Asymptotic solution for the equation for the linear one-dimensional surface waves with surface tension

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Introduction.

Euqation for the free surface elevation can be obtained from the Cauchy-Poisson problem for the potential of the velocity (**Witham**, *Linear and nonlinear waves*).

$$\Phi_{yy} + h^2\Phi_{xx} = 0,\ D(x) < y < 0,$$
$$h^2\Phi_{tt} + (1 - \tau\frac{\partial^2}{\partial x^2})\Phi_y = 0,\ y = 0,$$
$$\Phi_y + h^2\Phi_x D_x(x) = 0,\ y = -D(x),$$
$$\Phi|_{y=0,\,t=0} = \phi^0(\frac{x}{\mu}),\ h\Phi_t|_{y=0,\,t=0} = \phi^1(\frac{x}{\mu}) \equiv -V(\frac{x}{\mu}).$$

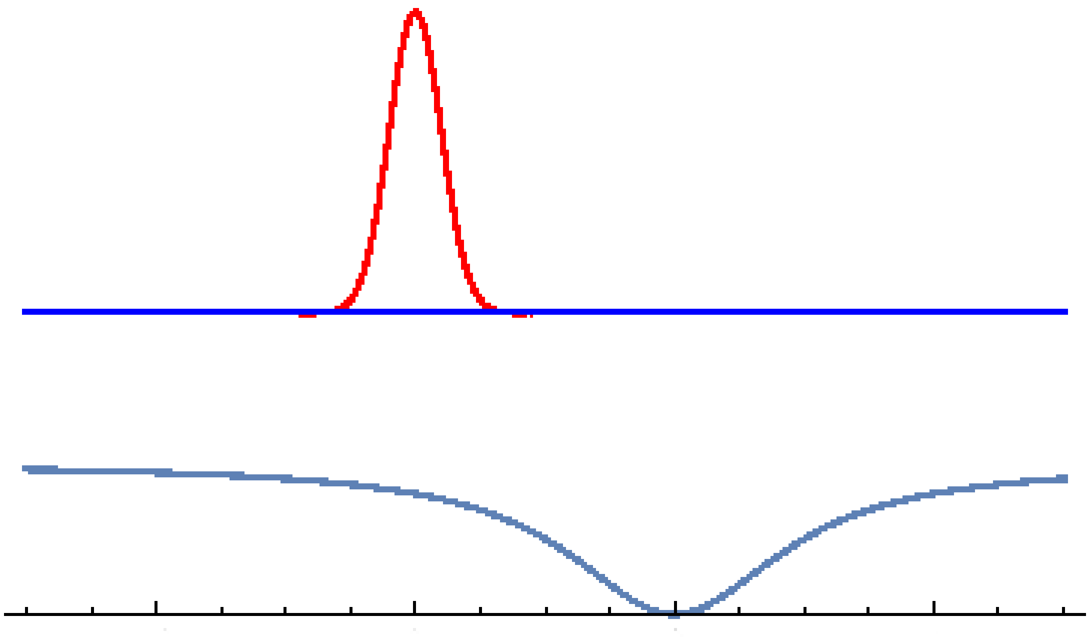
Model parameters and illustration for the initial moment.

Depth function $D(x)$ has the form of the slow-varying underwater bank of average depth $d_0 = 1,\ cm$

$$D(x) = \frac{1}{d_0}\left(d_0 + \frac{a}{1 + (x-c)^2/b}\right),$$

where $a = 0.1$, $b = 200$, $c = 5$.

On the left picture shown the scheme of the initial situation. Coefficient of the surface tension is choosen for the water.



The Cauchy problem with self-ajoined operator.

