Dynamical symmetries, coherent states and nonlinear realizations: The SO(2, 4) case

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Nonlinear realizations of the SO(2,4) group are discussed from the point of view of symmetries. Dynamical symmetry breaking is introduced. One linear and one quadratic model in curvature are constructed. Coherent states of the Klauder–Perelomov type are defined for both cases taking into account the coset geometry. A new spontaneous compactification mechanism is defined in the subspace invariant under the stability subgroup. The physical implications of the symmetry rupture in the context of nonlinear realizations and direct gauging are analyzed and briefly discussed.

International Journal of Geometric Methods in Modern Physics Vol. 15, No. 1 (2018) 1850005

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Introduction

UTIYAMA `56

- Yang-Mills extension to any Lie symmetry
- Problem of general cov. transf. and pseudoriemannian metric

Ne`eman-Regge`78, Hashashi-Shirafuji `81

- Tetrad gravitational field as a gauge field of the translation subgroup of Poincare
- As gauge potentials in YMT

Shirafuji-Suzuki`88, Ivanov-Niederle`82, Stelle-West`80, Tseytlin`82

- Poincare as IW contraction of SO(2,3), SO(1,4) (SU(2,2)subgroups)
- Fiber bundles not natural

Volkov-Soroka, Arnowitt-Pran-Nath

- Gravity as gauge theory in a pure geometrical context
- The problem to determine which fields transform as gauge fields and which not, as well as which fields are physical ones and which are redundant nonetheless remains.

Mansouri-MacDowell`77

- principal fiber bundle imposing a condition of orthogonality of the generators of the fiber and base manifold.
- Such conditions that break the symmetry of the original group are implemented by means of a particular choice of the metric tensor.
- This approach was implemented in a supergroup structure obtaining a gauge theory of supergravity.
- Note that the underlying geometry must be reductive (in the Cartan sense) or weakly reductive in the case of supergravity

Coset coherent states

Let us remind the definition of coset coherent states

$$H_0 = \{g \in G \mid \mathcal{U}(g)V_0 = V_0\} \subset G.$$

Consequently the orbit is isomorphic to the coset, e.g.

$$\mathcal{O}(V_0) \simeq G/H_0.$$

Analogously, if we remit to the operators, e.g.

$$|V_0\rangle\langle V_0| \equiv \rho_0$$

then the orbit

$$\mathcal{O}(V_0) \simeq G/H$$

with

$$\begin{aligned} H &= \{g \in G \mid \mathcal{U}(g)V_0 = \theta V_0\} \\ &= \{g \in G \mid \mathcal{U}(g)\rho_0\mathcal{U}^\dagger(g) = \rho_0\} \subset G. \end{aligned}$$

The orbits are identified with cosets spaces of G with respect to the corresponding stability subgroups H_0 and H being the vectors V_0 in the second case defined within a phase. From the quantum viewpoint $|V_0\rangle \in H$ (the Hilbert space) and $\rho_0 \in F$ (the Fock space) are V_0 normalized fiducial vectors (embedded unit sphere in H).

Symmetry Breaking Mechanism: The SO(2,4) Case

i) Let a,b,c=1,2,3,4,5 and i,j,k=1,2,3,4 (in the six-matrix representation) then the Lie algebra of SO(2,4) is

$$i[J_{ij}, J_{kl}] = \eta_{ik}J_{jl} + \eta_{jl}J_{ik} - \eta_{il}J_{jk} - \eta_{jk}J_{il},$$

$$i[J_{5i}, J_{jk}] = \eta_{ik}J_{5j} - \eta_{ij}J_{5k},$$

$$i[J_{5i}, J_{5j}] = -J_{ij},$$

$$i[J_{6a}, J_{bc}] = \eta_{ac}J_{6b} - \eta_{ab}J_{6c},$$

$$i[J_{6a}, J_{6b}] = -J_{ab}.$$

ii) Identifying the first set of commutation relations as the lie algebra of the SO(1,3) with generators *Jik=-Jki*

iii) The 1st commutation relations plus 2nd and 3rd are identified as the Lie algebra SO(2,3) with the additional generators *J5i* and *ηij=(1,-1,-1,-1).*

iv) The commutation relations 1st to 5th is the Lie algebra SO(2,4) written in terms of the Lorentz group SO(1,3) with the additional generators *J5i*, *J6b*, and *Jab=-Jba*, where ηab=(1,-1,-1,-1,1). It follows that the embedding is given by the chain SO(1,3)⊂SO(2,3)⊂SO(2,4)

From the six dimensional matrix representation parameterizing the coset any element G of SO(2,4) is written as

$$SO(2,4) \approx \frac{SO(2,4)}{SO(2,3)} \times \frac{SO(2,3)}{SO(1,3)} \times SO(1,3),$$

$$\begin{aligned} G &= e^{-iz^a(x)J_a} G(H) \\ &= e^{-iz^a(x)J_a} e^{-i\varepsilon^k(x)P_k} H(\Lambda). \end{aligned}$$

Consequently we have *G(H):H→G* is an embedding of an element of SO(2,3) into SO(2,4) where *Ja≡(1/λ)J6a* and *H(Λ):Λ→H* is an embedding of an element of SO(1,3) into SO(2,3) where *Pk≡(1/m)J5k* as follows

$$G = e^{-iz^a(x)J_a} \underbrace{e^{-i\varepsilon^k(x)P_k} \underbrace{\begin{pmatrix} SO(3,1) & 0 \\ 0 & I_{2x2} \end{pmatrix}}_{H(\Lambda)}}_{G(H)}$$

any element G of SO(2,4) is written as the product of an SO(2,4) boost, an ADS boost, and a Lorentz rotation.

