# Stochastic Nonlinear Perron-Frobenius Theorem

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## Perron-Frobenius theorem

Oskar Perron (1907), Georg Frobenius (1912).

Let $A \geq 0$ be a matrix $n \times n$.

Suppose $A^l > 0$ for some $l$.

For a vector $(x^1, \ldots, x^n)$, define $|x| = |x^1| + \ldots + |x^n|$.

**Theorem**. *There exist a number* $\alpha > 0$ *and a vector* $x > 0$ *such that*

$$\alpha x = Ax, \ |x| = 1.$$

*The pair* $(x, \alpha) \geq 0$ *satisfying the above two equations is unique.*

The vector $x$ and the number $\alpha$ are called *the Perron-Frobenius eigenvector and eigenvalue* of the matrix $A$.

**Properties of the eigenvector $x$ and the eigenvalue $\alpha$.**

**Theorem** (stability). *For any non-zero vector $y \geq 0$, we have*

$$\frac{A^k y}{|A^k y|} \to x.$$

**Theorem** (optimality). *The pair $(x, \alpha)$ is a unique solution to the maximization problem:*

*(M) maximize $\beta$ over all $(y, \beta) \geq 0$ such that*

$$\beta y \leq Ay, \ |y| = 1.$$

**Stochastic Perron-Frobenius theorem**

$(\Omega, \mathrm{F}, P)$ a probability space;

$T : \Omega \to \Omega$ its *automorphism*: a one-to-one mapping such that $T$ and $T^{-1}$ are measurable and preserve $P$;

$A(\omega)$ a measurable function taking values in the set of nonnegative $n \times n$ matrices.

Define

$$C(t,\omega) = A(T^{t-1}\omega)A(T^{t-2}\omega)\ldots A(\omega),\ \ t = 1,2,\ldots$$

This is a matrix-valued *cocycle*:

$$C(t,T^s\omega)C(s,\omega) = C(t+s,\omega),\ \ t,s \geq 0.$$

**Assumption**:

(**C**) For all $\omega$ there is $l(\omega)$ such that $C(l,\omega) > 0$.

**Theorem**. *There exists a measurable vector function* $x(\omega) > 0$ *and a measurable scalar function* $\alpha(\omega) > 0$ *such that*

$$\alpha(\omega)x(T\omega) = A(\omega)x(\omega), \quad |x(\omega)| = 1 \quad (a.s.). \tag{1}$$

*The pair of functions* $(\alpha(\cdot), x(\cdot)) \geq 0$ *satisfying (1) is unique up to the equivalence with respect to* $P$.

$x(\cdot)$ and $\alpha(\cdot)$ are an "eigenvector" and an "eigenvalue" of $A(\omega)$ w. r. t. the dynamical system $(\Omega, \mathrm{F}, P, T)$.

**Remark**. Put $A_t(\omega) = A(T^{t-1}\omega)$ and denote by $\mathrm{F}_t$ the $\sigma$-algebra generated by

$$\ldots, \; A_{t-2}(\omega), A_{t-1}(\omega), A_t(\omega)$$

and completed by all events of measure zero. Then $x(\omega)$ is $\mathrm{F}_0$-measurable and $\alpha(\omega)$ is $\mathrm{F}_1$-measurable.

**Properties of the "eigenvector" $x(\omega)$ and the "eigenvalue" $\alpha(\omega)$.**

**Theorem** (stability). *If $t \to \infty$, then*

$$\frac{C(t, T^{-t}\omega)a}{|C(t, T^{-t}\omega)a|} \to x(\omega) \text{ (a.s.)},$$

*where convergence is uniform in $a \geq 0$, $a \neq 0$.*

Here,

$$C(t, T^{-t}\omega) = A(T^{-1}\omega)A(T^{-2}\omega)\ldots A(T^{-t}\omega) = A_0(\omega)A_{-1}(\omega)\ldots A_{-t+1}(\omega).$$

**Theorem** (optimality). *The pair $(x, \alpha)$ is a unique solution to the maximization problem:*

*(M) maximize $E \ln \beta$ over all $(y, \beta) \geq 0$ such that*

$$\beta(\omega)y(T\omega) \leq A(\omega)y(\omega), \ |y(\omega)| = 1 \text{ (a.s.)},$$

*$y$ is $\mathrm{F}_0$-measurable, $\beta$ is $\mathrm{F}_1$-measurable.*

## Some references

I. Evstigneev (1974) Positive matrix-valued cocycles over dynamical systems. *Uspekhi Matem. Nauk (Russian Mathematical Surveys)* **29**: 219–220. (In Russian.)

