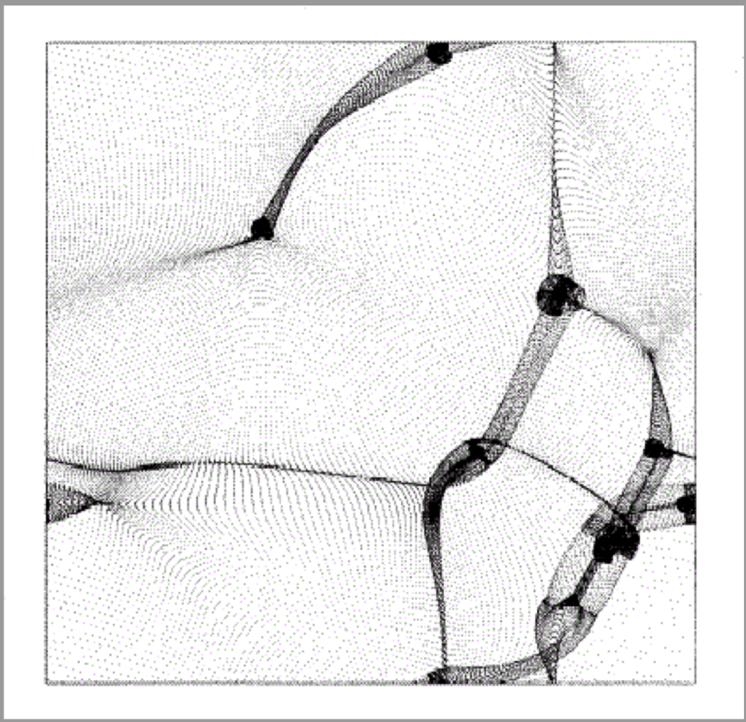
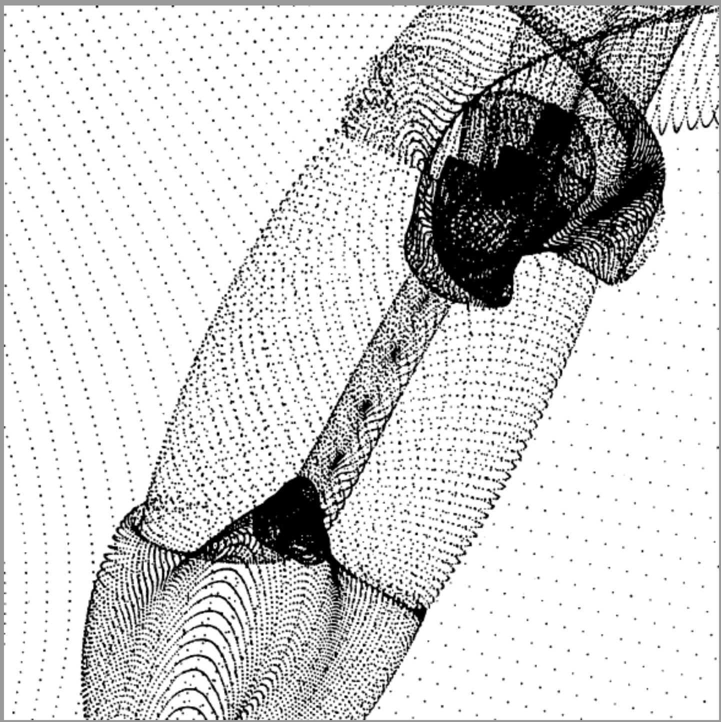


Inner structure of mass concentrations ($d = 2$)