Goldstone Fields and Symmetries

i) Our starting point is to introduce two 6-dimensional vectors V_1 and V_2 being invariant under SO(3,1) in a canonical form

$$\left.\underbrace{\begin{pmatrix}0\\0\\0\\0\\A\\0\end{pmatrix}}_{V_1}+\underbrace{\begin{pmatrix}0\\0\\0\\0\\0\\-B\end{pmatrix}}_{V_2}=\underbrace{\begin{pmatrix}0\\0\\0\\0\\A\\-B\end{pmatrix}}_{V_0}\right\}\text{invariant under } SO(3,1)$$

ii) Now we take an element of $Sp(2)\subset Mp(2)$ embedded in the 6-dimensional matrix representation operating over V as follows

$$\mathcal{M}V\rightarrow\underbrace{\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&a&b\\0&0&0&0&c&d\end{pmatrix}}_{Sp(2)\subset Mp(2)}\underbrace{\begin{pmatrix}0\\0\\0\\0\\A\\-B\end{pmatrix}}_{V_0}=\begin{pmatrix}0\\0\\0\\0\\A'\\-B'\end{pmatrix}=V'$$

$A'=aA-bB, -B'=cA-dB$

consequently we obtain a Klauder-Perelomov generalized coherent state with the fiducial vector V_0.

ii) The specific task to be made by the vectors is to perform the breakdown to SO(3,1). Using the transformed vectors above (Sp(2)~ Mp(2) CS) the symmetry of G can be extended to an internal symmetry as SU(1,1) given by G below (notice $|\lambda|^2-|\mu|^2=1$):

$$\widetilde{G}V' = e^{-iz^a(x)J_a}\underbrace{e^{-i\varepsilon^k(x)P_k}\underbrace{\begin{pmatrix} SO(3,1) & \mathbf{0} \\ \mathbf{0} & \begin{matrix} \lambda & \mu \\ \mu^* & \lambda^* \end{matrix} \end{pmatrix}}_{\widetilde{H}(\Lambda)}}_{\widetilde{G}(H)} V' = \underbrace{e^{-iz^a(x)J_a} e^{-i\varepsilon^k(x)P_k}\underbrace{\begin{pmatrix} SO(3,1) & \mathbf{0} \\ \mathbf{0} & \begin{matrix} \alpha & 0 \\ 0 & \beta \end{matrix} \end{pmatrix}}_{H(\Lambda)}}_{G(H)} V_0 = GV_0,$$

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^*\alpha & -\mu\beta \\ 0 & 0 & 0 & 0 & -\mu^*\alpha & \lambda\beta \end{pmatrix}$$

and if we also ask for DetM=1 then $\alpha\beta=1$, e.g. the additional phase: it will bring us the 10^{th} Goldstone field. The other nine are given by z(x) and ε(x) (a,b,c=1,2,3,4,5 and i,j,k=1,2,3,4) coming from the parameterizations of the cosets C=((SO(2,4))/(SO(2,3))) and P=((SO(2,3))/(SO(1,3))).

Invariant SO(2, 4) Action and Breakdown Mechanism

- Linear in RAB

$$S = \int \mu_{AB} \wedge R^{AB}$$

in this case we note at first, that the tensor μ_{AB} SO(2,4)-valuated acts as multiplier in S

i) if we have two diffeomorphic (or gauge) nonequivalent SO(2,4)-valuated connections, namely Γ^{AB} and Γ^{AB}, their difference transforms as a second rank six-tensor under the action of SO(2,4)

$$\kappa^{AB} = G^A_{\ C} G^B_{\ D} \kappa^{CD},$$
$$\kappa^{AB} \equiv \tilde{\Gamma}^{AB} - \Gamma^{AB}.$$

ii) now calculate the modified curvature

$$\tilde{R}^{AB} = R^{AB} + \mathcal{D}\kappa^{AB}$$

where the SO(2,4) covariant derivative is defined in the usual way

$$\mathcal{D}\kappa^{AB} = d\kappa^{AB} + \Gamma^A_{\ C} \wedge \kappa^{CB} + \Gamma^B_{\ D} \wedge \kappa^{AD}.$$

iii) Redefining the SO(2,4) six vectors as ψ_A and φ_B (in order to put all in standard notation), the 2-form κ can be constructed as

$$\kappa^{AB} \to \psi^{[A} \varphi^{B]} dU.$$

Consequently (U scalar function) and get

$$\tilde{R}^{AB} = R^{AB} + \mathcal{D}(\psi^{[A} \varphi^{B]} dU)$$
$$= R^{AB} + (\psi^{[A} \mathcal{D}\varphi^{B]} - \varphi^{[A} \mathcal{D}\psi^{B]}) \wedge dU.$$