L. Arnold, V. Gundlach, L. Demetrius (1994) Evolutionary formalism for products of positive random matrices. *Annals of Applied Probability* **4**: 859–901.

Y. Kifer (1996) Perron-Frobenius theorem, large deviations, and random perturbations in random environments. *Mathematische Zeitschrift* **222**: 677-698.

I. Evstigneev and S. Pirogov (2010) Stochastic nonlinear Perron-Frobenius theorem, *Positivity*, v. 14, 43-57.

E. Babaei, I. V. Evstigneev, and S. A. Pirogov, Stochastic fixed points and nonlinear Perron-Frobenius theorem, *Proceedings of the American Mathematical Society*, 2018, forthcoming. [Extension from $R^n_+$ *random* cones $X(\omega)$.]

## **Strict positivity is essential**: **a counterexample**

In the deterministic case, any non-negative matrix $A$ has a non-negative eigenvector. The assumption $A^l > 0$ for some $l$ is not needed for the existence. This is not the case in the stochastic setting.

**Example**. Suppose that

$$A(\omega) = \begin{pmatrix} 0 & \gamma(\omega) \\ 1 & 0 \end{pmatrix},$$

where $\gamma(\omega) \geq 1$ is a measurable function. Assume $\Theta := T^2$ is ergodic and there is no measurable function $\beta(\omega) > 0$ such that

$$\gamma(\omega) = \beta(T\omega)\beta(\omega) \text{ (a.s.).}$$

[For example, let $T$ be the Bernoulli shift associated with a sequence $\omega = (\ldots, s_{-1}, s_0, s_1, \ldots)$ of i.i.d. random variables taking values $1$ and $2$ with probability $1/2$, and $\gamma(\omega) = s_0$.] Then *the equations*

$$\alpha(\omega)x(T\omega) = A(\omega)x(\omega), \quad |x(\omega)| = 1 \quad \text{(a.s.).}$$

*do not have solutions in the class of measurable* $x(\cdot) \geq 0$ *and* $\alpha(\cdot) \geq 0$.

Compare with: Ochs, G., Oseledets, V.I. (1999) Topological fixed point theorems do not hold for random dynamical systems, J. Dyn. Diff. Eqs. 11, 583–593.

**Strategy of proof**. Define

$$f(\omega,x) = \frac{A(\omega)x}{|A(\omega)x|}.$$

This is a random mapping of the unit simplex $X := \{x \geq 0 : |x|= 1\}$ into itself.

The problem reduces to the analysis of the equation

$$x(T\omega) = f(\omega,x(\omega)) \text{ (a.s.)},$$

where $x(\omega)$ is a measurable function with values in $X$.

Existence, uniqueness and stability of a solution in this setting are equivalent to those in the Perron-Frobenius one.

Define

$$f_t(\omega,x) := \frac{A_t(\omega)x}{|A_t(\omega)x|},\ A_t(\omega) = A(T^{t-1}\omega).$$

The *stability* of the stochastic P-F eigenvector $x(\omega)$ is equivalent to:

$$f_0(\omega)f_{-1}(\omega)\ldots f_{-t}(\omega)y \to x(\omega) \text{ (a.s.)}$$

uniformly in $y \in X$ [Product means the composition of maps!].

Moreover, the above convergence implies the *existence* and uniqueness of the P-F eigenvector (define $x(\omega)$ as the above limit).

**Hilbert-Birkhoff metric**. Let $Y$ denote the (relative) interior of the unit simplex: $Y := \{y > 0 : |y| = 1\}$. For $x, y \in Y$ put

$$\rho(x,y) = \ln[\max_i \frac{x_i}{y_i} \cdot \max_j \frac{y_j}{x_j}].$$

This formula defines a complete metric on $Y$ (the *Hilbert-Birkhoff metric*), and the topology induced by $\rho$ on $Y$ coincides with the Euclidean topology on $Y$.

Note that

$$\max_i \frac{x_i}{y_i} = \min\{r : x \le ry\}, \ \max_i \frac{y_i}{x_i} = \min\{r : y \le rx\}.$$

*Any strictly positive matrix $A$ generates a mapping*

$$f(y) := \frac{Ay}{|Ay|}$$

*of $Y$ into itself which is (uniformly) contracting in the H-B metric!*

This means $\rho(f(x),f(y)) \le \kappa\rho(x,y)$ for all $x,y \in Y$, where $\kappa < 1$.