Following **Dobrokhotov and Zhevandrov**, *RJMP, 2003, vol. 10, N.1, pp. 1-31* and **Zhevandorov**, *USSR Computational Mathematics and Mathematical Physics, 1987, 27:6, 151-158* we obtain the reduced equation with self-ajoined operator.

$$\mu^2 u_{tt} + \frac{1}{\delta}\sqrt{\left(1+\frac{\tau}{\mu^2}\overset{3}{\hat{p}^2}\right)\frac{1}{2}(\overset{1}{\hat{p}}+\overset{3}{\hat{p}})\tanh(\overset{2}{D}(x)\ \delta\frac{1}{2}(\overset{1}{\hat{p}}+\overset{3}{\hat{p}}))\times}$$

$$\times\sqrt{\left(1+\frac{\tau}{\mu^2}\overset{1}{\hat{p}^2}\right)}u = 0,$$

$$u|_{t=0} \equiv u_0(\frac{x}{\mu}) = \frac{1}{\sqrt{1+\frac{\tau}{\mu^2}\hat{p}^2}}V(\frac{x}{\mu}),\ u_t|_{t=0} = 0.$$

Here τ is the parameter for the surface tension, $\delta = h/\mu$, and h is the depth parameter.

Initial conditions.

Initial conditions for the free-surface elevation η are choosen in the form

$$\eta|_{t=0} = V\left(\frac{x}{\mu}\right), \ \eta_t|_{t=0} = 0.$$

Here $\mu \ll 1$ is the localization parameter. Function $V(y)$ is a smooth function which is decaing on the infinity with derivations.

Localized function $V((x-\xi)/\mu)$ can be represented via the Maslov's canonical operator. **Dobrokhotov, Tirozzi, Shafarevich**, *Representations of rapidly decaying functions by the Maslov canonical operator*, Mathematical Notes November 2007, Volume 82, Issue 5-6, pp 713-717

$$V\left(\frac{x}{\mu}\right) = \frac{1}{\sqrt{2\pi}}\int e^{\frac{i}{\mu}px}\tilde{V}(p)dp = \sqrt{\frac{\mu}{i}}K^{\mu}_{\Lambda_0}[\tilde{V}], \ \sqrt{i} = e^{i\pi/4}.$$

Here $\tilde{V}(p)$ is the Fourier transform of the function $V(x)$. Initial manifold

$$\Lambda_0 = \{x = 0, \ p = \alpha; \ \alpha \in \mathbb{R}\}$$

is the vertical line.

Some history.

The problem with surface tension for the constant depth for the initial function of the form $V(x) = 1$, $|x| \leq \mu$, $V(x) = 0$, $|x| \geq \mu$ was studied in the book by **Jeffreys, Swirles**, *Methods of Mathematical Physics*.

Dobrokhotov, Nazaikinskii, *Punctured Lagrangian manifolds and asympto solutions of the linear water wave equations with localized initial conditions*, Mathematical Notes, 2017, Volume 101, Issue 5-6, pp 1053-1060

We are interested in the asymptotics of the leading wave wich arises after the localized pertrubation. (**Sergeev**, Math. Notes, 2018, Vol. 103, Issue 3, pp. 166-171).

The Hamilton system.

Let functions $X^{\pm}$ and $P^{\pm}$ are the solution of the following Hamiltonian system

$$\dot{x} = H_p^{\pm},\ \dot{p} = -H_x^{\pm},\ x|_{t=0} = 0,\ p|_{t=0} = \alpha \in \mathbb{R},$$

$$H^{\pm}(x,\ p) = \pm\frac{p}{\sqrt{\delta}}\sqrt{\left(1+\frac{\tau}{\mu^2}p^2\right)\frac{\tanh(D(x)\delta p)}{p}}.$$

Denote the following action functions

$$S^{\pm}(\alpha,\ t) =$$

$$= \int\limits_0^t \Big(P^{\pm}(\alpha,\ \tau)H_p^{\pm}(X^{\pm}(\alpha,\ \tau), P^{\pm}(\alpha,\ \tau)) - H^{\pm}(X^{\pm}(\alpha,\ \tau), P^{\pm}(\alpha,\ \tau))\Big)d\tau.$$

Main part of the asymptotics for the wave profile.

The following theorem describes the main part of the asymptotics of the waves.

Theorem. The main part of the asymptotics for the leading wave has the form

$$\eta_{as}(x,\,t) =$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{\pm} \frac{1}{2} \int\limits_{-\infty}^{+\infty} \tilde{V}(\alpha) \sqrt{|P^{\pm}_{\alpha}(\alpha,\,t)|} e^{\frac{i}{\mu}(S^{\pm}(\alpha,t)+P^{\pm}(\alpha,t)(x-X^{\pm}(\alpha,t)))} e(\alpha) d\alpha.$$

Where $e(\alpha)$ is the cutoff function near point $\alpha = 0$.