The next step is to find the specific form of μ_{AB} (such that will be invariant under tilde transformation) in order to make the splitting of the transformed action S reductive as follows

iv) Let us define

$$\tilde{\theta}^A = \tilde{D}\varphi^A$$

with the connection Γ+κ, then

$$\tilde{\theta}^A = \underbrace{D\varphi^A}_{\theta^A} + \kappa^A_{\ B}\varphi^B,$$

$$\tilde{\theta}^A = \theta^A + \left[\psi^A(\varphi^B)^2 - \varphi^A(\psi\cdot\varphi)\right]\wedge dU,$$

where $(\varphi^B)^2 = \left(\varphi_B\varphi^B\right)$ and $(\psi\cdot\varphi) = \psi_B\varphi^B$ etc

In the same manner we also define

$$\tilde{\eta}^A = \tilde{D}\psi^A,$$

$$\tilde{\eta}^A = \eta^A + \left[\psi^A_2(\psi\cdot\varphi) - \varphi^A(\psi^B)^2\right]\wedge dU.$$

v) To determine μ_{AB} we propose to cast it in the form

$$\mu_{AB} \propto \rho_s[a\psi^F\varphi^E\epsilon_{ABCDEF}(\theta^C\wedge\eta^D + \theta^C\wedge\theta^D + \eta^C\wedge\eta^D) + b\kappa^{AB}]$$

$$\tilde{\mu}_{AB} \propto \mu_{AB} - \frac{1}{2}\rho_s a\psi^F\varphi^E\epsilon_{ABEF}d\xi\wedge dU,$$

where $\xi = (\psi^A)^2(\varphi^B)^2 - (\psi\cdot\varphi)^2$.

vi) Finally we must see the behaviour of the transformed action

$$\tilde{S} = \int \tilde{\mu}_{AB} \wedge \tilde{R}^{AB}$$
$$= S + \int \frac{1}{2}\rho_s a \kappa_{AB} \wedge R^{AB} \wedge d\xi + \int \mu_{AB} \wedge \mathcal{D}\kappa^{AB}.$$

We see that till this point, the SO(2,4)-valuated six-vectors ψ^{F} and φ^{E} are in principle arbitrary. However, under the conditions discussed in the first Section the vectors go to the fiducial ones modulo a phase. Consequently

$$\xi \rightarrow A^2B^2$$

and the bivector comes to

$$\kappa^{AB} \rightarrow \psi^{[A}\varphi^{B]}dU \rightarrow \Delta(AB)\epsilon^{\alpha\beta} = \alpha\beta AB\epsilon^{\alpha\beta} = AB\epsilon^{\alpha\beta}, \qquad \alpha, \beta : 5, 6,$$

where we define the 2nd rank antisymmetric tensor $\varepsilon\alpha\beta$ and

$$\Delta = Det\begin{pmatrix} \lambda^*\alpha & -\mu\beta \\ -\mu^*\alpha & \lambda\beta \end{pmatrix} = \alpha\beta = 1(\text{unitary transformation})$$

A=m and B=λ

- If the coefficients A=m and B=λ play the role of constant parameters we have

$$d\xi \rightarrow d(\lambda^2 m^2) = 0$$

$$\mathcal{D}\kappa^{AB} \rightarrow d(\lambda m)\epsilon^{\alpha\beta} \wedge dU = 0$$

making the original action S invariant e.g.:

$$\widetilde{S}\Big|_{V_0} = \int \widetilde{\mu}_{AB} \wedge \widetilde{R}^{AB} = \int \mu_{AB} \wedge R^{AB} = S$$

being $\widetilde{S}\Big|_{V_0}$ the restriction of $\widetilde{S}$ under the subspace generated by V_0

and consequently breaking the symmetry from *SO(2,4)→SO(1,3).*

The connections after the symmetry breaking (when the mentioned conditions with λ and m constants are fulfilled) become

$$\tilde{\Gamma}^{AB} = \Gamma^{AB} + \kappa^{AB} \Rightarrow b.o.s. \rightarrow \tilde{\Gamma}^{ij} = \Gamma^{ij}; \ \tilde{\Gamma}^{i5} = \Gamma^{i5}, \qquad \tilde{\Gamma}^{i6} = \Gamma^{i6},$$

$$\text{but} \qquad \tilde{\Gamma}^{56} = \Gamma^{56} - (\lambda m) dU.$$

Vectors $\tilde{\theta}^A$ and $\tilde{\eta}^A$ after the symmetry breaking and under the same conditions become

$$\tilde{\theta}^A = \underbrace{d\varphi^A + \Gamma^A{}_C \wedge \varphi^C}_{\theta^A} + \kappa^A{}_B \varphi^B \Rightarrow \text{b.o.s.},$$

$$\tilde{\theta}^i = \theta^i = 0 + \Gamma^i{}_5 m + 0 \Rightarrow \theta^i = \Gamma^i{}_5 m,$$

$$\tilde{\theta}^5 = 0 = 0 + 0 = 0,$$

$$\tilde{\eta}^A = \underbrace{d\psi^A + \Gamma^A{}_C \wedge \psi^C}_{\theta^A} + \kappa^A{}_B \psi^B \Rightarrow \text{b.o.s.},$$

$$\tilde{\eta}^i = \eta^i = 0 - \Gamma^i{}_6 \lambda + 0 \Rightarrow \eta^i = -\Gamma^i{}_6 \lambda,$$

$$\tilde{\eta}^6 = \eta^6 = 0$$

and evidently $\mu_{i5} = \mu_{i6} = 0$.