G. Birkhoff. Extensions of Jentzsch's theorem. *Trans. Amer. Math. Soc.* **84** (1957), 219-227.

M.A. Krasnoselskii, E.A. Lifshitz, A.V. Sobolev, Positive linear systems: the method of positive operators, Berlin, Heldermann, 1989.

## Nonlinear generalizations of the Perron-Frobenius theorem

Large literature. Various directions of studies aimed at different applications.

R. M. Solow and P. A. Samuelson (1953). Balanced growth under constant returns to scale. *Econometrica* 21 , 412-424.

Survey:

S. Gaubert and J. Gunawardena (2004). The Perron-Frobenius theorem for homogeneous, monotone functions. *Transactions of the American Mathematical Society* 356, 4931-4950.

An important role was played by the following short note, which opened the way for using the H-B metric in the nonlinear context:

E. Kohlberg (1982). The Perron-Frobenius theorem without additivity. *Journal of Mathematical Economics* 10, 299-303.

The stochastic nonlinear version of the P-F theorem obtained in this work develops Kohlberg's considerations (which he used in the deterministic case).

**Results in the nonlinear case**. We obtain results for a class of nonlinear random mappings $A(\omega,x) : \Omega \times R_+^n \to R_+^n$ which are analogous to the results on stochastic Perron-Frobenius eigenvectors and eigenvalues in the linear case.

For two vectors $x = (x^1,\ldots,x^n)$ and $y = (y^1,\ldots,y^n)$, we write $x \prec y$ if $x \leq y$ and $x \neq y$.

A mapping $A : R_+^n \to R_+^n$ is called *monotone* if $Ax \leq Ay$ for any vectors $x,y \in R_+^n$ satisfying $x \leq y$. It is called *completely monotone* if it preserves each of the relations $x \leq y$, $x \prec y$ and $x < y$ between two vectors $x,y \in R_+^n$ (clearly, if $A$ preserves the second relation, it preserves the first). A mapping $A$ is termed *strictly monotone* if the relation $x \prec y$ implies $A(x) < A(y)$. Consider the (nonlinear) cocycle

$$C(t,\omega) = A(T^{t-1}\omega)A(T^{t-2}\omega)\ldots A(\omega),\ \ t = 1,2,\ldots,$$

[We write for convenience $A(\omega)x = A(\omega,x)$, and the product means the composition of maps.]

**Assumptions**. The mapping $A(\omega,x)$ is measurable in $\omega$ for each $x$ and it is completely monotone, homogeneous and continuous in $x$ for each $\omega$. For almost all $\omega \in \Omega$, there is a natural number $l$ (depending on $\omega$) such that the mapping $C(l,\omega)$ is strictly monotone.

**Theorem**. *There exists a measurable vector function $x(\omega) > 0$ and a measurable scalar function $\alpha(\omega) > 0$ such that*

$$\alpha(\omega)x(T\omega) = A(\omega)x(\omega), \ |x(\omega)| = 1 \quad (a.s.). \qquad (*)$$

*The pair of functions $(\alpha(\cdot), x(\cdot)) \geq 0$ satisfying $(*)$ is unique up to the equivalence with respect to $P$.*

**Properties**:

**Theorem** (stability). *If $t \to \infty$, then*

$$\frac{C(t, T^{-t}\omega)a}{|C(t, T^{-t}\omega)a|} \to x(\omega) \text{ (a.s.)},$$

*where convergence is uniform in $a \geq 0$, $a \neq 0$.*

Here, $C(t, T^{-t}\omega) = A(T^{-1}\omega)A(T^{-2}\omega)\ldots A(T^{-t}\omega) = A_0(\omega)A_{-1}(\omega)\ldots A_{-t+1}(\omega)$.

**Theorem** (optimality). *The pair $(x, \alpha)$ is a unique solution to the maximization problem:*

*(M) maximize $E \ln \beta$ over all $(y, \beta) \geq 0$ such that*

$$\beta(\omega)y(T\omega) \leq A(\omega)y(\omega), \ |y(\omega)| = 1 \text{ (a.s.)},$$

$$y \text{ is } \mathrm{F}_0\text{-measurable}, \ \beta \text{ is } \mathrm{F}_1\text{-measurable}.$$

## Concave homogeneous mappings

Let $A : R_+^n \to R_+^n$ be a *concave* mapping i.e.

