START WITH A FORM OF

MAX. PRINC FOR ELLIPTIC OPERATORS OF SECOND ORDER.

IN DOMAIN Ω (OPEN CONNECTED SET) IN $\mathbb{R}^n$, SAY BOUNDED,

ELLIPTIC OPERATOR

$$Lu = a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i(x) \frac{\partial u}{\partial x_i} + c(x) u$$

0

A GEOMETRIC PROBLEM

& THE HOPF LEMMA

L. NIRENBERG & YANYAN LI

REPORT OF 2 PAPERS:

Y.Y. LI & L NIRENBERG,

A GEOMETRIC PROBLEM AND THE HOPF LEMMA

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UNIFORMLY ELLIPTIC:

$\frac{|\xi|^2}{C} \le a_{ij}(x)\xi_i\xi_j \le C|\xi|^2, \; C>0$

$\xi \in \mathbb{R}^n$

$|b_i(x)|, |c| \le C.$

1) STRONG MAX PRINC)

IF $w \ge 0$ IN Ω AND

$Lw \le 0$,

AND $w(x_0) = 0$ for some $x_0 \in \Omega$

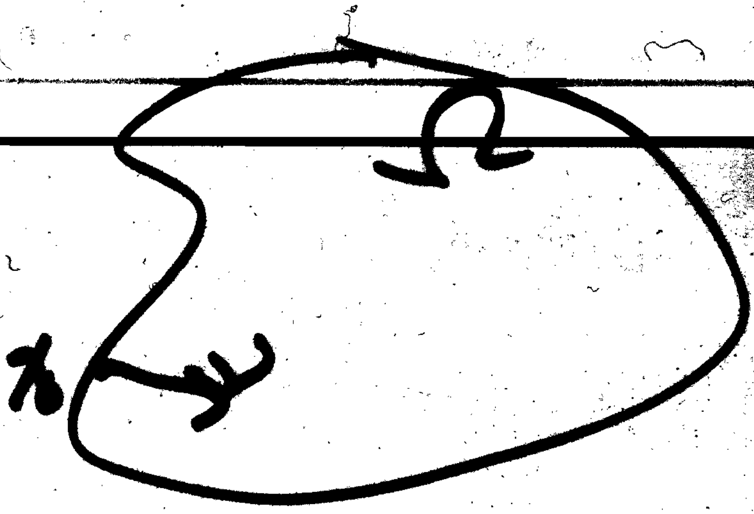
THEN $w \equiv 0$

2) HOPF LEMMA: $w \ge 0$ in Ω

($\partial\Omega$ SMOOTH), $Lw \le 0$ AND

AT SOME $x_0 \in \partial\Omega$, $w(x_0) = 0$

THEN EITHER $w \equiv 0$, OR $w_\nu(x_0) > 0$.



NONLINEAR COROLLARIES:

- FOR NONLINEAR ELLIPTIC OPERATOR

$$F(x, u, Du, D^2u)$$

UNIFORMLY ELLIPTIC,

$$\frac{|\xi|^2}{C} \le F_{u_{ij}} \xi_i \xi_j \le C|\xi|^2$$

$$|F_u|, |F_{u_i}| \le C$$

1') SUPPOSE $u \geq v$ ON Ω

$F(x,u,Du,D^2u) \leq F(x,v,Dv,D^2v)$.

IF AT SOME $x_0 \in \Omega$,

$$u(x_0) = v(x_0) \qquad (4.1)$$

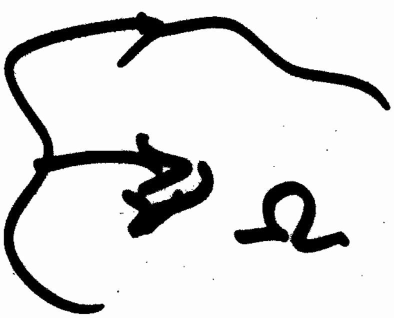
THEN

$$u \equiv v.$$

2') $u \geq v$ SAME AS ABOVE EXCEPT THAT (4.1) HOLDS AT SOME $x_0 \in \partial\Omega$. THEN

EITHER $u \equiv v$

or $u_\nu(x_0) > v_\nu(x_0)$



WE CONSIDER A GEOMETRIC PROBLEM WHICH SUGGESTS NEED FOR MORE GENERAL KINDS OF HOPF LEMMA

THEOREM (A.D. ALEXANDROFF 1955)

