Ruijsenaars type deformation of hyperbolic BC_n Sutherland model : An illustration of Action-angle duality arising from Hamiltonian reduction

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Duality

Suppose that we have an Integrable system $\{H_1, \ldots, H_n\}$ on (M, ω), expressed typically in terms of some physically meaningful variables (q, p); then there exist action-angle coordinates (θ, λ) on M wrt which $\omega = \sum d\theta_i \wedge d\lambda_i$ and such that each function H_k is independent of θ. That is

$$(q, p) \longleftrightarrow (\theta, \lambda)$$

and

$$H_k(q, p) = F_k(\lambda(q, p))$$

We might now just express (q, p) in terms of (θ, λ), and then define another set of Hamiltonians by $\hat{H}_i(\theta, \lambda) = \Phi_i(p)$; meaning

$$\hat{H}_i(\theta, \lambda) = \Phi_i(p(\theta, \lambda)),$$

and of course this set of Hamiltonians will constitute an integrable system, and we may call it "dual" to the orginal one.

All of this looks a bit trivial. *However*, if we demand that both systems have some specified structure, and say—for example—that they should both be "of Calogero type", then the problem is nontrivial, even if it remains somehow informal in nature.
If we further introduce the technical requirement that the construction be *global* on the whole of M, and not just depend on the local coordinates (q, p) and (λ, θ), then we see that the possibilities are considerably restricted, and dual systems turn out to be a rarity.
Duality is the Classical Mechanics analogue of *Bispectrality* in Quantum Mechanics.

Example: Calogero–Moser

$$H(q,p) = \frac{1}{2}p^2 + \frac{1}{2}\beta^2 \sum_{i \neq j} \frac{1}{(q_i - q_j)^2}$$

with Lax representation

$$\dot{L} = [M, L].$$

where L and M are both in $u(n)$ and

$$L_{diag} = i\text{diag}(p_1, \dots, p_n).$$

A recipe for finding action–angle variables is as follows:
— Define $Q = i\text{diag}(q_1, \dots, q_n)$
— Look for $U(q,p) \in U(n)$ s.t. $\tilde{L} := ULU^\dagger$ be diagonal and consider $\tilde{Q} := UQU^\dagger$.
— Then the diagonal parts of $\tilde{Q}$ and of $\tilde{L}$ will be canonically conjugate, and they generate action–angle variables.

It happens—apparently by a sort of miracle—that $\tilde{Q}$ has the same form as L and so the “dual system” is the same as the system we started with.
This is not always the case: otherwise, the whole exercise would not be so very interesting...

$$L = \begin{pmatrix} ip_1 & & -\beta(q_k - q_l)^{-1} \\ & \ddots & \\ * & & ip_n \end{pmatrix} = \mathcal{L}(q,p), \quad \text{say},$$

$$Q = \begin{pmatrix} iq_1 & & \\ & \ddots & \\ & & iq_n \end{pmatrix}$$

Look for $g \in U(n)$ s.t. $\tilde{L} := gLg^\dagger$ be diagonal, and define $\tilde{Q} = gQg^\dagger$. Denoting the diagonal parts of $\tilde{L}$ and $\tilde{Q}$ by λ and θ respectively, we find that $dq \wedge dp = d\theta \wedge d\lambda$ and

$$\tilde{Q} = \mathcal{L}(\lambda, \theta)$$

The CM system is generated by invariants of L, and the dual system is generated by invariants of $\tilde{Q}$. Hence we see that the "dual system" to CM is again CM.

History

1. Olshanetsky and Perelomov discovered the hyperbolic BC_n Sutherland model by a reduction/projection procedure, but it had only 2 independent parameters
2. Inozemtsev and Meshcheryakov proved integrability of the three parameter version
3. Feher-Pusztai obtained it with all three parameters by reduction: Reduce $M = T^*K$ by canonical left and right actions of subgroups $K_+ \prec K$, with $K = SU(n,n)$ and K_+ the compact subgroup satisfying $gg^\dagger = I$. Fix the generators of right- and left-actions : J^r is chosen to be a character, and J^l an analog of the so-called Kazhdan-Kostant-Sternberg element.

$$2H = \sum_{i=1}^{n} p_i^2 + a \sum_{i=1}^{n} \sinh^{-2}(2q_i) + b \sum_{i=1}^{n} \sinh^{-2}(q_i) + c \sum_{i,j \; i \neq j} \left[\sinh^{-2}(q_i + q_j) + \sinh^{-2}(q_i - q_j) \right].$$

Relativistic versions of Calogero type models were introduced by Ruijsenaars. It was proposed in several papers of Gorsky and others that Poisson Lie group reduction should be the appropriate setting for these systems.