$$A(\theta x + (1-\theta)y) \geq \theta A(x) + (1-\theta)A(y)$$

for all $x, y \in R_+^n$ and $\theta \in [0,1]$. Clearly, if $A$ is homogeneous, then $A$ is concave if and only if it is *superadditive*:

$$A(x+y) \geq A(x) + A(y).$$

A superadditive mapping $A : R_+^n \to R_+^n$ preserves the relation $\succ$ if and only if

(**M.1**) $A(h) \succ 0$ for all $h \succ 0$; it preserves the relation $>$ if and only if (**M.2**) $A(h) > 0$ for all $h > 0$; and it is strictly monotone if and only if (**M.3**) $A(h) > 0$ for all $h \succ 0$.

Any superadditive mapping is monotone.

Thus, *for a concave homogeneous mapping, its complete monotonicity is equivalent to the validity of* (**M.1**) *and* (**M.2**), *and its strict monotonicity is equivalent to* (**M.3**).

If $A(x)$ is linear, i.e., defined by a non-negative matrix $A$, then (**M.1**) means that $A$ does not have zero columns. (**M.2**) holds if and only if $A$ does not have zero rows. Complete monotonicity means that $A$ has no zero rows and columns. Such mappings are strictly monotone when $A > 0$.

A key role in the proofs is played by the following fact. Let $A$ be a mapping $R_+^n \to R_+^n$ such that $A(x) \neq 0$ for $x \in Y$. Define

$$f(x) = \frac{A(x)}{|A(x)|}, \;\; x \in Y.$$

(Recall that $Y := \{y > 0 : |y| = 1\}$.)

**Theorem**. *If $A(x)$ is homogeneous and strictly monotone, then $f(x)$ is contracting on $Y$ in the H-B metric $\rho$, i.e.*

$$\rho(f(x), f(y)) \;< \rho(x, y)$$

*for $x, y \in Y$ with $x \neq y$.*

This result is essentially contained in above-mentioned Kohlberg's (1982) paper.

Note that the above contraction property is not uniform. The conventional Banach contraction principle does not hold. Instead, (a stochastic version of) the following fact is used.

**Theorem**. *Let $Z$ be a compact space with a metric $\rho$ and let $f : Z \to Z$ be a mapping satisfying $\rho(f(x), f(y)) \;< \rho(x, y)$ for all $x \neq y$. Then $f$ has a unique fixed point $z$, and $f^k(x) \to z$ for each $x \in Z$.*

## Stochastic contraction principle

$(\Omega, \mathrm{F}, P)$ probability space;

$T : \Omega \to \Omega$ automorphism;

$\mathrm{X}$ standard Borel space;

$f(\omega, x) : \Omega \times X \to X$ jointly measurable mapping.

We provide conditions under which the equation

$$\xi(T\omega) = f(\omega, \xi(\omega)) \text{ (a.s.)}$$

has a measurable solution $\xi(\omega)$.

Define

$$f_k(\omega, x) := f(T^{k-1}\omega, x) \ (k = 0, \pm 1, \pm 2, \ldots),$$

$$f^{(k)}(\omega, x) := f_0(\omega) f_{-1}(\omega) \ldots f_{-k}(\omega)(x) \ (k = 0, 1, 2, \ldots),$$

where the product means the composition of maps.

**Assumptions**. Let $Y$ be a measurable subset of $X$ equipped with a metric $\rho$ such that $Y$ is separable with respect to this metric and the Borel measurable structure on $Y$ coincides with the measurable structure induced from $X$. Suppose that $f(\omega,x)$ satisfies the following requirements.

(f.1) For each $\omega \in \Omega$, the mapping $f(\omega,x)$ transforms $Y$ into itself and is continuous on $Y$ with respect to the metric $\rho$.

(f.2) There is a sequence of F-measurable sets $\Omega_0 \subseteq \Omega_1 \subseteq \ldots \subseteq \Omega$ such that $P(\Omega_m) \to 1$ and for each $m = 0, 1, 2, \ldots,$ and $\omega \in \Omega_m$ the following conditions hold:

(a) the set

$$X^{(m)}(\omega) := f^{(m)}(\omega, X)$$

is contained in $Y$ and is compact with respect to the metric $\rho$;