$M^n \subset \mathbb{R}^{n+1}$ SMOOTH EMBEDDED COMPACT HYPERSURFACE (NO BOUNDARY)

IF MEAN CURVATURE $H(x) = \text{CONST.}$ $\forall x \in M$

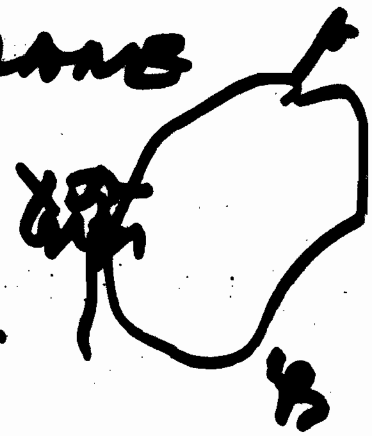
THEN M IS A SPHERE

(1986, WENTE.)

PROOF PROCEEDS BY SHOWING

THAT FOR ANY DIRECTION SAY x_{n+1}, M IS SYMMETRIC ABOUT SOME HYPERPLANE

$$x_{n+1} = \lambda_0 .$$



I'LL SHOW THE PROOF

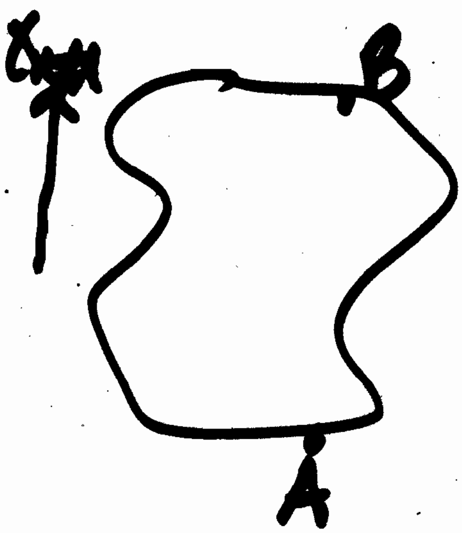
IN 1997 Y.Y LI CONSIDERS M^n WITH PROPERTY:

IF A, B ARE ON M, WITH A ABOVE B, i.e. $A_{n+1} > B_{n+1}$, $A_i = B_i$, $i \le n$

THEN $H(B) \ge H(A)$.

QUESTION: IS H THEN SYMMETRIC ABOUT SOME PLANE $x_{n+1} = \lambda_0$?

THM (LI) YES PROVIDED H CAN BE EXTENDED AS A LIPSCHITZ FUNCTION DE~~IN~~CREASING IN x_{n+1}.