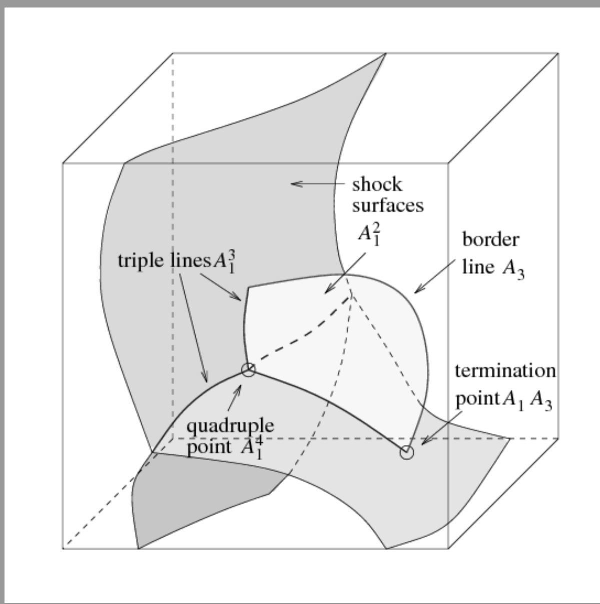


A.L. Melott & S. Shandarin, *Astrophys. J* (1989) doi:10.1086/167681

The adhesion model

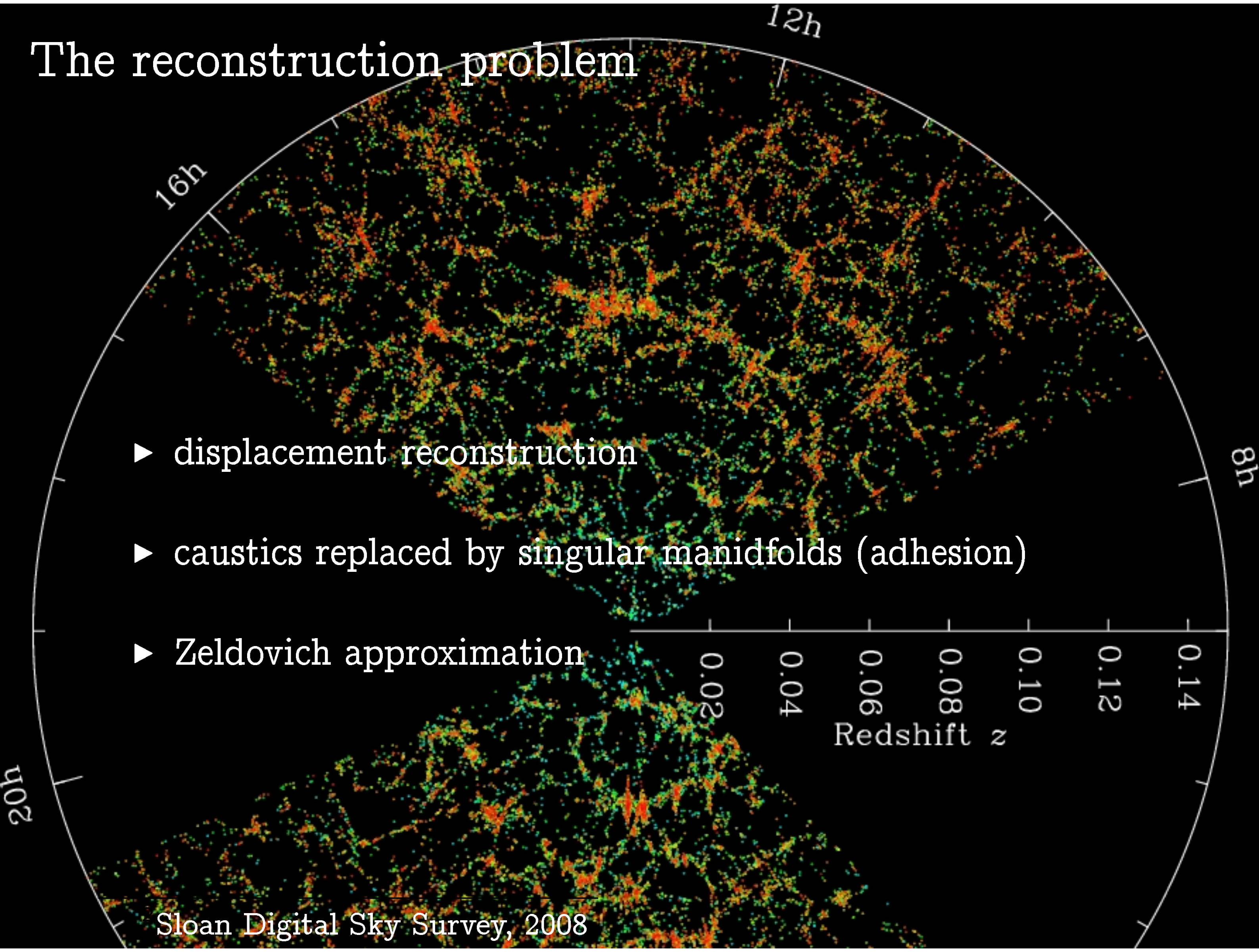
$$\frac{\partial \delta}{\partial a} + \nabla_x \cdot ((1+\delta)\boldsymbol{u}) = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial a} + (\boldsymbol{u} \cdot \nabla_x)\boldsymbol{u} = \epsilon \nabla_x^2 \boldsymbol{u}, \quad \epsilon \to 0$$



The reconstruction problem

- displacement reconstruction
- caustics replaced by singular manidfolds (adhesion)
- Zeldovich approximation



Sloan Digital Sky Survey, 2008

MAK (Monge, Ampère, Kantorovich) reconstruction

- $x(a, q) = q + a u(0, q) = q - a \nabla_q \psi(0, q)$
- $x(a, q)$ gradient (of convex potential for small a)
- $x(a, q)$ maps uniform ρ_0 to highly non-uniform $\rho(a, x)$

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- $\boldsymbol{x}(a, \boldsymbol{q})$ gradient (of convex potential for small a)
- $\boldsymbol{x}(a, \boldsymbol{q})$ maps uniform ρ_0 to highly non-uniform $\rho(a, \boldsymbol{x})$

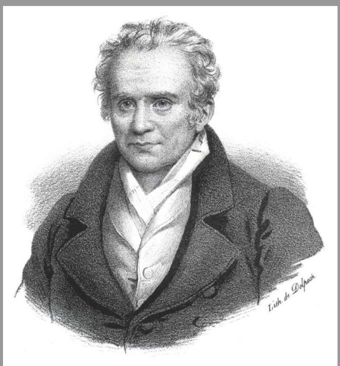
Theorem (Y. Brenier, circa 1990)

The unique solution of

$$I = \int |\boldsymbol{x}(a, \boldsymbol{q}) - \boldsymbol{q}|^2 \rho_0 \mathrm{d}\boldsymbol{q} = \int |\boldsymbol{x} - \boldsymbol{q}(a, \boldsymbol{x})|^2 (1 + \delta(a, \boldsymbol{x})) \mathrm{d}\boldsymbol{x} \to \min$$

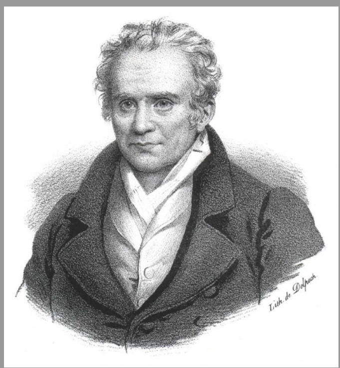
under conditions $\rho_0 \, \mathrm{d}\boldsymbol{q} = (1 + \delta(a, \boldsymbol{x})) \, \mathrm{d}\boldsymbol{x}$ *is gradient of a convex potential.*