Dynamic of the long waves.

The dynamic of the trajectories corresponding to the long waves, can be described in the terms of the wave equation (**Dobrokhotov, Nazaikinskii**). Near the point $\alpha = 0$ the solution of the Hamilton system (we fixed the $+$ sign) can be presented in the expnasion

$$X(\alpha, t) = X_0(t) + \alpha^2 X_1(t) + \dots, \; P(\alpha, t) = \alpha P_0(t) + \dots.$$

Functions $X_0(t)$ and $P_0(t)$ are the solution of the Hamilton system

$$\dot{x}_0 = H_{0p}(x_0, p_0), \; \dot{p}_0 = -H_{0x}(x_0, p_0), \; x_0|_{t=0} = 0, \; p_0|_{t=0} = 1,$$

where $H_0(x_0, p_0) = |p_0|\sqrt{D(x_0)}$. Such Hamilton function corresponds to the wave equation.

The dispersion effects and surface tension are included in the equation for the function $X_1(t)$. Such representation leads to the reduced equation for the long waves in the form of the linearized Boussinesq-type equation (**Nazakinskii:** equation is incorrect, so it is good for asymptotics but bad for numerical solution).

Properties of the Hamilton function.

The surface tension dramatically changes the properties of the derivation H_p, which corresponds to the group velocity: $H_p \to +\infty$, when $|p| \to \infty$.

Moreover in the point $p = 0$ this derivation has either local maximum or global minimum, depending of the sing of the function

$$\Theta(t) = \sqrt{D_0} \int\limits_0^t P_0^2(s) \left(\frac{\tau}{\mu^2} - \frac{1}{3}\delta^2 D^2(X_0(s)) \right) ds.$$

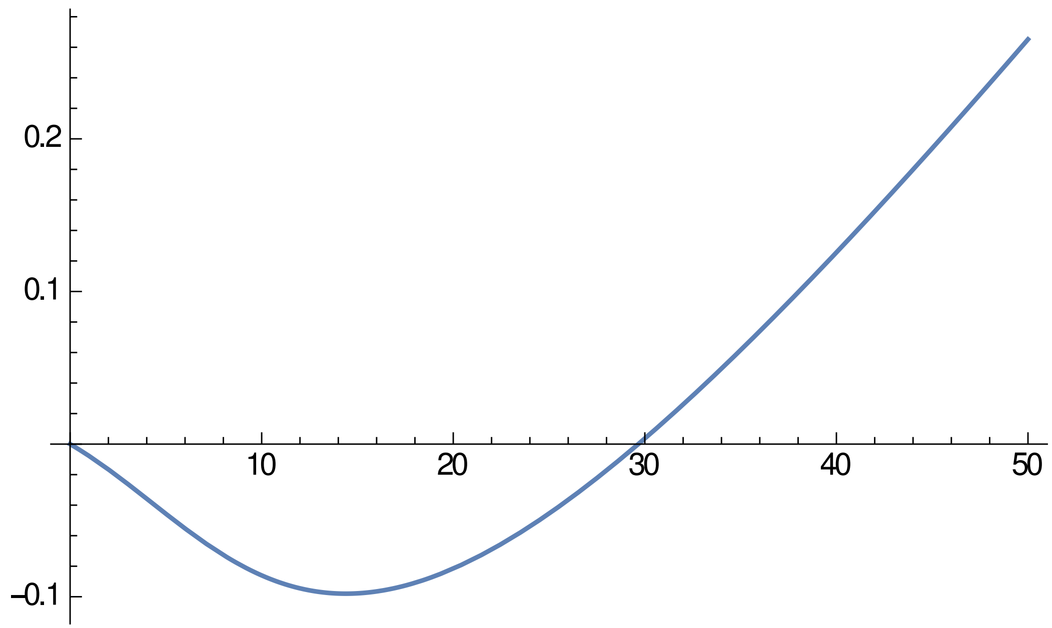
This function describes the influence of the dispersion effects and surface tension.