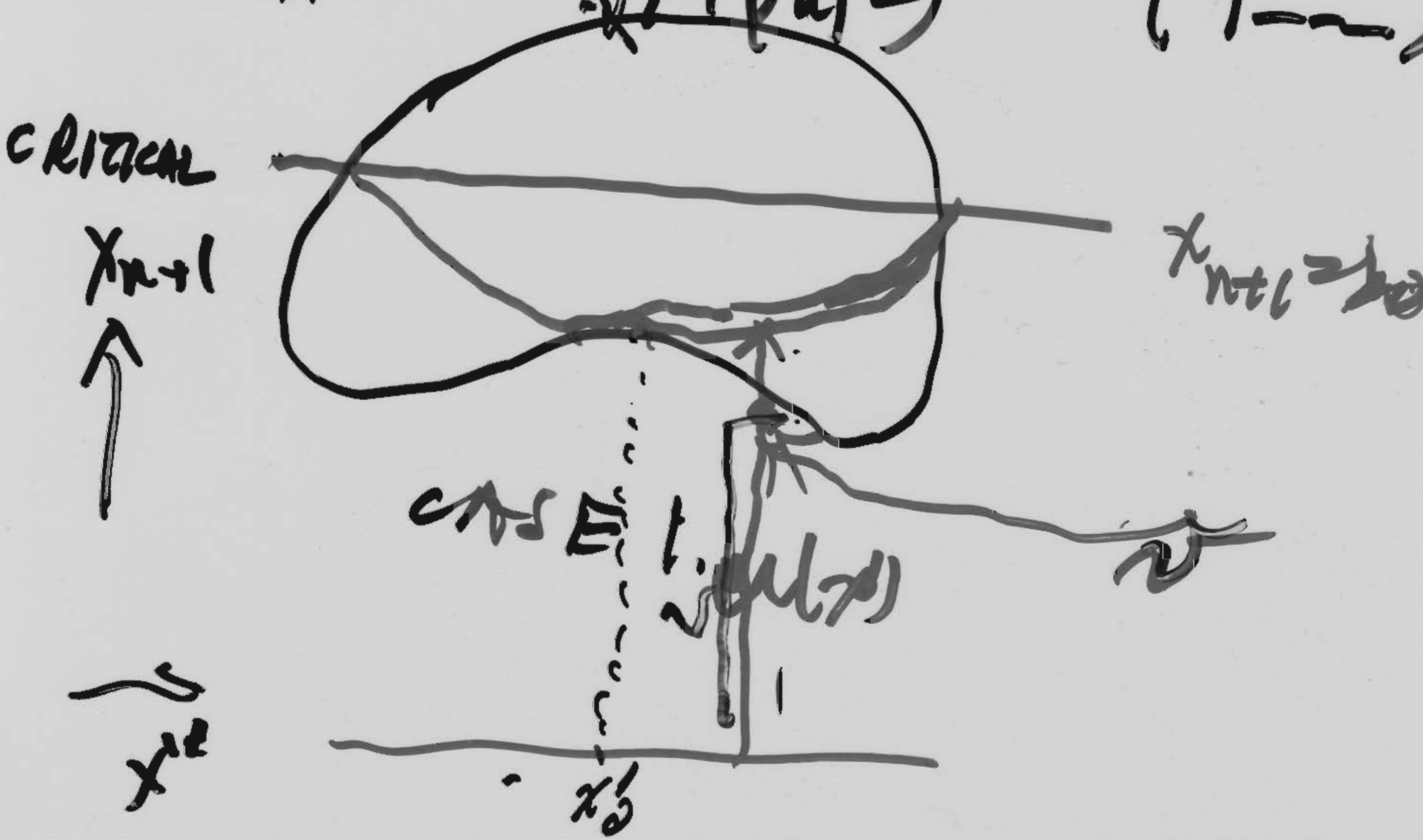
WHAT IF WE DROP THE CONDITION ON EXTENDABILITY OF