curvatures becomes

$$R^{\bar{g}} = R^{\bar{g}}_{\{\ \}} + m^{-2}\theta^i \wedge \theta^j + \lambda^{-2}\eta^i \wedge \eta^j,$$

$$R^{i5} = m^{-1}\left[\overbrace{d\theta^i + \omega^i_{\ j} \wedge \theta^j}^{D\theta^i} + \left(\frac{m}{\lambda}\right)\eta^i \wedge \Gamma^{65}\right] = m^{-1}\left[D\theta^i - \frac{m}{\lambda}\eta^i \wedge \Gamma^{65}\right],$$

$$R^{i6} = -\lambda^{-1}\left[D\eta^i - \left(\frac{m}{\lambda}\right)^{-1}\theta^i \wedge \Gamma^{56}\right],$$

$$R^{56} = d\Gamma^{56} + (m\lambda)^{-1}\theta_i \wedge \eta^i,$$

D is the SO(1,3) covariant derivative.

The tensor responsible of the symmetry breaking becomes to

$$\mu_{\bar{g}} = -2\rho_s a\lambda m\epsilon_{\bar{g}kl}(\theta^k \wedge \eta^l + \theta^k \wedge \theta^l + \eta^k \wedge \eta^l)$$

$$\mu_{56} = -\rho_s b\epsilon_{56}\lambda m dU.$$

Consequently, with all ingredients at hand, the action will be

$$S = \int \mu_{AB} \wedge R^{AB} = \underbrace{\int \mu_{ij} \wedge R^{ij}}_{S_1} + \underbrace{\int \mu_{56} \wedge R^{56}}_{S_2},$$

$$\begin{aligned}
S_1 &= -2\int \rho_s a \epsilon_{ijkl}(\theta^k \wedge \eta^l + \theta^k \wedge \theta^l + \eta^k \wedge \eta^l) \wedge \left(\lambda m R^{ij}_{\{\ \}} + \frac{\lambda}{m}\theta^i \wedge \theta^j + \frac{m}{\lambda}\eta^i \wedge \eta^j\right) \\
&= -2\int \rho_s a \epsilon_{ijkl}\left(\theta^k \wedge \eta^l \wedge \lambda m R^{ij}_{\{\ \}} + \theta^k \wedge \theta^l \wedge \lambda m R^{ij}_{\{\ \}} + \eta^k \wedge \eta^l \wedge \lambda m R^{ij}_{\{\ \}}\right) \\
&\quad -2\int \rho_s a \epsilon_{ijkl}\left(\theta^k \wedge \eta^l \wedge \frac{\lambda}{m}\theta^i \wedge \theta^j + \theta^k \wedge \theta^l \wedge \frac{\lambda}{m}\theta^i \wedge \theta^j + \eta^k \wedge \eta^l \wedge \frac{\lambda}{m}\theta^i \wedge \theta^j\right) \\
&\quad -2\int \rho_s a \epsilon_{ijkl}\left(\theta^k \wedge \eta^l \wedge \frac{m}{\lambda}\eta^i \wedge \eta^j + \theta^k \wedge \theta^l \wedge \frac{m}{\lambda}\eta^i \wedge \eta^j + \eta^k \wedge \eta^l \wedge \frac{m}{\lambda}\eta^i \wedge \eta^j\right)
\end{aligned}$$

$$S_2 = -\lambda m \int \rho_s b \epsilon_{56} \wedge \left(d\Gamma^{56} + (m\lambda)^{-1}\theta_i \wedge \eta^i\right)$$

At this point we can naturally associate the tetrad field with the θ-form

$$\theta^k \sim e^k_a \omega^a$$

$$\eta_{ab} = g_{jk} e^j_a e^k_b, \qquad g_{jk} = \eta_{ab} e^a_j e^b_k, \qquad e^k_a e^b_k = \delta^a_b, \qquad \text{etc.},$$

where ηjk is the Minkowski metric. That allows us to up and to down indices, and η^{i} with the following symmetry typical of a SU(2,2) Clifford structure

$$\eta^k \sim f^k_a \omega^a,$$
$$e^a_j f^k_a g_{lk} = f_{lj} = -f_{jl}$$

that consequently allows us to introduce the electromagnetic field (that will be proportional to fij) into the model.