Feher and Klimcik worked on this project and found PLG reduction interpretations for several known Ruijsenaars type systems.

My result is a PLG reduction construction of the same kind. It imitates the result of Feher and Pusztai, essentially amounting to the replacement of T^*K by the Heisenberg double of K. The product of this procedure is a *new* integrable system.

This is roughly speaking the whole story. The rest is a lot of tricky computations.

However, van Diejen demonstrated quantum integrability for the same kinds of systems, and extended to a wider class than has been proved for the classical analogues. The result is the curious situation in which we have a collection of systems which are "known to be integrable" but are lacking solid proofs, and for which—in particular—it would be interesting to discover Lax representations.

Reduction

Suppose we have a group G acting on a differentiable manifold M. The conerstone of reduction is the

decision to restrict from the ring $C^\infty(M)$ of all smooth functions on M to the subring of invariant functions $C^\infty(M)^G$.

This means that we shall be interested only in invariant flows. That is, for $v \in Vect(M)$, for any $\varphi \in C^\infty(M)$ we require

$$v(\varphi \circ g)(x) = v(\varphi)(g \cdot x), \qquad \text{where} \quad (\varphi \circ g)(x) = \varphi(g \cdot x). \quad (1)$$

Suppose that G be a Lie group.
With $\hat{\xi}(\varphi)(x) := \left.\frac{d}{dt}\right|_{t=0} \varphi\left(e^{t\xi} \cdot x\right)$, then infinitesimally (1) amounts to $\mathcal{L}_{\hat{\xi}} v = 0 \quad \forall \xi \in \mathfrak{g}$. Interpret v as a vector field on M/G, by restricting it to act only on $C^\infty(M)^G$.

Poisson case : M, G

Suppose now that M is a Poisson space, and that G is a Lie group acting on M in a so-called "admissible fashion". That is,

$$F, K \in C^{\infty}(M)^{G} \;\Rightarrow\; \{F, K\} \in C^{\infty}(M)^{G}. \tag{2}$$

In this case, further reduction may be possible.

Poisson reduction I : $\mathcal{F}$ and $\mathcal{F}_{gen}(\nu)$

Suppose that $\mathcal{F} \subset C^\infty(M)$ is a ring of functions, all of whose Hamiltonian vector fields are tangent to the vector fields in $\hat{\mathfrak{g}}$, i.e., $\varphi \in \mathcal{F}, F \in C^\infty(M)^G \;\Rightarrow\; 0 = \mathbb{X}_\varphi(F) = -\mathbb{X}_F(\varphi)$.
Restricting to G-invariant functions $C^\infty(M)^G$, all the functions in $\mathcal{F}$ may be viewed as constants :
(i) Introduce $\mathcal{F}_{gen} := \{\varphi_\alpha \mid \alpha \in \mathcal{A}\}$, for which $\mathcal{F} = \langle \mathcal{F}_{gen} \rangle$, but for any strict subset $\hat{\mathcal{F}} \subset \mathcal{F}_{gen}$, $\mathcal{F} \neq \langle \hat{\mathcal{F}} \rangle$.
(ii) For $\nu := \{\nu_\alpha \in \mathbb{R} \mid \alpha \in \mathcal{A}\}$, define

$$\mathcal{F}_{gen}(\nu) := \{\varphi_\alpha - \nu_\alpha \mid \alpha \in \mathcal{A}\}.$$

(iii) Restrict to the submanifold

$$N_\nu := \bigcap_{\psi \in \mathcal{F}_{gen}(\nu)} \psi^{-1}(0) = \bigcap_{\alpha \in \mathcal{A}} \{x \in M \mid \varphi_\alpha(x) = \nu_\alpha\}.$$

Poisson reduction II : factor by G_ν

Suppose that $G_\nu \subset G$ is the maximal subgroup which acts on N_ν.