H? IS THERE STILL SYMMETRY?

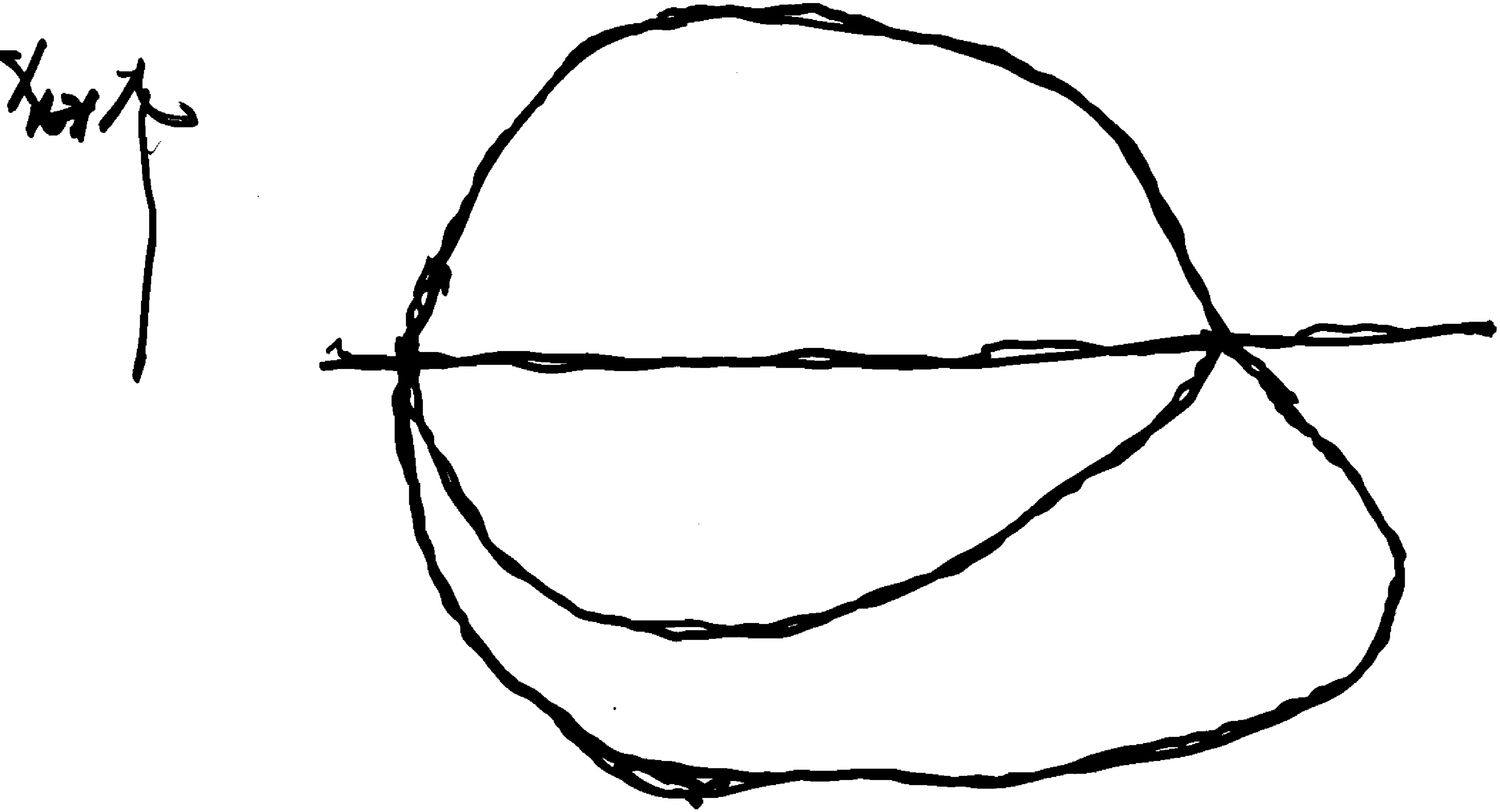
ALEXANDROFF's PROOF.