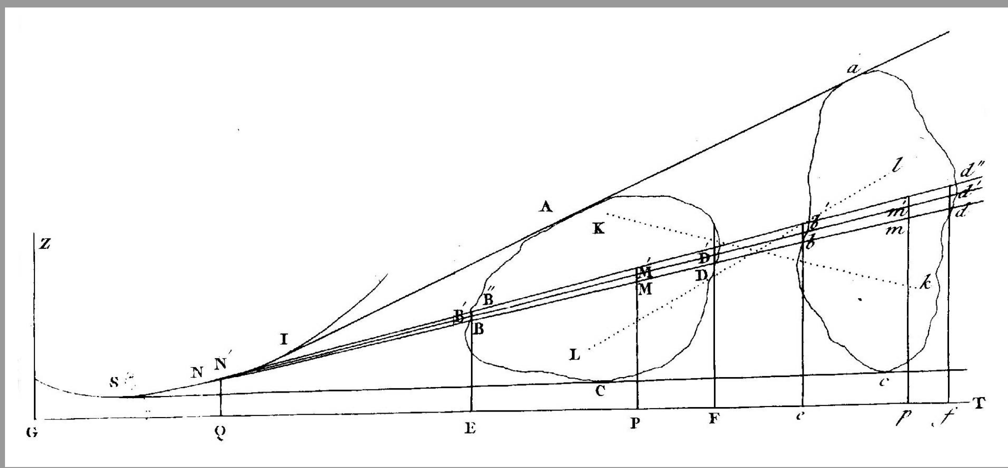
Monge's mass transportation problem



Il n'est pas indifférent que telle molécule de déblai soit transportée dans tel ou tel autre endroit de remblai, mais qu'il y a une certaine distribution à faire des molécules du premier dans le second, d'après laquelle la somme de ces produits sera la moindre possible, et le prix du transport total sera un minimum

Monge's mass transportation problem





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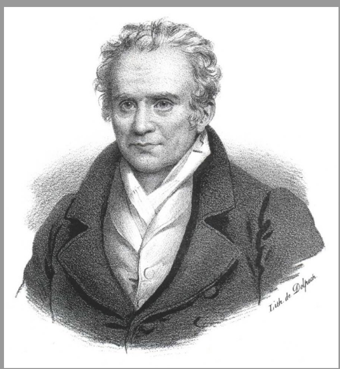
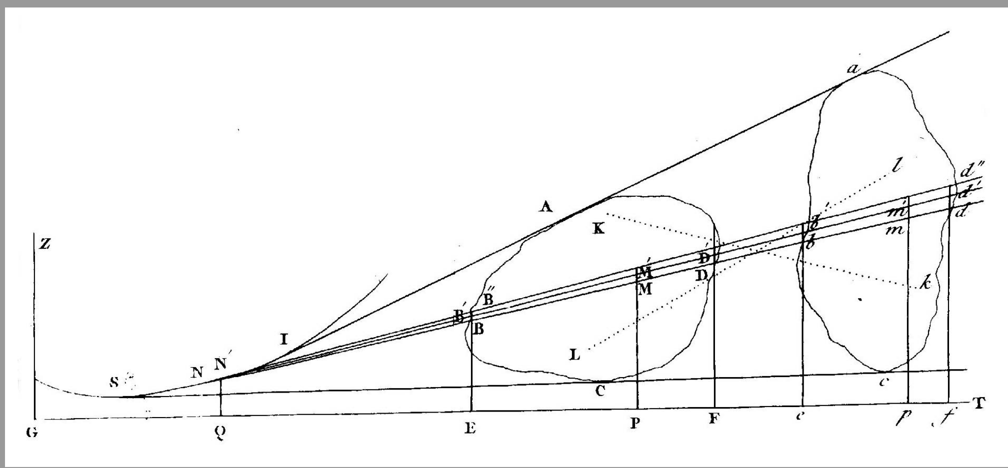




For given $\delta(a, x)$ minimize

$$I = \int |x(a, q) - q|\, \rho_0 \,\mathrm{d}q = \int |x - q(a, x)|\, (1 + \delta(a, x))\, \mathrm{d}x$$

under conditions $\rho_0 \,\mathrm{d}q = (1 + \delta(a, x))\, \mathrm{d}x$

The numerical least-squares method (MAK)