Despite the short waves propagate faster then long waves they can be neglected due to the effects of viscosity. So we are interested in the propagation of the long waves.

Variable depth.

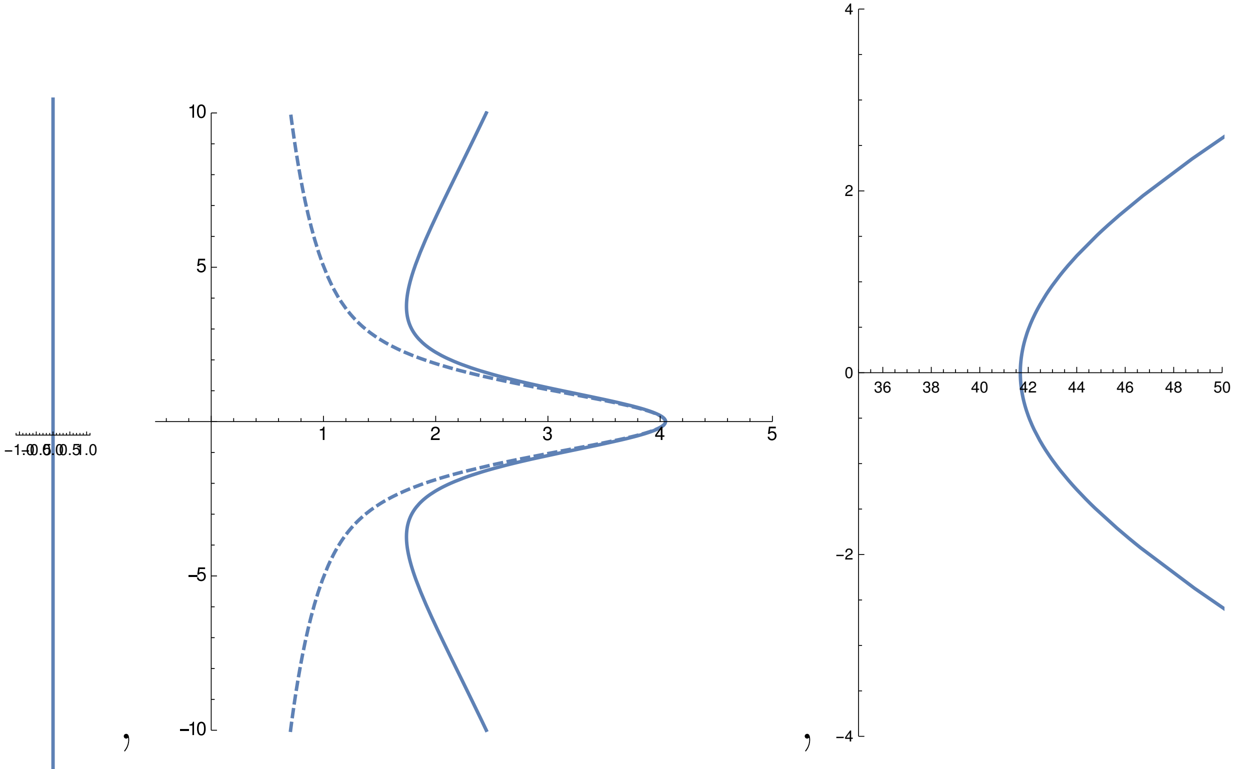
If the depth is constant then can be one of the following situations: either surface tension prevail over the dispersion effects or vise versa. It corresponds to the minimum or maximum.

In the case of variable depth such situation can change over time. The graph of the function $\Theta(t)$ is on the picture. The negative values correspond to the dispersion, positive values are for the surface tension.



Lagrangian manifolds.