METHOD OF MOVING PLANES.

M

$$H = \nabla\cdot\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \nabla\left(\frac{\nabla v}{1-\!-\!-}\right)$$





CASE 2

EN CASE 1 WE HAVE

$$u(x') \geq v(x')$$
$$u(x'_0) = v(x'_0)$$

AND

$$H = \nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \nabla\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right)$$

THE NONLINEAR OPERATOR
IS ELLIPTIC.

BY STRONG MAX PRINC

$u \equiv v$ NEAR x'_0.

THIS THEN HOLDS FOR ALL x'

BY STRONG MAX PRINC

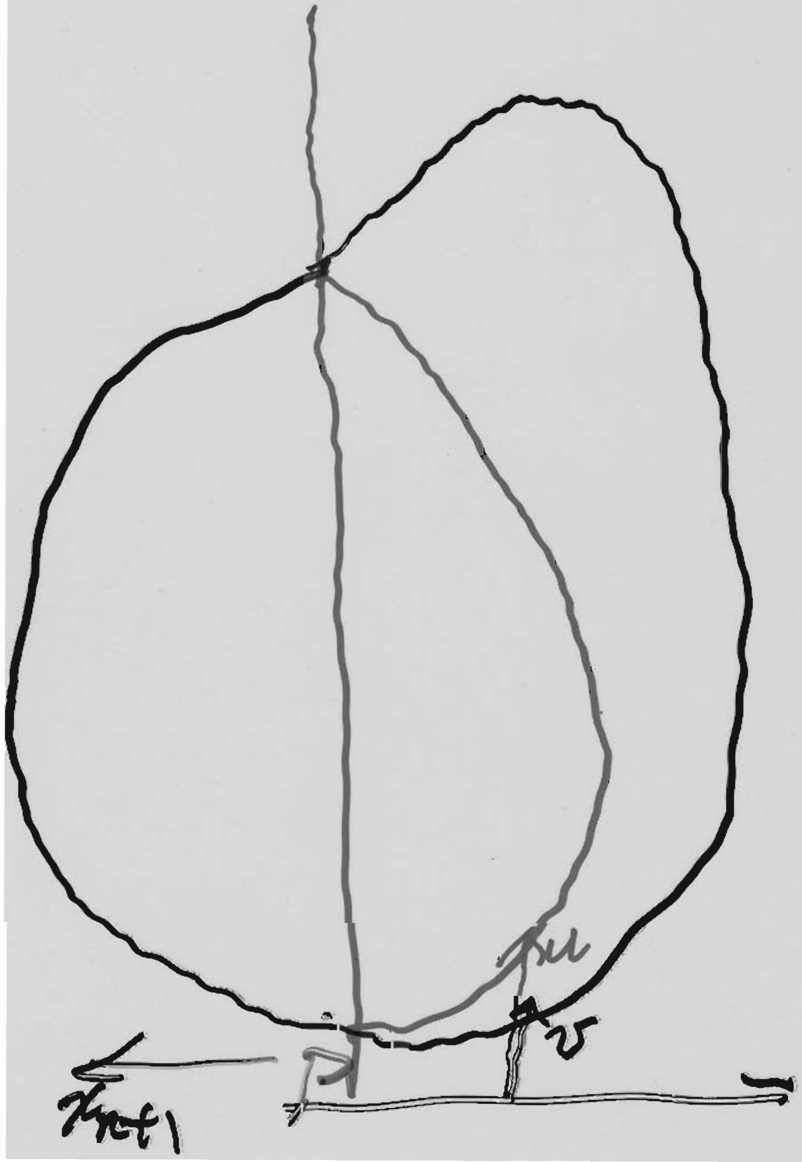
SO THE REFLECTED SURFACE

COINCIDES WITH THE UNREFLECTED

ONE. $\Longrightarrow$ SYMMETRY

CASE 2 IS MORE INTERESTING

TURN THE PICTURE AROUND



IN NEW COORD'S

u AND v

REPRESENT THE

GRAPHS. IN

$x_{n+1} \le \lambda_0$.

$u \ge v$

$u(P) = v(P)$

AT A BOUNDARY POINT P

AND $\nabla u = \nabla v$ AT P.

BY HOPF LEMMA APPLIED

TO SAM NONLINEAR EQUATION

$H = H[u] = H[v]$,

$u \equiv v$ NEAR P

THEN, BY STRONG MAX PRINC.

REFLECTED SURFACE

AGREES WITH UNREFLECTED.

THAT ENDS THE PROOF.

VERY SIMPLE AND

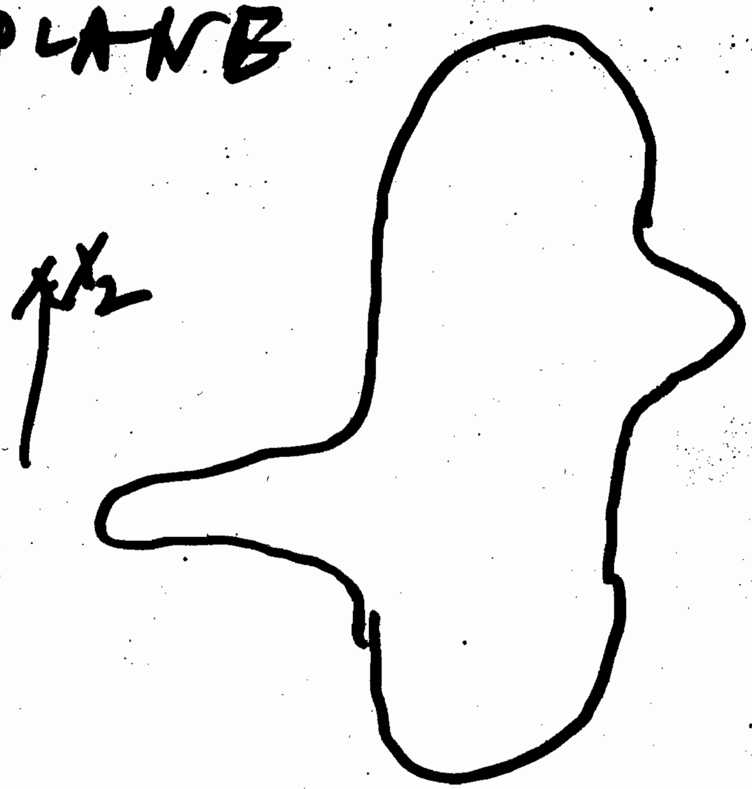
BEAUTIFUL.

Question: Does just

(HC) $\{ H(A) \le H(B)$

for A above B imply SYMMMETRY?

<u>EXAMPLE</u>. <u>NO</u>. CURVE IN PLANE



ADD CONDITION

S: ANY VERTICAL (CONTAINING LINE PARALLEL TO X_{n+1} AXIS) HYPERPLANE TANGENT TO M IS A PLANE OF SUPPORT.