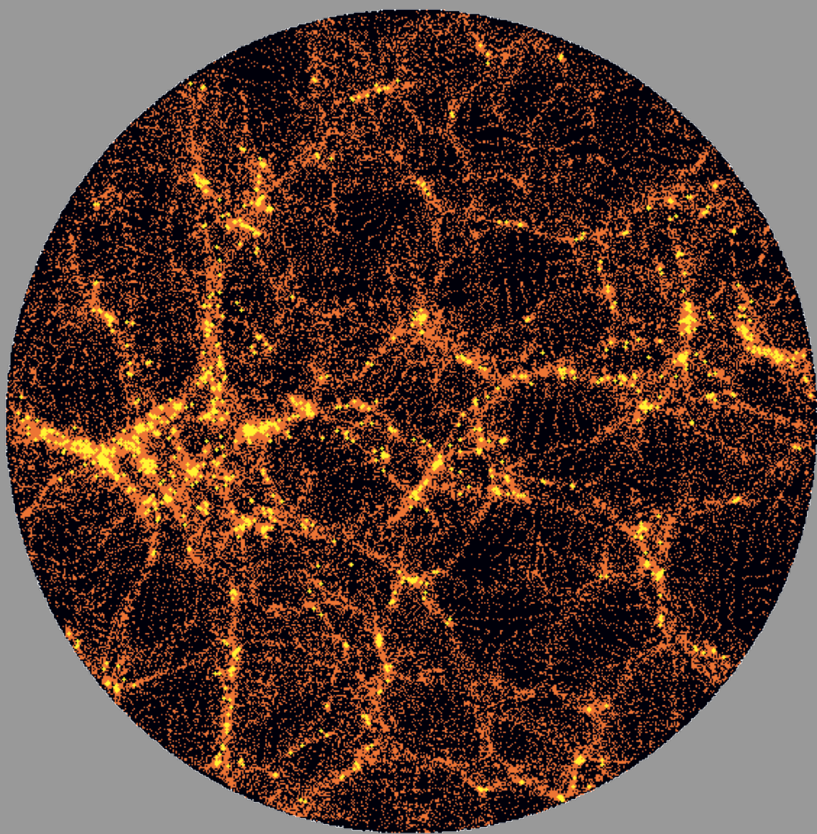
- observed density: galaxy catalogue $\sum_i m_i \, \delta(\boldsymbol{x} - \boldsymbol{x}_i)$
- initial density: uniform grid $\sum_j \mu \, \delta(\boldsymbol{q} - \boldsymbol{q}_j)$
- Assignment Problem: minimize

  $$\frac{1}{2} \sum_{i,j} \gamma_{ij} \, |\boldsymbol{x}_i - \boldsymbol{q}_j|^2$$

  under conditions $\gamma_{ij} \geq 0$, $\sum_j \gamma_{ij} = m_i$, $\sum_i \gamma_{ij} = \mu$
- U. Frisch, S. Matarrese, R. Mohayaee, A.Sobolevski, *Nature* **417** (2002) 260–262
- Y. Brenier, U. Frisch, M. Hénon, G. Loeper, S. Matarrese, R. Mohayaee, A. Sobolevskiĭ, *MNRAS* **346** (2003) 501–524

Testing the MAK reconstruction

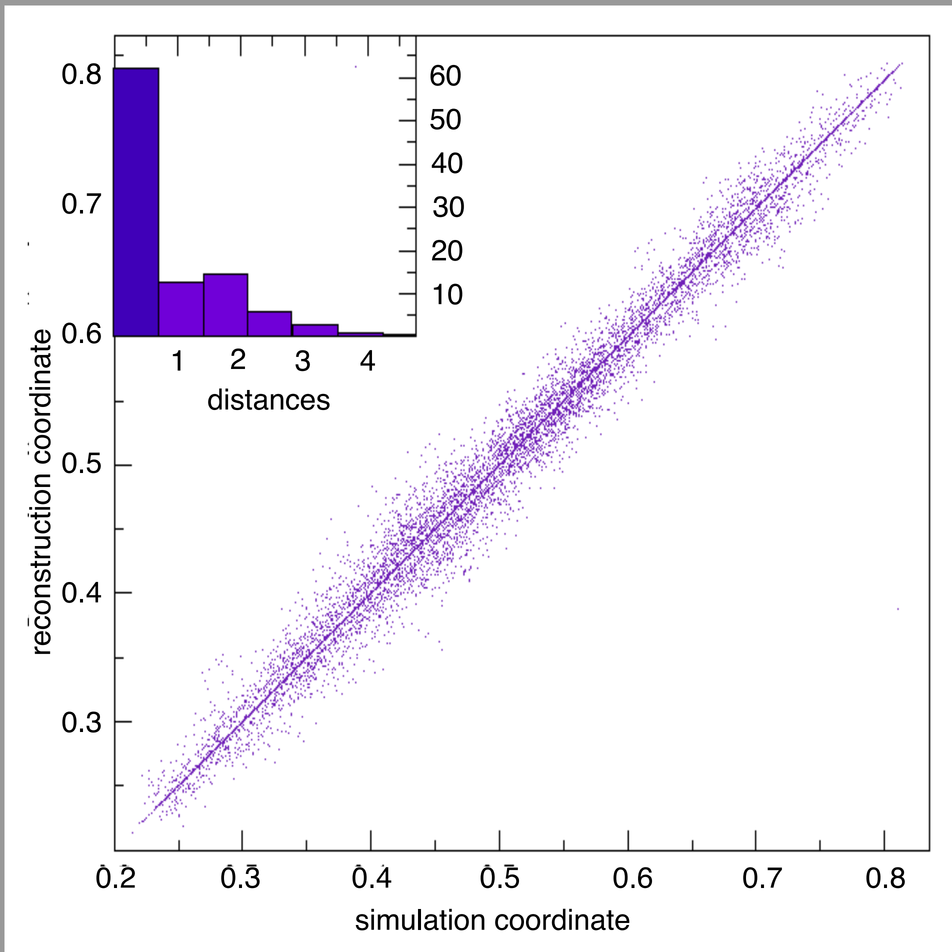
- $N = 128^3$ particles
- $200 \times 200 \times 200\,\text{Mpc}$
- periodic boundary conditions
- inscribed sphere contains 17178 particles from a subgrid of 32^3 nodes