we can re-write the action as

$$S_1 = -2\int \rho_s a \epsilon_{ijkl}(\theta^k \wedge \eta^l + \theta^k \wedge \theta^l + \eta^k \wedge \eta^l) \wedge \left(\lambda m R^{ij}_{\{\ \}} + \frac{\lambda}{m}\theta^i \wedge \theta^j + \frac{m}{\lambda}\eta^i \wedge \eta^j\right)$$

$$= -2\int \rho_s a\Big[\lambda m\left(f_{ij} R^{ij}_{\{\ \}} + \left(g_{ij} + f^k_i f_{kj}\right) R^{ij}_{\{\ \}}\right) + \left(\frac{\lambda}{m} + \frac{m}{\lambda}\right) f^{kj} f_{kj}$$

$$+\left(\frac{\lambda}{m}\sqrt{g} + \frac{m}{\lambda}\sqrt{f}\right)\Big] d^4x.$$

i) terms $\sim \eta\wedge\eta\wedge\eta\wedge\theta$ and $\eta\wedge\theta\wedge\theta\wedge\theta$ vanish;

ii) terms $\sim \eta \wedge \eta \wedge \theta \wedge \theta$ and $\eta \wedge \eta \wedge \theta \wedge \theta$ lead to $\rightarrow f^{kj} f_{kj}$;

iii) term $\sim \epsilon_{\bar{i}\bar{j}kl}\theta^k \wedge \eta^l \wedge R^{\bar{i}\bar{j}}_{\{\ \}}$ leads $\rightarrow f_{\bar{i}\bar{j}} R^{\bar{i}\bar{j}}_{\{\ \}}$

iv) term $\sim \epsilon_{ijkl}(\theta^k \wedge \theta^l + \eta^k \wedge \eta^l) R^{ij}_{\{\ \}}$ leads to $\rightarrow \left(g_{ij} + f^k_i f_{kj}\right) R^{ij}_{\{\ \}}$
picking the symmetric part of the generalized Ricci tensor (containing Einstein-Hilbert plus quadratic torsion term)

v) terms $\sim \eta \wedge \eta \wedge \eta \wedge \eta$ and $\theta \wedge \theta \wedge \theta \wedge \theta$ lead to the volume elements $\sqrt{f}$ and $\sqrt{g}$

where we defined as usual $g \equiv Det(g_{lk})$ and $f \equiv Det(f_{lk}) = (f^*_{lk} f^{lk})^2$.

A=m(x) and B=λ(x): Spontaneous subspace

If the coefficients A=m(x) and B=λ(x) are not constants but functions of the coordinates we have

$$d\xi \rightarrow d(\lambda^2 m^2) = 2d(\lambda m)$$

$$\mathcal{D}\kappa^{AB} \rightarrow d(\lambda m)\epsilon^{\alpha\beta} \wedge dU$$

Consequently

$$\widetilde{S} = \int \widetilde{\mu}_{AB} \wedge \widetilde{R}^{AB} = S + \int \tfrac{1}{2}\rho_s a \kappa_{AB} \wedge R^{AB} \wedge d\xi + \int \mu_{AB} \wedge \mathcal{D}\kappa^{AB}$$

$$\widetilde{S} = S + \int [\mu_{\alpha\beta} + \rho_s a R_{\alpha\beta}\lambda m]\epsilon^{\alpha\beta} d(\lambda m) \wedge dU.$$

we obtain the required condition:

$$\widetilde{S} = S \quad \text{if}$$

$$\mu_{\alpha\beta} = -\rho_s a R_{\alpha\beta}\lambda m,$$

then we see that μ_{AB} takes the place of induced metric and is proportional to the curvature

$$R_{\alpha\beta} = \Lambda\mu_{\alpha\beta}$$

$$\text{with} \quad \Lambda = -(\rho_s a \lambda m)^{-1}.$$

Note that we have now a four-dimensional spacetime plus the above "internal" space of a constant curvature. This point is very important as a new compactification-like mechanism

Supergravity as a gauge theory and topological QFT

we have shown, by means of a toy model, that there exists a supersymmetric analog of the above symmetry breaking mechanism coming from the topological QFT. Here we recall some of the above ideas in order to see clearly the analogy between the group structures of the simplest supersymmetric case, Osp(4), and of the classical conformal group SO(2,4)

The starting point is the super SL(2C) superalgebra (strictly speaking Osp(4))

$$[M_{AB}, M_{CD}] = \epsilon_{C(A} M_{B)D} + \epsilon_{D(A} M_{B)C},$$
$$[M_{AB}, Q_C] = \epsilon_{C(A} Q_{B)}, \qquad \{Q_A, Q_B\} = 2M_{AB}.$$