$$C^\infty(M)|_{N_\nu} \supset C^\infty(M)^G|_{N_\nu} = \left(C^\infty(M)|_{N_\nu}\right)^{G_\nu}$$

We arrive at a Poisson algebra consisting of the functions

$$C^\infty(N_\nu)^{G_\nu},$$

or, in other words, to the reduced space

$$N_\nu/G_\nu =: M_{red}(\nu) \quad \text{say.}$$

If $\mathbf{F} := \{F_i \mid i = 1, 2, \dots\}$ is a collection of invariant functions with the property $\{F_i, F_j\} = 0$, then their restrictions to N_ν will still have zero Poisson bracket, and they define commuting functions on the reduced space $M_{red}(\nu)$.

KKS example

$M = su(n) \times su(n)$, $\omega(X,Y) = -\text{tr } dX \wedge dY$. $G = SU(n)$, with $G \circlearrowright M$ given by $g \cdot (X,Y) = (gXg^{-1}, gYg^{-1})$.
The functions $\Phi_k = -\frac{1}{2k}\text{tr } Y^{2k}$ are invariant and are in involution with one another.
The Hamiltonian vector fields $\mathbb{X}_k$ generated by the functions Φ_k are given by $(\dot{X}, \dot{Y}) = (Y^{2k-1}, 0)$ and are trivially integrated.
Fix any vector $\hat{v} \in \mathbb{C}^n$ with $|\hat{v}|^2 = n$, then, for $\xi \in \mathfrak{g} = su(n)$, define $F_\xi(X,Y) = \text{tr}\left([X,Y] - i(\mathbf{I} - \hat{v}\hat{v}^\dagger)\right)\xi$.
$\mathcal{F}_{gen}(\nu) = \{F_\xi \mid \xi \in \mathfrak{g}\}$, and $N_\nu = \{(X,Y) \mid [X,Y] = i(\mathbf{I} - \hat{v}\hat{v}^\dagger)\}$.
Writing $(X,Y) = g^\dagger \cdot (\hat{X}, \hat{Y})$, with $\hat{X}$ diagonal, observe that g is defined modulo left-multiplication by $\mathbb{T}$.

$(X,Y) \in N_\nu \Rightarrow [\hat{X},\hat{Y}] = i(\mathbf{I} - vv^T)$, with $v = g\hat{v}$, and w.l.o.g. the $\mathbb{T}$-ambiguity in the definition of g permits the assumption $v_k \in \mathbb{R}_{\geq 0}\ \forall k$. Solving the condition on $(\hat{X},\hat{Y})$ yields
$\hat{X}_{kk} = ix_k$, and $k \neq l \Rightarrow x_k \neq x_l$
$\hat{Y}_{kl} = -(x_k - x_l)^{-1}$ if $k \neq l$, $\hat{Y}_{kk} = iy_k$
– $M_{red}(\nu)$ is represented by the subspace $\widehat{M}(\nu)$ in M of all $(\hat{X},\hat{Y})$ of this form. Restricting ω to $\widehat{M}(\nu)$ gives $\omega_{red} = \sum_k dx_k \wedge dy_k$, and restricting Φ_1 gives the standard Calogero-Moser Hamiltonian

$$H = \frac{1}{2}\sum y_k^2 + \sum\sum_{k\neq l}(x_k - x_l)^{-2}$$

$\mathcal{M}$: Heisenberg double, and $\mathcal{K}$ — our symmetry group

Let $I(= I_{nn}) = \begin{pmatrix} \mathbf{I_n} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I_n} \end{pmatrix}$. $\mathcal{M} = G$ denotes $SL(2n, \mathbb{C})$.
K denotes $SU(2n) = \{g \in SL(2n, \mathbb{C}) \mid g^\dagger g = id\}$.
B denotes the set of all upper triangular matrices in $SL(2n, \mathbb{C})$ with real, positive diagonal entries, and B_n denotes the same set in $GL(n, \mathbb{C})$.

$K \supset K_+ = \left\{ \begin{pmatrix} p & \mathbf{0} \\ \mathbf{0} & q \end{pmatrix} \right\}, \quad p, q, \in U(n), \qquad \mathcal{K} = K_+ \times K_+$ acts on $\mathcal{M}$ by ordinary left and right multiplication.