(b) for all $x, y \in Y$ with $x \neq y$, we have

$$\rho(f^{(m)}(\omega,x), f^{(m)}(\omega,y)) < \rho(x,y).$$

**Theorem** (Stochastic Contraction Principle).

*(i) There exists a measurable mapping $\xi : \Omega \to Y$ for which equation*

$$\xi(T\omega) = f(\omega, \xi(\omega)) \text{ (a.s.)} \tag{1}$$

*holds and*

$$\lim_{k\to\infty} \sup_{x\in X} \rho(\xi(\omega), f_0(\omega)\ldots f_{-k}(\omega)(x)) = 0$$

*with probability one.*

*(ii) If $\eta : \Omega \to X$ is any (not necessarily measurable) solution to (1), then $\eta = \xi$ (a.s.).*

*(iii) Let $\mathrm{F}_0 \subseteq \mathrm{F}$ be a $\sigma$-algebra such that the mappings $f_{-k}(\omega, x)$, $k = 0, 1, \ldots,$ of the space $\Omega \times X$ into $X$ are $\mathrm{F}_0 \times \mathrm{X}$-measurable and $\Omega_m \in \mathrm{F}_0$ for all $m \geq 0$. Then there exists an $\mathrm{F}_0$-measurable mapping $\xi$ possessing the properties described in (i) and (ii).*

I. Evstigneev and S. Pirogov (2007) A stochastic contraction principle, Random Operators and Stochastic Equations, v. 15, 155-162.

## An application: analysis of fixed-mix dynamic investment strategies in stationary asset markets

Consider a financial market with $n$ assets whose prices change in time and depend on random factors. Randomness is described as follows. There is a stochastic process $\ldots, s_{-1}, s_0, s_1, \ldots$ with values in a space $S$. The value of $s_t$ characterizes the "state of the economy" at time $t \in \{0, \pm 1, \pm 2, \ldots\}$. The vector of asset prices

$$p_t(s^t) = (p_t^1(s^t), \ldots, p_t^n(s^t)), \quad [p_t^k(s^t) > 0]$$

at time $t = 0, 1, \ldots$ depends on the history $s^t = (\ldots, s_{t-1}, s_t)$ of the process $(s_t)$.

An *investment strategy* (*trading strategy*) is a sequence of non-negative vector functions

$$h_t(s^t) = (h_t^1(s^t), \ldots, h_t^n(s^t)), \; t = 0, 1, 2, \ldots,$$

where the component $h_t^k(s^t)$ of the vector $h_t(s^t)$ represents the number of units of asset $k$ in the *portfolio* $h_t = h_t(s^t)$. The choice of the portfolio may depend on time and on information about the process $(s_t)$: therefore $h_t$ depends on $t$ and $s^t$. We assume that all the coordinates of $h_t(s^t)$ are non-negative (short sales are ruled out).

**Fixed-mix investment strategies**. Let $\gamma_{kj}$, $k,j \in \{1,\ldots,n\}$, be a (non-random) matrix satisfying

$$\gamma_{kj} > 0, \ \sum_{k=1}^{n} \gamma_{kj} = 1.$$

A strategy $h_t(s^t)$, $t \geq 0$, is called a *fixed-mix strategy associated with the matrix* $\gamma = (\gamma_{kj})$, or, for short, a $\gamma$*-strategy*, if

$$p_t^k h_t^k = \sum_{j=1}^{n} \gamma_{kj} p_t^j h_{t-1}^j, \tag{2}$$

holds for all $k, t$ and $s^t$. Clearly any $\gamma$-strategy is *self-financing*, i.e. $p_t h_t = p_t h_{t-1}$. In an important special case, $\gamma_{kj}$ does not depend on $j$:
$\gamma_{kj} = \gamma_k$ $[\gamma_k > 0, \ \gamma_1 + \ldots + \gamma_n = 1]$. Then (2) reduces to

$$p_t^k h_t^k = \gamma_k \sum_{j=1}^{n} p_t^j h_{t-1}^j = \gamma_k p_t h_{t-1}, \quad k \in \{1,\ldots,n\}.$$

An investor using a strategy of this kind divides the available wealth $p_t h_{t-1}$ according to the proportions $\gamma_1,\ldots,\gamma_n$ and spends the amount $\gamma_k p_t h_{t-1}$ for purchasing $\gamma_k p_t h_{t-1}/p_t^k$ units of asset $k$.