CONJECTURE (MC) AND S IMPLY SYMMETRY.

TRUE FOR $n=1$. FOR $n>1$, ADD CONDITIONS.

T: ANY LINE TANGENT TO M, PARALLEL TO X_{n+1} AXIS TOUCHES M OF FINITE ORDER

<u>QUESTION</u>: DO (MC), S AND T IMPLY SYMMETRY?

WE ADD A 3RD CONDITION

CONDITION (LC) (cornett)

IN PARTICULAR: IF M IS CONVEX AND ANALYTIC THEN

(MC) $\Longrightarrow$ SYMMETRY

METHOD OF ATTACK

MOVING PLANES

(p. 8-9)

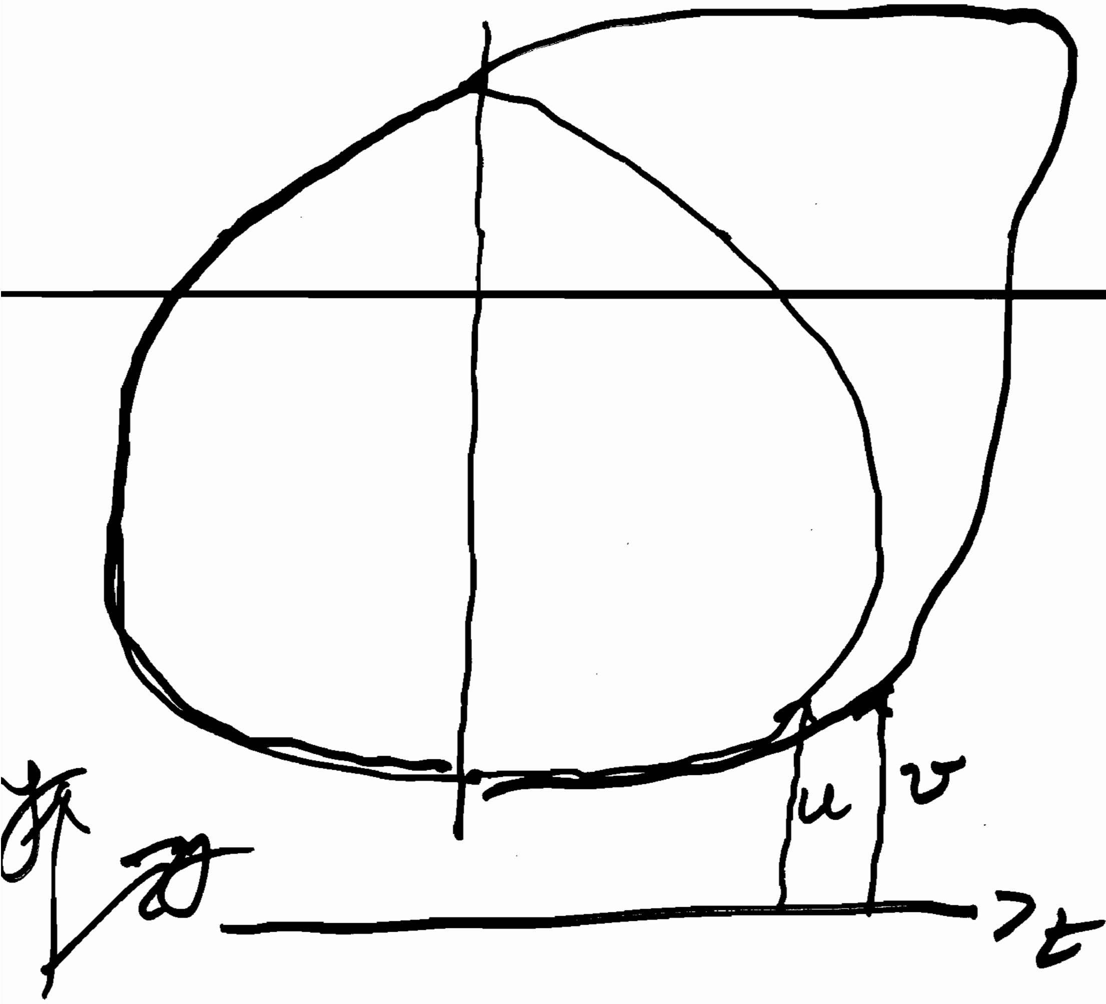
PROCEED AS BEFORE.