—decompositions on $\mathcal{M}$

Any matrix in $g \in SL(2n, \mathbb{C})$ can be written, uniquely, either in the form

$$g = k_L b_R, \qquad \text{with} \;\; k_L \in K, \; b_R \in B$$

or in the form

$$g = b_L k_R, \qquad \text{with} \;\; k_R \in K, \; b_L \in B$$

$\mathfrak{g} = Lie(G)$ can be decomposed as the sum $\mathfrak{g} = \mathfrak{k} + \mathfrak{b}$ of the two subalgebras $\mathfrak{k} = Lie(K)$ and $\mathfrak{b} = Lie(B)$, with respect to which the projections $P_{\mathfrak{k}} : \mathfrak{g} \to \mathfrak{k}$ and $P_{\mathfrak{b}} : \mathfrak{g} \to \mathfrak{b}$ are well-defined.

—Poisson structure

Let $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ denote the non-degenerate, invariant inner product defined by

$$\langle X, Y\rangle = \operatorname{Im} \operatorname{tr} XY.$$

Then $R := P_{\mathfrak{k}} - P_{\mathfrak{b}}$ defines a classical r-matrix on $\mathfrak{g}$, skew-symmetric with respect to $\langle\ ,\ \rangle$. For any function $F \in C^\infty(G)$, the left- and right-derivatives, $D^{l,r}F : G \to \mathfrak{g} \sim \mathfrak{g}^*$, of F are defined by

$$\left.\frac{d}{dt}\right|_{t=0} F(e^{tX} g e^{tY}) = \langle D^l F(g), X\rangle + \langle D^r F(g), Y\rangle \qquad \forall X, Y \in \mathfrak{g}.$$

The Poisson structure on G, viewed as the Heisenberg double based on the bi-algebra $\mathfrak{g} = \mathfrak{k} + \mathfrak{b}$, is defined by

$$\{F, H\} = \langle D^l F, R(D^l H)\rangle + \langle D^r F, R(D^r H)\rangle.$$

—and symplectic structure

Let $c_1, c_2 \in G$ and define the subspace $M(c_1, c_2) \subset G$ by

$$M(c_1, c_2) = \{bc_1k \mid b \in B, k \in K\} \cap \{kc_2b \mid b \in B, k \in K\}.$$

Introduce "coordinates" (b_L, k_L, b_R, k_R) on $M(c_1, c_2)$ (not independent) by

$$M(c_1, c_2) \ni g = b_L c_1 k_R = k_L c_2 b_R.$$

Proposition (Alekseev and Malkin – CMP1994)
$M(c_1, c_2)$ is a symplectic leaf, and all symplectic leaves are of this form. The symplectic structure on $M(c_1, c_2)$ can be written

$$[Symp](g) = \langle db_L b_L^{-1} \overset{\wedge}{,} dk_L k_L^{-1} \rangle \; + \; \langle b_R^{-1} db_R \overset{\wedge}{,} k_R^{-1} dk_R \rangle.$$

Momentum maps

The maps

$$g = k_L b_R \mapsto b_R \in B$$

and

$$g = b_L k_R \mapsto b_L \in B$$

generate the left and right actions of K on $\mathcal{M}$. The compositions of these maps with the projection $B \to B/N$ defines generators of the actions of K_+ on $\mathcal{M}$, where

$$N := \left\{ \begin{pmatrix} \mathbf{I} & X \\ 0 & \mathbf{I} \end{pmatrix} \right\} \subset B$$

some useful gadgets

Introduce $\hat{w}$ s.t. $I\hat{w} = \hat{w}$; that is $\hat{w} = \begin{pmatrix} \hat{v} \\ 0 \end{pmatrix}$ for $\hat{v} \in \mathbb{C}^n$, then:

$$\Omega(kb) = bb^\dagger,$$
$$L(kb) = k^\dagger I k I,$$
$$w(kb) = k^\dagger \hat{w}$$

Ω and L are invariant wrt the action of K_+ on the left. The action of K_+ on the right becomes

$$\Omega(gf^\dagger) = \tilde{f}\Omega(g)\tilde{f}^\dagger,$$
$$L(gf^\dagger) = \tilde{f}L(g)\tilde{f}^\dagger,$$
$$w(gf^\dagger) = \tilde{f}w(g).$$

Key observation : $LIw = w$.