We define the superconnection A due the following "gauging"

$$A^p T_p \equiv \omega^{\alpha\dot{\beta}} M_{\alpha\dot{\beta}} + \omega^{\alpha\beta} M_{\alpha\beta} + \omega^{\dot{\alpha}\dot{\beta}} M_{\dot{\alpha}\dot{\beta}} + \omega^{\alpha} Q_{\alpha} - \omega^{\dot{\alpha}} \overline{Q}_{\dot{\alpha}},$$

where (ω·M) define a ten dimensional bosonic manifold (Corresponding to the number of generators of SO(4,1) or SO(3,2) that define the group manifold) and p≡multi-index, as usual. Analogically the super-curvature is defined by F≡F·T with the following detailed structure

$$F(M)^{AB} - R^{AB} + \omega^A \wedge \omega^B - 0,$$
$$F(Q)^A - d\omega^A + \omega^A_{\ C} \wedge \omega^C - d_\omega \omega^A - 0,$$

indices $A, B, C...$ stay for $\alpha, \beta, \gamma ...$ $(\dot{\alpha}, \dot{\beta}, \dot{\gamma} ...)$ spinor indices: $\alpha, \beta(\dot{\alpha}, \dot{\beta}) = 1,2(\dot{1}, \dot{2})$ in the Van der Werden spinor notation

There are a bosonic part and a fermionic one associated with the even and odd generators of the superalgebra. Our proposal for the "toy" action was (as before for SO(2,4)) as follows

$$S = \int F^p \wedge \mu_p$$

where the tensor μ_p (that plays the role of a Osp(4) diagonal metric as in the Mansouri proposal) is defined as

$$\mu_{a\dot{\beta}} = \zeta_a \wedge \bar{\zeta}_{\dot{\beta}} \quad \mu_{a\beta} = \zeta_a \wedge \zeta_\beta \quad \mu_a = \nu\zeta_a \ \ etc.$$

with ζ_α anti-commuting spinors (suitable basis: In general this tensor has the same structure that the Cartan-Killing metric of the group under consideration) and ν the parameter of the breaking of super SL(2C) (Osp(4)) to SL(2C) symmetry of μ_p. Notice that the introduction of the parameter ν means that we are not take care in the particular dynamics to break the symmetry.

<u>Dynamical equations</u>

$$\begin{aligned}\delta S &= \int \delta F^p \wedge \mu_p + F^p \wedge \delta\mu_p \\ &= \int d_A\mu_p \wedge \delta A^p + F^p \wedge \delta\mu_p,\end{aligned}$$

where d_A is the exterior derivative with respect to the super-SL(2C) connection and: $\delta F = d_A \delta A$ have been used.

Then, as the result, the dynamics are described by

$$d_A \mu = 0, \qquad F = 0$$

1)The first equation claims that μ is covariantly constant with respect to the super SL(2C) connection.

2) SL(2C) symmetry breaks down to SL(2C)

$$d_A \mu = d_A \mu_{AB} + d_A \mu_A = 0$$

3)super Cartan connection to be flat

$$A = \omega^{AB} + \omega^A$$

$$F(M)^{AB} = R^{AB} + \omega^A \wedge \omega^B = 0,$$
$$F(Q)^A = d\omega^A + \omega^A{}_C \wedge \omega^C = d_\omega \omega^A = 0,$$

d_ω is the exterior derivative with respect to the $SL(2C)$ connection and $R^{AB} \equiv d\omega^{AB} + \omega^A{}_C \wedge \omega^{CB}$ is the $SL(2C)$ curvature

$$F = 0 \Leftrightarrow R^{AB} + \omega^A \wedge \omega^B = 0 \qquad \text{and} \qquad d_\omega \omega^A = 0$$

The second condition says that the SL(2C) connection is super-torsion free.
The first says not that the SL(2C) connection is flat but that it is homogeneous with a cosmological constant related to the explicit structure of the Cartan forms ω^{A},

Quadratic in R_{AB}

The previous action, linear in the generalized curvature, has some drawbacks that make necessary the introduction of additional "subsidiary conditions" due that the curvatures Ri5 and Ri6 play not role into the linear/first order action. Such curvatures have very important information about the dynamics of θ and η fields. In order to simplify the equations of motion we define

$$\Gamma^{56} \equiv A,$$
$$m^{-1}\theta^i \equiv \tilde{\theta}^i,$$
$$\lambda^{-1}\eta^i \equiv \tilde{\eta}^i,$$

and as always

$$R^{ij} = R^{ij}_{\{\ \}} + m^{-2}\theta^i \wedge \theta^j + \lambda^{-2}\eta^i \wedge \eta^j$$

with the $SO(1,3)$ curvature $R^{ij}_{\{\ \}} = d\omega^{ij} + \omega^i{}_\lambda \wedge \omega^{\lambda j}$.

Consequently from the quadratic Lagrangian density

$$S = \int R_{AB} \wedge R^{AB}$$

we obtain the following equations of motion:

$$\frac{\delta(R_{AB} \wedge R^{AB})}{\delta\theta^i} \rightarrow D\left(D\tilde{\theta}_j\right) + 2R_{ij} \wedge \tilde{\theta}^i - \tilde{\theta}^i \wedge \tilde{\eta}_i \wedge \tilde{\eta}_j + \tilde{\theta}_j \wedge A \wedge A = 0,$$

$$\frac{\delta(R_{AB} \wedge R^{AB})}{\delta\eta^i} \rightarrow D\left(D\tilde{\eta}_j\right) + 2R_{jk} \wedge \tilde{\eta}^k - \tilde{\theta}^i \wedge \tilde{\eta}_i \wedge \tilde{\theta}_j + \tilde{\eta}_j \wedge A \wedge A = 0,$$