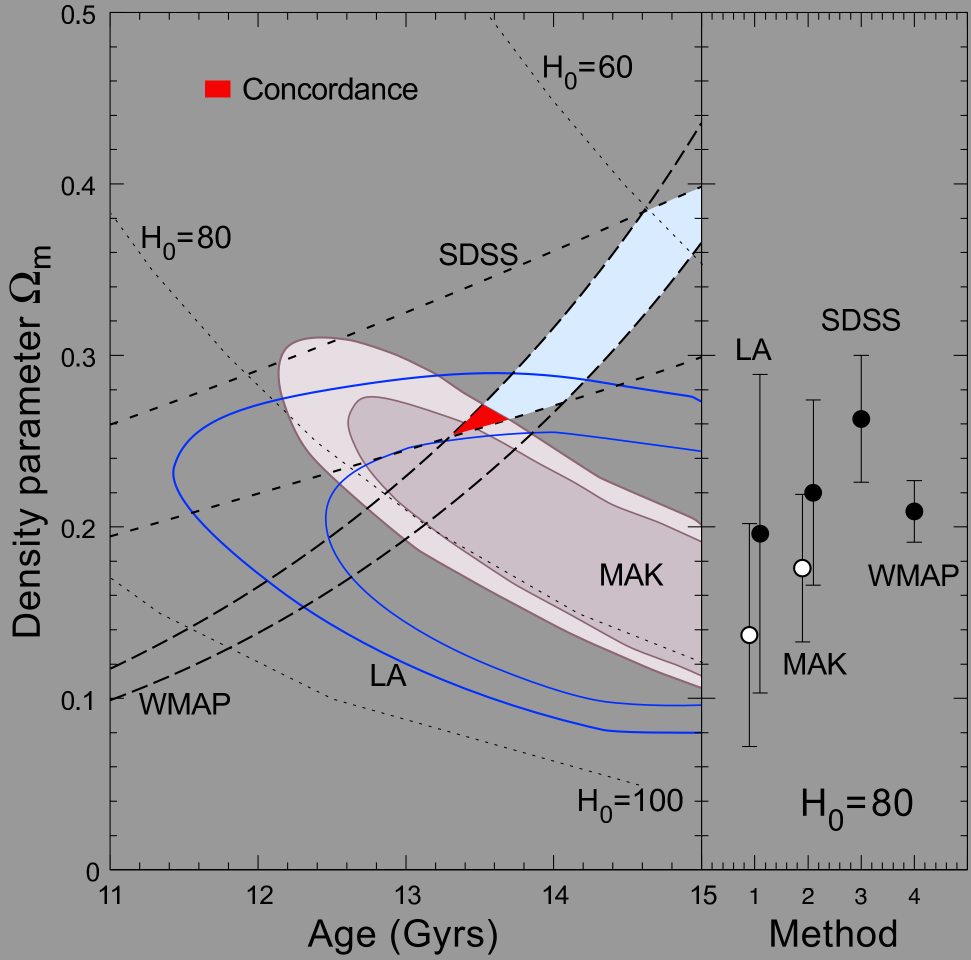
Testing the MAK reconstruction (continued)

"quasi-periodic projection":

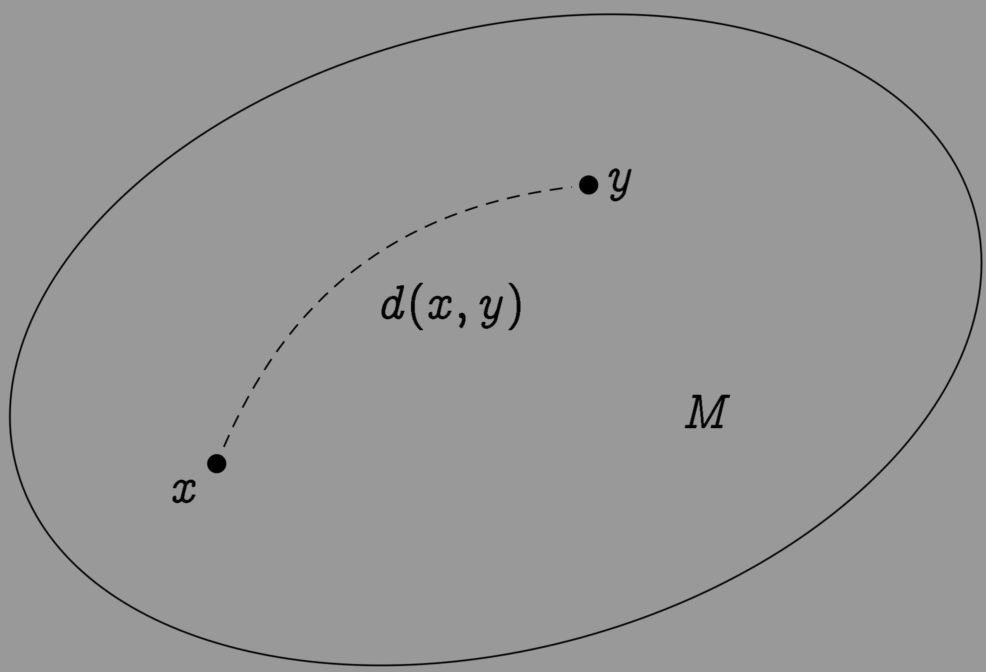
$$\tilde{q} = \frac{q_1 + \sqrt{2}q_2 + \sqrt{3}q_3}{1 + \sqrt{2} + \sqrt{3}}$$