“Geodesic flows”

Proposition The functions F_l and Φ_l, defined by

$$\begin{aligned} F_l(g) &= \frac{1}{2l}\operatorname{tr}\Omega(g)^l, \\ \Phi_l(g) &= \frac{1}{2l}\operatorname{tr}L(g)^l, \end{aligned} \qquad l = 1, 2, \ldots$$

are all invariant, and are separately in involution on $\mathcal{M}$. The Hamiltonian vector fields are

$$\mathbb{X}_{F_l}(g) \;:\; \begin{cases} \dot{k} = \mathrm{i}k\,[\Omega^l - \nu_l\, id\,] \\ \dot{b} = 0 \end{cases}, \qquad \text{with} \quad \nu_l = (2n)^{-1}\operatorname{tr}\Omega^l, \tag{3}$$

$$\mathbb{X}_{\Phi_l}(g) \;:\; \begin{cases} \dot{k} = \frac{1}{2}\mathrm{i}k(IL^{l-1} - L^{l+1}I - IL^l + L^lI) \\ \dot{b} = \frac{1}{2}\mathrm{i}(id + I)L^l(id - I)b. \end{cases} \tag{4}$$

Each of these vector fields generates a complete flow on $\mathcal{M}$.

Constraints : momentum map formulation

Fixing $\sigma \in B_n$ and $x, y \in \mathbb{R}_+$, the constraints are as follows: suppose that when written in the form $G \ni g = k_L b_R$, the block-diagonal part of b_R is fixed:

$$b_R = \begin{pmatrix} x\mathbf{I} & \omega \\ \mathbf{0} & x^{-1}\mathbf{I} \end{pmatrix}$$

and that, when written in the form $g = b_L k_R$, the block-diagonal part of b_L is fixed:

$$b_L = \begin{pmatrix} y^{-1}\sigma & y^{-1}\nu \\ \mathbf{0} & y\mathbf{I} \end{pmatrix},$$

with $\det(\sigma) = 1$, and with both ω and ν undetermined in $gl(n)$. ($\sigma \in B_n$ will be the PLG analog of the KKS element)

Equivalently, using $\mathcal{F}_{gen}(\nu)$

For $\xi, \eta \in \mathfrak{k}_+$, define

$$F_{\xi,\eta}(g) = \mathrm{Im}\ \mathrm{tr}\, \xi \left[gg^\dagger - y^{-2} gg^\dagger \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{I} \end{pmatrix} gg^\dagger - y^{-2} \begin{pmatrix} \sigma\sigma^\dagger & 0 \\ 0 & 0 \end{pmatrix} \right]$$

$$\cdot$$

$$+ \mathrm{Im}\ \mathrm{tr}\, \eta \left[g^\dagger g - x^{-2} g^\dagger g \begin{pmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{pmatrix} g^\dagger g - x^{-2} \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{I} \end{pmatrix} \right],$$

and then the constraint submanifold may be defined equivalently by

$$N_{x,y,\sigma} := \{ g \in \mathcal{M} \mid F_{\xi,\eta}(g) = 0 \quad \forall \xi, \eta \in \mathfrak{k}_+ \}.$$

diagonalizations

The diagonal subgroup in $U(n)$ will be denoted by $\mathbb{T}$.

For any $k \in K$ we may write

$$k = \begin{pmatrix} \rho & \mathbf{0} \\ \mathbf{0} & m \end{pmatrix} \begin{pmatrix} \Gamma & \Sigma \\ \Sigma & -\Gamma \end{pmatrix} \begin{pmatrix} p & \mathbf{0} \\ \mathbf{0} & q \end{pmatrix}, \quad \text{with } \rho, m, p, q \in SU(n),$$

.

. $\Gamma = diag(\cos(\Delta_i))$, $\Sigma = diag(\sin(\Delta_i))$

For any $b \in B$, satisfying the right-constraint we may write

$$b = \begin{pmatrix} p & \mathbf{0} \\ \mathbf{0} & q \end{pmatrix} \begin{pmatrix} x & \beta \\ \mathbf{0} & x^{-1} \end{pmatrix} \begin{pmatrix} p^\dagger & \mathbf{0} \\ \mathbf{0} & q^\dagger \end{pmatrix}, \quad \text{with } p, q \in SU(n),$$

.