$$\frac{\delta(R_{AB} \wedge R^{AB})}{\delta\Gamma^{56}} \rightarrow \tilde{\theta}^i \wedge \tilde{\theta}_i - \tilde{\eta}^i \wedge \tilde{\eta}_i,$$

$$\frac{\delta(R_{AB} \wedge R^{AB})}{\delta\omega^i_j} \rightarrow -DR_{kl} + D\tilde{\theta}_k \wedge \tilde{\theta}_l + D\tilde{\eta}_k \wedge \tilde{\eta}_l + \tilde{\theta}_k \wedge \tilde{\eta}_l \wedge A = 0.$$

Maxwell equations and the electromagnetic field

we can identify

$$\theta^i \equiv e^i_\mu dx^\mu,$$
$$\eta^i \equiv f^i_\mu dx^\mu$$

with the symmetries

$$e^i_\mu e^\nu_i = \delta^\nu_\mu, e^i_\mu e_{i\nu} = g_{\mu\nu} = g_{\nu\mu}$$

$$f^i_\mu f^\nu_i = \delta^\nu_\mu, \qquad e_{i\nu} f^i_\mu = f_{\mu\nu} = -f_{\nu\mu}$$

such that the geometrical (Bianchi) condition

$$\nabla_{[\rho} f_{\mu\nu]} = \nabla^*_\rho f^{\rho\nu} = 0$$

$$D\left(\tilde{\theta}^i \wedge \tilde{\eta}_i\right) = 0$$

enforce to the curvatures R^{i6} and R^{i5} to be null. And conversely if R^{i6} and R^{i5} are zero then $D\left(\tilde{\theta}^i \wedge \tilde{\eta}_i\right) = 0$

or equivalently $\nabla_{[\rho} f_{\mu\nu]} = \nabla^*_\rho f^{\rho\nu} = 0.$

Proof

From: $R^{i5} = \left[D\tilde{\theta}^i - \tilde{\eta}^i \wedge \Gamma^{65}\right]$ and $R^{i6} = \left[-D\tilde{\eta}^i + \tilde{\theta}^i \wedge \Gamma^{56}\right]$ we make

$$R^{i5} \wedge \tilde{\eta}_i + \tilde{\theta}_i \wedge R^{i6} = D\left(\tilde{\theta}^i \wedge \tilde{\eta}_i\right) + (\tilde{\eta}^i \wedge \Gamma^{56}) \wedge \tilde{\eta}_i + \tilde{\theta}_i \wedge \left(\tilde{\theta}^i \wedge \Gamma^{56}\right),$$
$$R^{i5} \wedge \tilde{\eta}_i + \tilde{\theta}_i \wedge R^{i6} = D\left(\tilde{\theta}^i \wedge \tilde{\eta}_i\right).$$

(In the last line we used the constraint given by eq $\tilde{\theta}^i \wedge \tilde{\theta}_i = \tilde{\eta}^i \wedge \tilde{\eta}_i$,

Consequently if R^{i6} and R^{i5} are zero then $D\left(\tilde{\theta}^i \wedge \tilde{\eta}_i\right) = 0$ or equivalently $\nabla_{[\rho} f_{\mu\nu]} = \nabla^*_\rho f^{\rho\nu} = 0$ and vice versa.

Corollary

Note that the vanishing of the R^{56} curvature (that transforms as a Lorentz scalar) does not modify the equation of motion for Γ^{56} and simultaneously defines the electromagnetic field as

$$R^{56} = d\Gamma^{56} + (m\lambda)^{-1}\theta_i \wedge \eta^i = 0,$$
$$\Rightarrow dA - F = 0.$$

Equations of motion in components and symmetries

Let us define

$$R^{ij}_{\{\ \}\mu\nu} = \partial_\mu \omega^{ij}_\nu - \partial_\nu \omega^{ij}_\mu + \omega^{i}_{\mu k}\omega^{kj}_\nu - \omega^{kj}_\mu \omega^{i}_{\nu k},$$

$$T^i_{\mu\nu} = \partial_\mu e^i_\nu - \partial_\nu e^i_\mu + \omega_{\mu\ k}^{\ i} e^k_\nu - \omega_{\nu\ k}^{\ i} e^k_\mu,$$

$$S^i_{\mu\nu} = \partial_\mu f^i_\nu - \partial_\nu f^i_\mu + \omega_{\mu\ k}^{\ i} f^k_\nu - \omega_{\nu\ k}^{\ i} f^k_\mu.$$