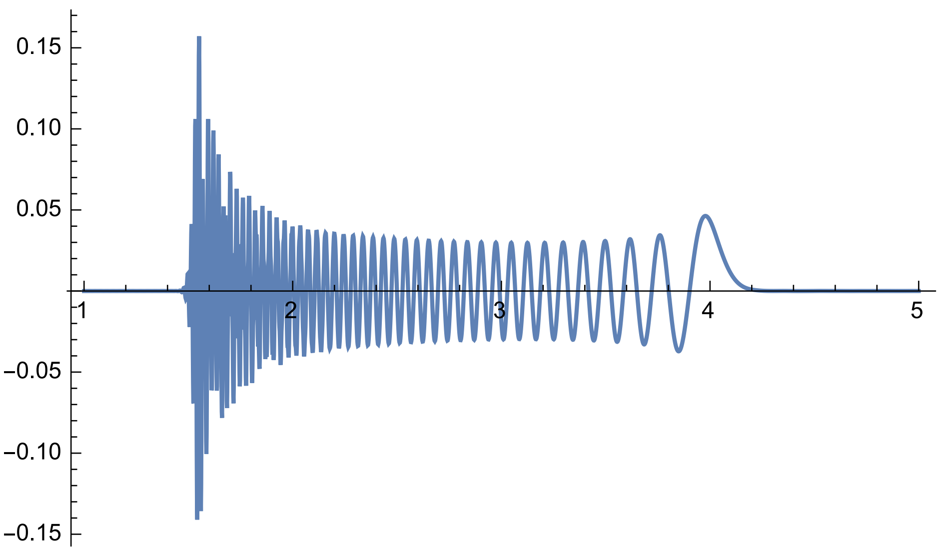
The initial manifold Λ_0 is the vertical line. The manifolds Λ_t at time t are the shifts of the initial manifold along the trajectories of the Hamiltoinain system.



On the left: initial manifold. In the centrum: solid line corresponds to the situation with surface tension ($\Theta(t) < 0$) and punctured line - to the situation without surface tension. On the right: $\Theta(t) > 0$.

Graphs of the asymptotics.

On the pictures the graps of the function η are shown at two time moments. The leftone corresponds to the situation $\Theta(t) < 0$ and the rightone is for the $\Theta(t) > 0$.



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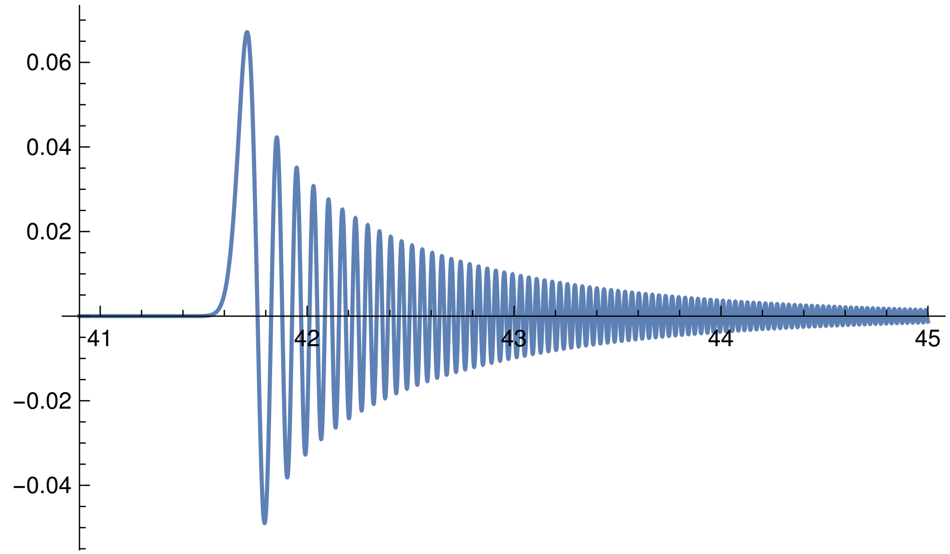


Case of the special initial function.

If $\Theta(t) > 0$ then only one focal point $\alpha = 0$ is left.