MAK reconstruction for real cosmic data



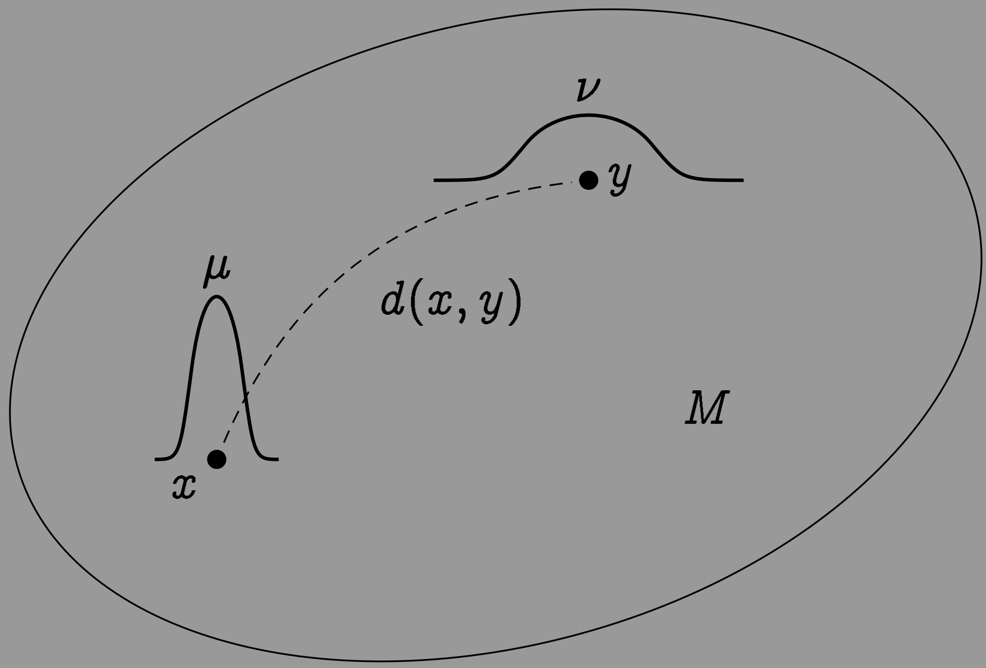
Probability measures as points of geodesic space



M locally compact geodesic space

$\mathcal{P}(M)$ space of probability measures on M

Probability measures as points of geodesic space



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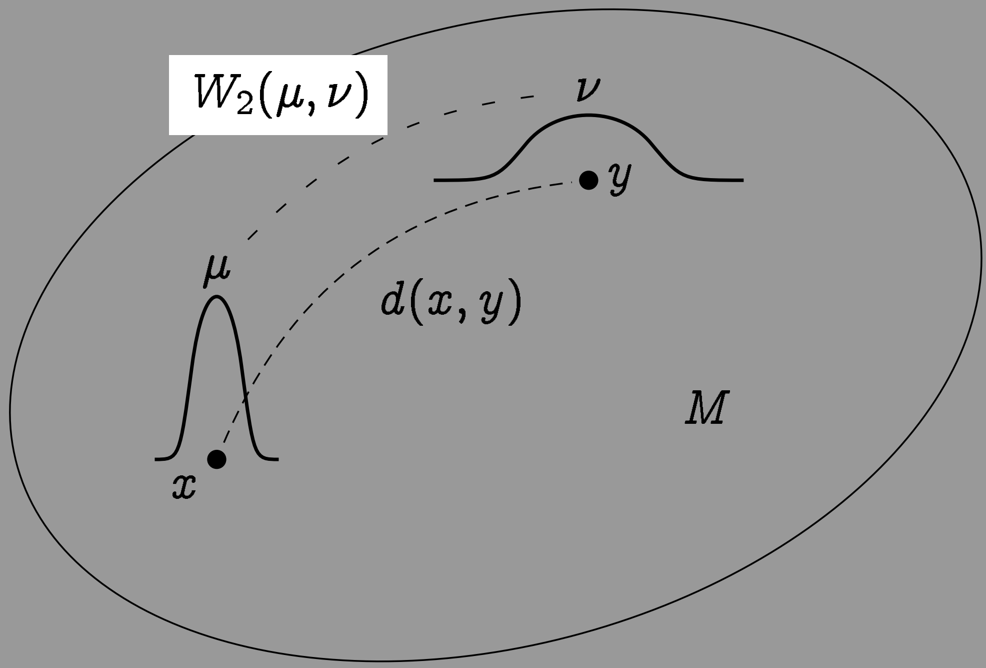

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Wasserstein distance $W_2(\mu, \nu)$

Let $M \subset \mathbf{R}^d$, $d(x, y) = |x - y|$

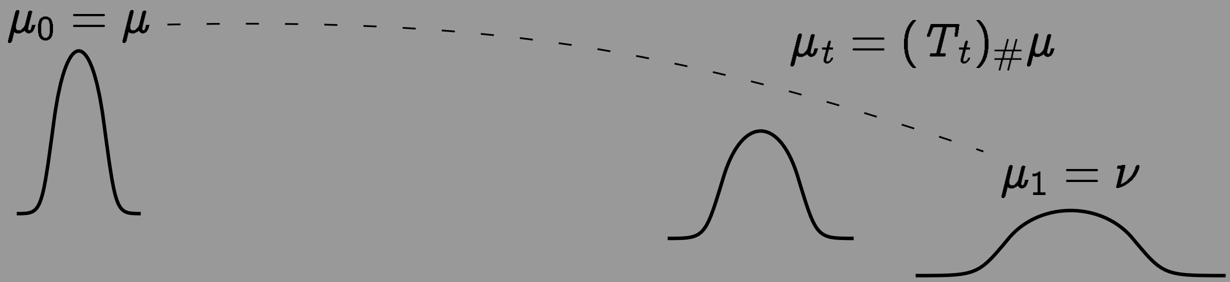
$$(W_2(\mu, \nu))^2 = \inf_{\gamma \in \mathcal{P}(M \times M)} \int |x - y|^2 \, \mathrm{d}\gamma(x, y)$$

$$\swarrow \qquad \searrow$$

$$\mu \qquad\qquad \nu$$

Infimum is attained at transport plan $\gamma^* = (\mathrm{id}, T^*)_{\#}\mu$
where $T^*_{\#}\mu = \nu$ and $T = \nabla\Phi$ with Φ convex [?]