IN CASE 1 (PAGE 8) WE ARGUE AS BEFORE. BUT NOW

$$H[u] = \nabla\left(\frac{\nabla u(x')}{\sqrt{1+|\nabla u|^2}}\right) \leq \nabla\left(\frac{\nabla v(x')}{\sqrt{1+|\nabla v|^2}}\right) \quad (12.1)$$

BUT THE STRONG MAX PRINC. STILL APPLIES AND WE GET SYMMETRY

CASE 2 IS THE INTERESTING ONE. SOMETHING NEW ARISES WITH u AND v AS IN THE PICTURE ON PAGE (9'')



WHERE $u(t,y) = v(s,y)$

WE HAVE

$$\nabla\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \le \nabla\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right)$$

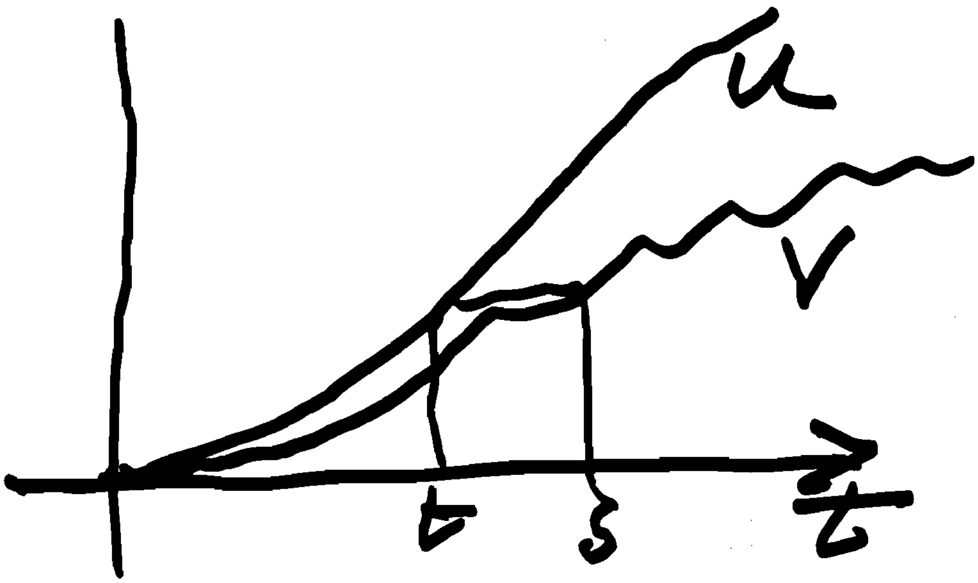
BUT WE ARE COMPARING AT DIFFERENT POINTS.

DO WE STILL HAVE $u \equiv v$?

NEED NEW KIND OF HOPF LEMMA.

OUR PROOF IS GLOBAL NOT LOCAL

1 DIM'L MODEL PROBLEM



$u_\delta \ge 0$, $u \ge v > 0$ ON $(0, \delta]$

BOTH IN C^2 ON $[0, \delta]$

$$u_\delta(0) = u(0) = v(0) = 0.$$

(MC) WHEREVER $u(t) = v(s)$, $t \le s$, $\dot{u}(t) \le \dot{v}(s)$.

<u>QUESTION</u>: IS $u \equiv v$?

THM A: YES!

ANALOGOUS QUESTION IN MORE DIM.

$$u(t,y) \geq v(t,y) > 0\,, \quad u_t > 0$$

$$\text{for } 0 < t < b,$$

$$|y| \leq A\,, \quad y \in \mathbb{R}^n$$

$$u(0,y) = v(0,0) = u_t(0,y) = 0 \quad \forall y$$

(XC) WHENEVER

$$u(t,y) = v(s,y) \qquad t < s$$

THEN $\Delta u \leq \Delta v$.

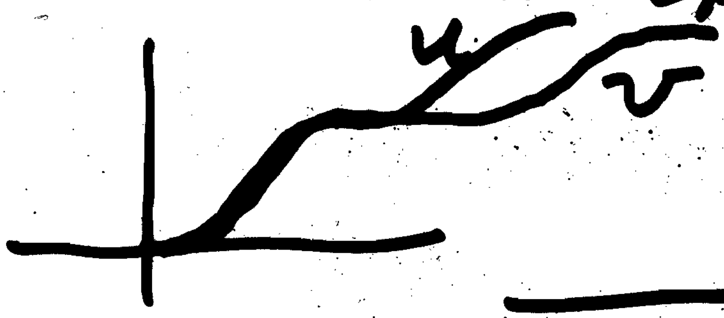
QUESTION: Is $u \equiv v$?