**Currency markets**. We examine fixed-mixed strategies, in particular, in the context of the modelling of currency markets. Consider a frictionless market where $n$ currencies are traded. The exchange rates $\pi_t^{kj} = \pi_t^{kj}(s^t) > 0$ fluctuate randomly in time, depending on the stochastic factors $(s_t)$. Here, $\pi_t^{kj}$ denotes the amount of currency $k$ which can be purchased by selling one unit of currency $j$ at time $t$. Assume the trader divides the amount $h_{t-1}^j \geq 0$ of currency $j$ available at the beginning of a time period $(t-1, t]$ according to the proportions $\gamma_{kj} > 0$ $(\sum_k \gamma_{kj} = 1)$ and exchanges $\gamma_{kj} h_{t-1}^j$ into currency $k$. Then the amount of currency $k$ obtained at time $t$ will be equal to

$$h_t^k = \sum_{j=1}^{n} \gamma_{kj} \pi_t^{kj} h_{t-1}^j. \tag{3}$$

By virtue of no-arbitrage considerations, exchange rates in a frictionless market must satisfy

$$\pi_t^{kj} = \pi_t^{km} \pi_t^{mj}$$

for all $k, m$ and $j$. This implies $\pi_t^{kj} = 1/\pi_t^{jk},\ \pi_t^{jj} = 1$. Let us regard currency $1$ as a numeraire and define $p_t^k = \pi_t^{1k}$. Then $\pi_t^{kj} = p_t^j / p_t^k$, and so (3) can be written

$$p_t^k h_t^k = \sum\nolimits_{j=1}^{n} \gamma_{kj} p_t^j h_{t-1}^j.$$

**Stationary markets**. We focus on *stationary* markets. We say that the market under consideration is stationary if the stochastic process $(s_t)$, $t = 0, \pm 1, \pm 2, \ldots$, is stationary, and the price vectors $p_t$ do not explicitly depend on $t$:

$$p_t = p(s^t). \tag{4}$$

In the above model of currency exchange, the counterpart of condition (4) is

$$\pi_t^{kj} = \pi^{kj}(s^t),$$

which implies (4), when $p_t$ is defined by $p_t^k = \pi_t^{1k}$.

**Question**. Assume that the trader systematically applies the rule of currency exchange specified by the matrix $(\gamma_{kj})$. How will the portfolio $h_t$ behave in the long run? Will it stabilize in one sense or another, will it grow or will it generally decrease?

**Assumptions**. The prices $p^k(s^t) > 0$ satisfy

$$E|\ln p^k(s^t)| < \infty, \ k \in \{1,\ldots,n\},$$

and the process $(s_t)$ is stationary and ergodic. Additionally, we impose the following requirement of non-degeneracy of the price process $p(s^t)$:

(A) The vector $\bar{p}(s^t) = (\bar{p}^1(s^t),\ldots,\bar{p}^n(s^t))$ of normalized prices

$$\bar{p}^j(s^t) := \frac{p^j(s^t)}{\sum_m p^m(s^t)}, \ j \in \{1,\ldots,n\},$$

is not constant a.s. with respect to $s^t$.

**Theorem**. *For each* $k \in \{1,2,\ldots,n\}$, *the limit*

$$\lim_{t\to\infty} \frac{1}{t} \ln h_t^k$$

*exists and is strictly positive almost surely. Furthermore, this limit does not depend on* $k$, *and we have*

$$\lim_{t\to\infty} \frac{1}{t} \ln h_t^k = \lim_{t\to\infty} \frac{1}{t} \ln p_t h_t > 0 \quad \text{(a.s.)}$$

Thus $h_t^k$ **tends to infinity at an exponential rate**! Further, the wealth $p_t h_t$ of **the investor grows with the same positive exponential rate**!

**Positive matrix cocycles associated with fixed-mix strategies**

Denote by $A_t = A(s^t) = (a^{kj}(s^t))$ the positive random $n \times n$ matrix defined by

$$a^{kj}(s^t) = \gamma_{kj} \frac{p^j(s^t)}{p^k(s^t)}.$$

Then a $\gamma$-strategy $(h_t)$ can be represented as

$$h_t(s^t) = A(s^t)A(s^{t-1})\ldots A(s^1)h_0(s^0).$$

The asymptotic behaviour of this strategy is completely determined by the Perron-Frobenius eigenvalue $\alpha(s^1)$:

$$\alpha(s^1)\bar{h}(s^1) = A(s^1)\bar{h}(s^0) \text{ (a.s.).}$$

It is proved that

$$E \ln \alpha(s^1) > 0.$$

M. A. H. Dempster, I.V. Evstigneev and K. R. Schenk-Hoppé, Exponential growth of fixed-mix strategies in stationary asset markets, *Finance and Stochastics*, 2003 ,v. 7, 263-276.