$S^i_{\mu\nu}$ is a totally antisymmetric torsion field due the symmetry of $f^i_\nu dx^\nu = \eta^i$

$$\nabla_\mu\left[\sqrt{|g|}R^{ij\mu\nu}\right] + \sqrt{|g|}(-m^{-2}T^{ji\nu} + \lambda^{-2}S^{ji\nu}) - \sqrt{|g|}(\lambda m)^{-1}f^{[i\nu}A^{i]} = 0,$$

$$\nabla_\mu\left[\sqrt{|g|}\left(R^{ij\mu\nu}_{\{\ \}} - m^{-2}e^{[i\mu}e^{j]\nu} + \lambda^{-2}f^{[i\mu}f^{j]\nu}\right)\right]$$
$$+\sqrt{|g|}(-m^{-2}T^{ji\nu} + \lambda^{-2}S^{ji\nu}) - \sqrt{|g|}(\lambda m)^{-1}f^{[i\nu}A^{i]} = 0,$$

$$\nabla_\mu\left(\sqrt{|g|}T^{j\mu\nu}\right) + \sqrt{|g|}\left(R^{j\nu}_{\{\ \}} - m^{-2}e^{j\nu} + A^iA^\nu\right) = 0,$$

$$\nabla_\mu\left(\sqrt{|g|}S^{j\mu i}\right) + \sqrt{|g|}\left(R^{ij}_{\{\ \}} - \lambda^{-2}f^{ij} + A^{[i}A^{j]}\right) = 0,$$

$$\nabla_{[\mu}A_{\nu]} = F_{\mu\nu} = (\lambda m)^{-1}F_{\mu\nu},$$

$$\nabla_{[\rho}F_{\mu\nu]} = 0.$$

Nonlinear realizations viewpoint

Notice that in our case identify θ~e and η~f being the table below completely understood. Also the Γ^{65} is identified with the g of E. Ivanov and J. Niederle

	this work	cite: Ivanov:1981wn,Ivanov:1981wm
R^{ij}	$R^{ij}_{\{\ \}} + m^{-2}\theta^i \wedge \theta^j + \lambda^{-2}\eta^i \wedge \eta^j$	$R^{ij}_{\{\ \}} + 4ge^i \wedge f^j$
R^{i5}	$m^{-1}[D\theta^i - \frac{m}{\lambda}\eta^i \wedge \Gamma^{65}]$	$De^i + 2ge^i \wedge \mathbf{g}$
R^{i6}	$-\lambda^{-1}\left[D\eta^i - \left(\frac{m}{\lambda}\right)^{-1}\theta^i \wedge \Gamma^{56}\right]$	$Df^i - 2gf^i \wedge \mathbf{g}$
R^{56}	$d\Gamma^{56} + (m\lambda)^{-1}\theta_i \wedge \eta^i$	$d\mathbf{g} + 4ge_i \wedge f^i$
DS/ADS reduction	Yes	No

Algebra and transformations in the case of the work of Ivanov and Niederle are different due different definitions of the generators of the SO(2,4) algebra, however the meaning of g that it is associated to the connection Γ^{65} remains obscure for us because the second Cartan structure equations R^{i5} and R^{i6}. Notice that, although the group theoretical viewpoint in the case of the simoultaneous nonlinear realization of the affine and conformal group Borisov:1974 to obtain Einstein gravity are more or less clear, the pure geometrical picture is still hard to recognize due the factorization problem and the orthogonality between coset elements and the corresponding elements of the stability subgroup

A.B.Borisov and V.I.Ogievetsky, Theor.Math.Phys.21, 1179 (1975) [Teor.Mat. Fiz.21, 329 (1974)];
E. A. Ivanov and J. Niederle, Phys. Rev. D 25, 976 (1982), Phys. Rev. D 25, 988 (1982).

Discussion

1. In this work, we introduced two geometrical models: one linear and another one quadratic in curvature.
2. Both models are based on the *SO(2, 4) group.*
3. *Dynamical* breaking of this symmetry was considered. In both cases, we introduced coherent states of the Klauder–Perelomov type, which as defined by the action of a group (generally a Lie group) are invariant with respect to the stability subgroup of the corresponding coset being related to the possible extension of the connection which maintains the proposed action invariant.
4. The linear action, unlike the cases of West, Kerrick or even McDowell and Mansouri [41], uses a symmetry breaking tensor that is dynamic and unrelated (in principle) to a particular metric.
5. Such a tensor depends on the introduced vectors (i.e. the coherent states) that intervene in the extension of the permissible symmetries of the original connection.
6. Only some components of the curvature, defined by the second structure equation of Cartan, are involved in the action, leaving the remaining ones as a system of independent or ignorable equations in the final dynamics.
7. The quadratic action, however, is independent of any additional structure or geometric artifacts and all the curvatures (e.g. all the geometrical equations for the fields) play a role in the final action (Lagrangian of the theory).
8. With regard to the parameters that come into play *λ and m (they play the role of* a cosmological constant and a mass, respectively), we saw that in the case of linear action if they are taken dependent on the coordinates and under the conditions of the action invariance, a new spontaneous compactification mechanism is defined in the subspace invariant under the stability subgroup.
9. Following this line of research with respect to possible physical applications, we consider scenarios of the Grand Unified Theory, derivation of the symmetries of the Standard Model together with the gravitational ones. The general aim is to obtain in a precisely established way the underlying fundamental theory.
10. This will be important, in particular, to solve the problem of hierarchies and fundamentalconstants, the masses of physical states, and their interaction.