OPEN PROBLEM.

PROOF OF THM A:

SINCE $u_t > 0$

this is needed. Ex:



WE MAY THINK OF t AS FCN OF u. THEN

$$\ddot{u}(t) = f(u), \quad f \text{ unknown}$$

BUT IT IS CONT. ON $[0, k]$ AND LOCALLY LIPSCHITZ FOR $u > 0$.

(MC) MEANS

$$\ddot{v}(s) \geq f(v(s)),$$

v IS A <u>SUBSOLUTION</u>

PAVLOV REACTION: BETWEEN SUB & SUPER SOLN'S THERE IS A SOLUTION W. SUPPOSING

$v(q) < u(q)$ FOR SOME q, WE MAY TAKE

$$w(0) = q, \quad w(\omega) = v(\omega).$$

So

$$\dot{w} = f(w).$$

$$v \le w \le u$$
$$\ne$$

CAREFUL: $f(p)$ not Lipschitz, JUST LOCALLY LIP for $p > 0$. NEVERTHELESS, W EXISTS

THE DESIRED CONTRADICTION THEN FOLLOWS FROM FOLLOWING EXERCISE

<u>EXERCISE</u>. LET u, w BE POSITIVE C^{2}_{loc} FUNCTIONS ON $(0, a]$, IN $C^1([0,a])$ SATISFYING

$$\ddot{u} = f(u), \quad \ddot{w} = f(w).$$

AND

$$u(0) = w(0) = 0, \quad \dot{u}(0) = \dot{w}(0)$$

{SAME CAUCHY DATA}.

ASSUME $f(\rho)$ LOCALLY LIPSCHITZ FOR $\rho > 0$.

<u>THEN</u> $u \equiv w$.

Ex: $u = t^3$, $\ddot{u} = 6u^{1/3}$.

HIGHER DIMENSION?

UNIQUENESS OF POSITIVE SOL'NS OF CAUCHY PROBLEM?

$u(t,y)$, $w(t,y)$ POSITIVE FOR $t > 0$, $|y| < a$. $y \in \mathbb{R}^n$

, $u(0,y) = w(0,y) = 0$

$u_t(0,y) = w_t(0,y)$

BOTH SATISFY

$$\Delta u = f(y, u)$$

WHERE f IS JUST LOCALLY LIPSCHITZ FOR $u > 0$

IS $u \equiv w$?

OPEN

HIGHER DIMENSION?

UNIQUENESS OF POSITIVE SOL'NS OF CAUCHY PROBLEM?

$u(t,y)$, $w(t,y)$ POSITIVE FOR $t>0$, $|y|<a$. $y \in \mathbb{R}^n$

, $u(0,y) = w(0,y) = 0$

$$u_t(0,y) = w_t(0,y)$$

BOTH SATISFY

$$\Delta u = f(y,u)$$

WHERE f IS JUST LOCALLY LIPSCHITZ FOR $p>0$

IS $u \equiv w$?

OPEN

AN EARLIER PROBLEM LED TO FOLLOWING:

$g(t,y)$ is smooth for

$0 \le t \le a$, $|y| \le a$, $y \in \mathbb{R}^n$

AND SATISFIES

$$|\Delta g| \le \frac{C|g|}{t^2}$$

AND g VANISHES OF INFINITE ORDER ON $t=0$

Is $g \equiv 0$?

OPEN. also

IF $g \ge 0$ AND SATISFIES

$$\Delta g \le \frac{C}{t^2} g$$

AND VANISHES OF INF ORDER ON $t=0$

THEN $g \equiv 0$.

(KIND OF HOPF LEMMA)

REMARK:

THE FOLLOWING IS TRUE,

$g(x)$ DEFINED IN NEIGH'D OF ORIGIN IN $\mathbb{R}^n$, VANISHES OF INFINITE ORDER AT ORIGIN, AND SATISFIES

$$|\Delta g| \leq \frac{C}{|x|^2}|g|.$$

THEN

$$g \equiv 0.$$