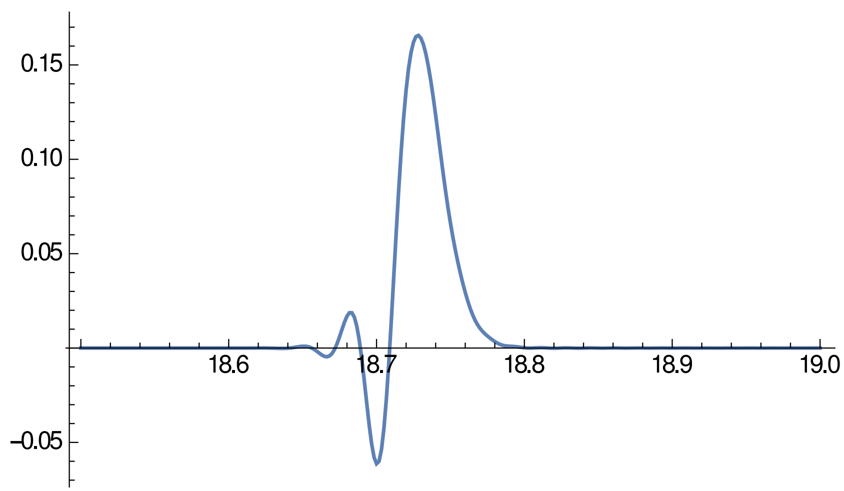
Corollary In the case $\Theta(t) > 0$ and if the initial function V is choosen in the form of Gaussian exponential then main part of the asymptotic solution has the following form

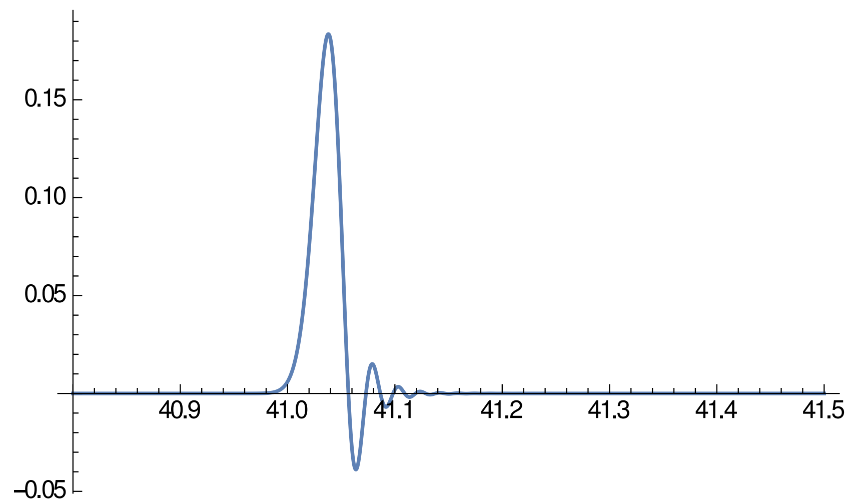
$$\eta_{as} = \frac{1}{2}\sqrt[3]{\frac{2\mu}{3\Theta(t)}}\sqrt{|P_0(t)|}exp\left(\frac{\mu}{3\Theta(t)}\left(\frac{1}{9}\frac{\mu}{\Theta(t)} + P_0(t)\frac{x - X_0(t)}{\mu}\right)\right) \times$$
$$\times Ai\left(\sqrt[3]{\frac{2}{3}}\left(\frac{1}{6}\left(\frac{\mu}{\Theta(t)}\right)^{4/3} + P_0(t)\frac{(x - X_0(t))}{\Theta^{1/3}(t)\mu^{2/3}}\right)\right).$$

Wave profiles for special case.

If the dispersion effects and surface tension are of the same order $O(\mu)$ then the Hamiltonian system reduced to the system for the wave equation. And all effects can be taken into account in the transport equation.

Here we combined the small order of the effects and the graphs for the Gaussian exponential. Situation on the left picture corresponds to $\Theta(t) < 0$, on the right — to $\Theta(t) > 0$





The proof of the theorem. Non-standard charateristics.

Dobrokhotov, Zhevandrov, Maslov, Shafarevich, *Mathematical notes of the Academy of Sciences of the USSR, 1991, Volume 49, Issue 4, pp 355-365.*

Main part of the asymptotics can be described with the help of the parametrix. More preciesely we need the first term $\hat{T}_0$ so the right-hand side of the euqation is the smooth function of order $O(\mu^2)$ after inserting the function $T_0\eta_0$:

$$\eta = \hat{T}_0\eta_0 + w,\ T_0|_{t=0} = 1,\ T_{0t}|_{t=0} = 0,$$

where $\|w\|_{W_2^1} \leq const\,\|\eta_0\|_{W_2^1}$. Operator $\hat{T}_0 = T_0(x,\,t,\,\hat{\omega})$ can be written in the terms of the Maslov's canonical operator based on the system of non-standard characteristics.