. $\beta = diag(\beta_1, \ldots, \beta_n)$, $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n \geq 0$

Factoring on the right by K_+ and on the left by the subgroup $\left\{\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & p \end{pmatrix}\right\} \subset K_+$, let's assume that g is in the gauge

$$g = \begin{pmatrix} \rho\Gamma & \rho\Sigma \\ \Sigma & \Gamma \end{pmatrix} \begin{pmatrix} x\mathbf{I} & \omega \\ \mathbf{0} & x^{-1}\mathbf{I} \end{pmatrix} \quad \text{with } \omega \in gl(n, \mathbb{C}),\ \rho \in SU(n)$$

Compare the two versions of $gI_{nn}g^\dagger$ and reduce to a constraint condition for a certain $T \in U(n)$,

$$T^\dagger \Sigma^2 T = \Sigma \rho^\dagger \sigma \sigma^\dagger \rho \Sigma$$

Explanation of where we're up to:
Let us make the substitution $\Omega = \Sigma\omega + x^{-1}\Gamma$, so that

$$g = \begin{pmatrix} x\rho\Gamma & \rho\Sigma^{-1}(\Gamma\Omega - x^{-1}\mathbf{I}) \\ x\Sigma & \Omega \end{pmatrix}.$$

Now, from $gI_{nn}g^\dagger = b_L I_{nn} b_L^\dagger$ we get

$$\Omega\Omega^\dagger = y^2\mathbf{I} + x^2\Sigma^2 =: \Lambda^2 \;\Rightarrow\; \Omega = \Lambda T, \quad T \in U(n)$$

$$\nu = \rho\Sigma^{-1}(y^2\Gamma - x^{-1}\Omega^\dagger)$$

and

$$T^\dagger\Sigma^2 T = \Sigma\rho^\dagger\sigma\sigma^\dagger\rho\Sigma$$

Now specify the KKS element

The PLG KKS element is $\sigma \in B_n$, satisfying

$$\sigma\sigma^\dagger = \alpha^2 \mathbf{I} + \hat{v}\hat{v}^\dagger,$$

for $\alpha \in \mathbb{R}$ and $\hat{v} \in \mathbb{C}^n$ s.t. $\det \sigma = 1$, so the constraint reads

$$T^\dagger \Sigma^2 T = \alpha^2 \Sigma^2 + vv^T$$

$v := \Sigma \rho^\dagger \hat{v}$ and can be supposed real with all entries in $\mathbb{R}_{\geq 0}$.
The constraint condition can be solved(!)
After some work (quite a lot), restricting the Alekseev–Malkin formula to our constraint subspace,

$$[Symp] = \langle\, \rho^\dagger d\rho \overset{\wedge}{,} \Sigma^{-1} T^\dagger \Sigma d(\Sigma^{-1} T \Sigma) \,\rangle + \langle\, T^\dagger dT + dT T^\dagger \overset{\wedge}{,} \Sigma^{-1} d\Sigma \,\rangle.$$

After using a few more tricks, we arrive at

$$[Symp] = \sum_{i=1}^{n} dp_i \wedge \Sigma_i^{-1} d\Sigma_i$$

where, for $P := e^{ip} \in \mathbb{T}$, the general solution of the constraint condition was $T = P\tilde{T}$ and $\tilde{T}$ is a (complicated!) explicit, real matrix function of Σ.

The simplest of the commuting Hamiltonians produces

$$\Phi_1 = \tfrac{1}{2}(x^{-2} + y^2) \sum_{i=1}^{n} \Sigma_i^{-2}$$

$$- x^{-1} \sum_{i=1}^{n} (\cos p_i) \sqrt{1 + \Sigma_i^{-2}} \sqrt{x^2 + y^2 \Sigma_i^{-2}} \times$$

$$\prod_{k \neq i} \frac{\sqrt{\Sigma_k^2 - \alpha^2 \Sigma_i^2} \sqrt{\alpha^2 \Sigma_k^2 - \Sigma_i^2}}{\alpha(\Sigma_k^2 - \Sigma_i^2)}$$