Geodesics in $\mathcal{P}(M)$: Displacement interpolation



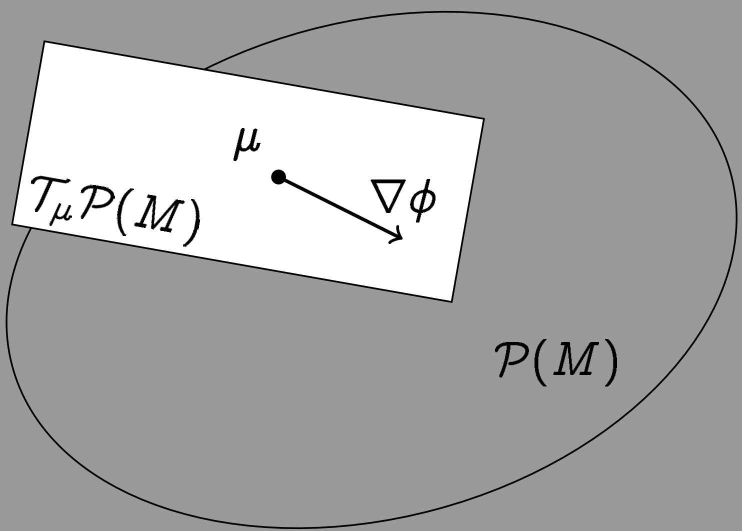
$$T_t : \quad x \mapsto (1-t)x + t\, T^* x$$
$$= x + t(T^* x - x)$$

Curl-free velocity field:

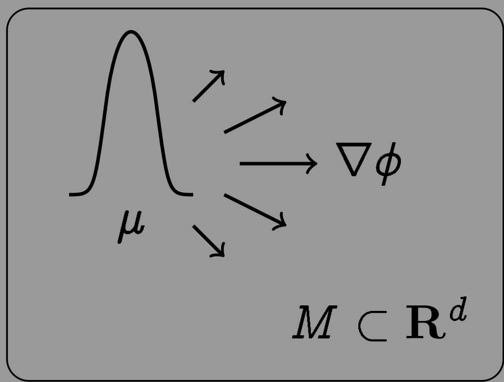
$$T^* x - x = \nabla(\Phi(x) - \frac{|x|^2}{2})n =: \nabla \phi(x)$$

[?]

Formal Riemannian structure of $\mathcal{P}(M)$



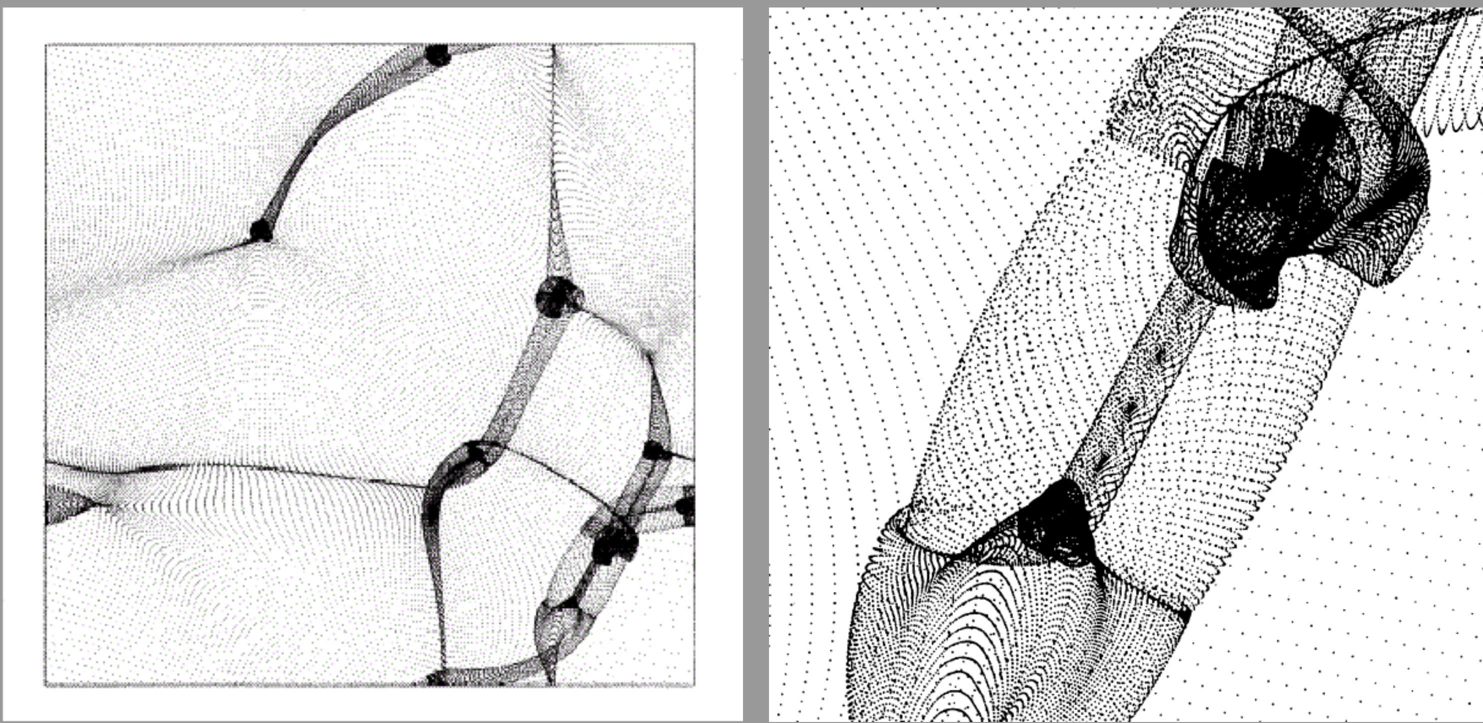
$\mathcal{T}_\mu\mathcal{P}(M)$ is formed by all curl-free vector fields $\nabla\phi$ on M



$$||\nabla\phi||^2_{\mathcal{T}_\mu\mathcal{P}(M)} = \int_M |\nabla\phi(x)|^2 \, \mathrm{d}\mu$$

[?]