Consider the following Hamiltonian system

$$\dot{x} = \mathcal{H}_p,\ \dot{p} = -\mathcal{H}_x,\ x|_{t=0} = \alpha,\ p|_{t=0} = 0,$$

where $\mathcal{H}(x,\,p,\,\omega) = \frac{1}{|\omega|}H(x,\,|\omega|(p + \frac{\omega}{|\omega|}))$.

Denote $\mathcal{X} = \mathcal{X}(\alpha,\,t,\,\omega)$ and $\mathcal{P} = \mathcal{P}(\alpha,\,t,\,\omega)$ the solution of this system.

The Maslov canonical operator. I.

Maslov, Fedoryuk, Dobrokhotov, Nazaikinskii, Shafarevich, Zhevandrov, Tirozzi and many others

We call the point α regular if $\mathcal{X}_\alpha \neq 0$ and focal otherwise. The manifold $\Lambda_t(\omega)$ can be presented as atlas with maps Ω_j. If Ω_j contains the focal point we call it focal, otherwise — regular. In the regular map Ω_j the canonical operator has the WKB-form

$$K(\Omega_j)\chi(\alpha) = \frac{e^{i|\xi|\mathcal{S}(\alpha,t,\omega)}}{\sqrt{|\mathcal{X}_\alpha(\alpha,\, t,\, \omega)|}}.$$

In the focal map Ω_j it has the following form

$$K(\Omega_j)\chi(\alpha) = e^{-i\pi/4}\sqrt{\frac{|\xi|}{2\pi}}\int\limits_{\mathbb{R}} \sqrt{|\mathcal{P}_\alpha(\alpha,\, t,\, \omega)|}e^{i|\xi|(\mathcal{S}(\alpha,t,\omega)+\mathcal{P}(x-\mathcal{X}))}d\alpha.$$

Here ξ is a big parameter and $\omega = \mu\xi$.

The Maslov canonical operator. II.

The Maslov's canonical operator has the form

$$K^{|\xi|}_{\Lambda_t(\omega)}\chi(\alpha) = \sum_j e^{-i\pi m(\Omega_j)} K(\Omega_j)\chi(\alpha).$$

Here $\mathcal{S} = \int \mathcal{P} d\mathcal{X}$, and $m(\Omega_j)$ is the Maslov index.

Then the operator $\hat{T}_0 = T_0(x,\, -i\frac{\partial}{\partial x},\, t)$ is of the form

$$T_0(x,\, \omega, t) = K^{|\omega|/\mu}_{\Lambda_t(\omega)} exp\Big(\frac{i}{\mu}|\omega| \int_0^t (\mathcal{P}\mathcal{H}_p - \mathcal{H}) d\tau\Big).$$

So the fuction $\hat{T}_0 V(\frac{x}{\mu})$ satisfies the initial conditions and the euqation with the presision of $\mu^2\kappa(x,\, t)$ where κ is a smooth function.

Representation of the $\hat{T}_0$ near the point $t = 0$.

As follows from the initial conditions for the funcitons $\mathcal{X}$ and $\mathcal{P}$ at the time moment $t = 0$ the following equality holds $\mathcal{X}_\alpha = 0$. This means that for some interval $t \in [0,\, \Delta t]$ we can present operator $\hat{T}_0$ in the WKB-form, so the function $\hat{T}V$ can be written as follows

$$\hat{T}V = \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \tilde{V}(\omega) \frac{exp\left(\frac{i}{\mu}|\omega| \int\limits_0^t (\mathcal{P}\mathcal{H}_p - \mathcal{H}) d\tau\right)}{\sqrt{|\mathcal{X}_\alpha(\alpha,\, t,\, \omega)|}} e^{\frac{i}{\mu} x\omega} d\omega.$$

Here $\tilde{V}(\omega)$ is the Fourier transform of the function $V(y)$.

Further transformation of this formula is a technical procedure which is omitted in this speech.

THANK YOU FOR YOUR ATTENTION!