The linearisation of the Hamiltonian - which means the degeneration of G to the cotangent bundle of T^*K, which is the same as the semi-direct product $K \ltimes \mathfrak{k}^* \sim K \ltimes \mathfrak{b}$ - yields the BC_n Hamiltonian of Feher and Pusztai

$$H_2 = \tfrac{1}{2}\sum_{i=1}^{n} \hat{p}_i^2 + c_1 \sum_{i=1}^{n} \frac{1}{\sinh^2 \hat{q}_i} + c_2 \sum_{i=1}^{n} \frac{1}{\sinh^2(2\hat{q}_i)} + c_3 \sum_{i,j \;\; i\neq j} \left[\frac{1}{\sinh^2(\hat{q}_i + \hat{q}_j)} + \frac{1}{\sinh^2(\hat{q}_i - \hat{q}_j)} \right].$$

— second model:

Now let's assume that g is in the gauge

$$g = k \begin{pmatrix} x & \beta \\ 0 & x^{-1} \end{pmatrix}$$

This time, we are *not* able to find k in terms of β: if we could, then we'd repeat the procedure adopted for the first model, and rerstrict the AM sympletic form to the resulting explicitly defined subspace. We *were* able, by a series of clever tricks, to uncover an explicit characterisation of the subspace, and to find its Poisson bracket.

Here's what we do:

Subsituting $g = kb$ in the constraint condition

$$gg^\dagger - gg^\dagger \begin{pmatrix} 0 & 0 \\ 0 & y^{-2} \end{pmatrix} gg^\dagger = \begin{pmatrix} y^{-2}\sigma\sigma^\dagger & 0 \\ 0 & 0 \end{pmatrix}, \tag{5}$$

then conjugating with $k^\dagger$ and multiplying by $2y^2$, we have

$$2y^2 bb^\dagger - bb^\dagger k^\dagger (id - I) kbb^\dagger = 2k^\dagger \begin{pmatrix} \sigma\sigma^\dagger & 0 \\ 0 & 0 \end{pmatrix} k. \tag{6}$$

Then, after using $\sigma\sigma^\dagger = \alpha^2 \mathbf{1}_n + \hat{v}\hat{v}^\dagger$ and rewriting the matrix on the right hand side accordingly, we obtain

$$2y^2\Omega - \Omega^2 + \Omega L I \Omega = \alpha^2 id + \alpha^2 LI + 2ww^\dagger. \tag{7}$$

Our objective is to find the general solution of (7) for L in terms of

$$\Omega = bb^\dagger = \begin{pmatrix} x^2\mathbf{1}_n + \beta^2 & x^{-1}\beta \\ x^{-1}\beta & x^{-2}\mathbf{1}_n \end{pmatrix}. \tag{8}$$

Remarkably, this last equation *can* be solved directly. To do this, we diagonalize Ω, writing

$$\Omega = \rho \Lambda \rho^{-1}$$

and then

$$Q := \rho L I \rho^{-1}, \qquad \tilde{w} := \rho w,$$

obtaining

$$2y^2 \Lambda - \Lambda^2 + \Lambda Q \Lambda = \alpha^2 id + \alpha^2 Q + 2\tilde{w}\tilde{w}^\dagger, \tag{9}$$

and thence

$$Q_{ab} = (\Lambda_a \Lambda_b - \alpha^2)^{-1} \Big[(\Lambda_a^2 - 2y^2 \Lambda_a + \alpha^2)\delta_{ab} + 2\tilde{w}_a \tilde{w}_b^* \Big],$$

with $a, b = 1, 2, \dots, 2n.$

In some ways this side of the reduction was more straightforward than the other one. The last equation can be solved, giving explicit formulae for Q and for $|\tilde{w}_a|^2$ in terms of $\Lambda_1, \dots, \Lambda_n$. The significant new challenge here was to find the PB on the reduced space, described in terms of the coordinates Λ_i, θ_j. Here the new variables, $\theta_1, \dots, \theta_n$ are introduced via

$$\tilde{w}_i \tilde{w}^*_{n+i} = |\tilde{w}_i \tilde{w}_{n+i}| \exp(\mathrm{i}\theta_i).$$