Beyond adhesion approximation. . .



A.L. Melott & S. Shandarin, *Astrophys. J.* (1989) doi:10.1086/167681

The Vlassov–Poisson system

- $f(a, x, u)$ distribution function in (x, u) space

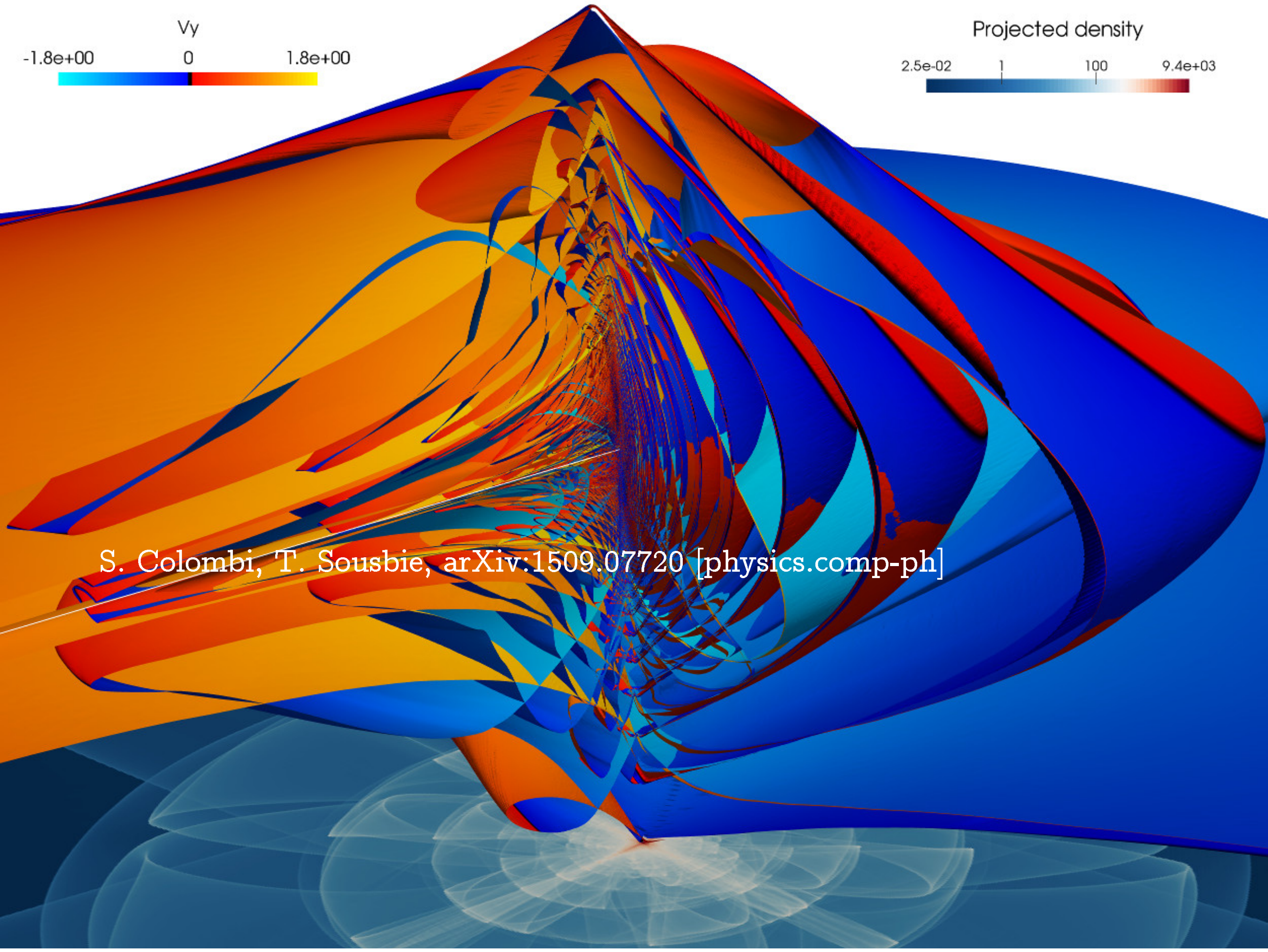
  $$\frac{\partial f}{\partial a} + u \cdot \nabla_x f - \frac{3}{2a}(u + \nabla_x \psi) \cdot \nabla_u f = 0,$$

  $$\nabla_x^2 \psi = \frac{\delta}{a},$$

  $$1 + \delta(a, x) = (ma^2 \dot{a})^3 \int f \, \mathrm{d}u$$

- monokinetic Ansatz: $f = \rho(a, x)\, \delta(u - u(a, x))$;

  becomes "multikinetic" in caustics


Vy
-1.8e+00
0
1.8e+00
Projected density
2.5e-02
1
100
9.4e+03
S. Colombi, T. Sousbie, arXiv:1509.07720 [physics.comp-ph]