New Perspective on the Kuhn-Tucker Theorem

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$$\min_X f(x) \tag{1}$$

subject to

$$g_i(x) \leq 0_m \,,$$

$$i = 1, \dots, m.$$

X – Banach space, $f: X \to \mathbb{R}^1$, $g: X \to \mathbb{R}^m$

$$g = (g_1, \dots, g_m)^\top$$

Projection Theorem

$$v \in X, \qquad Z \in X, \qquad p = \mathrm{Pr}_Z\, v$$

$$\|v - p\| = \inf_{z \in Z} \|v - z\|$$

Theorem. A point p is the projection of a point v onto a convex closed set $Z \in \mathbb{R}^n$ if and only if

$$\langle v - p, z - p \rangle \leq 0 \qquad (2)$$

for all $z \in Z$.

Let $v = 0_n$ then

$$\langle z, p \rangle \geq \|p\|^2 \qquad (3)$$

Kuhn-Tucker Theorem

Theorem. Let x^* be a solution of problem (1), and $f(x), g_i(x), i = 1, \dots, m,$ be continuously differentiable functions on X. Then there exist nonnegative multipliers $\lambda_0^*, \lambda_1^*, \dots, \lambda_m^*$ such that

$$\lambda_0^* f'(x^*) + \sum_{i=1}^{m} \lambda_i^* g_i'(x^*) = 0, \qquad (4)$$

and

$$\lambda_i^* \geq 0, \qquad i = 0, 1, \dots, m,$$

$$\lambda_0^* + \sum_{i=1}^{m} \lambda_i^* = 1. \qquad (5)$$

$$N(x) = \{f'(x), g_i'(x), i = 1, \dots, m\}$$

$$K \triangleq \mathrm{Conv}\, N(x^*) = \left\{ \xi \in \mathbb{R}^n \mid \xi = \lambda_0 f'(x^*) + \sum_{i=1}^{m} \lambda_i \ g_i'(x^*), \right.$$

$$\left. \lambda_i \ \geq 0, \qquad i = 0, \dots, m, \qquad \sum_{i=0}^{m} \lambda_i \ = 1 \right\}$$

Proof

1°. $X = \mathbb{R}^n$.

$$\lambda_0^* f'(x^*) + \sum_{i=1}^{m} \lambda_i^* g_i'(x^*) = 0,$$

$$\lambda_0^* + \sum_{i=1}^{m} \lambda_i^* = 1, \qquad \lambda_i^* \geq 0, \qquad i = 0, 1, \dots, m, \ .$$

$$\sim$$

$$0_n \in K$$

Let $0_n \notin K$ then for $z = f'(x^*) \in K$ (from (3): $\langle z, p\rangle \geq \|p\|^2$)

$$\langle f'(x^*), p\rangle \geq \|p\|^2 \qquad (6)$$

and for $z = g_i{}'(x^*) \in K,$ (from (3))

$$\langle g_i{}'(x^*), p\rangle \geq \|p\|^2, \qquad i = 1, \dots, m. \qquad (7)$$

We obtain

$$f(x^* - \alpha p) = f(x^*) - \alpha\langle f'(x^*), p\rangle + o(\alpha) < f(x^*) \qquad (8)$$

$$g_i(x^* - \alpha p) = g_i(x^*) - \alpha\langle g_i{}'(x^*), p\rangle + o(\alpha) < g_i(x^*) = 0 \quad (9)$$

$$\alpha > 0, \qquad i = 1, \dots, m.$$

$(x^* - \alpha p)$ – is a feasible point and $f(x^* - \alpha p) < f(x^*)$!

Therefore (4) holds!

2°. Banach case of X

X – B-space and X^* is a dual of X.

Auxiliary results

Lemma. Let X be a B-space and let $\xi_i \in X^*$, $i = 1, \dots, m$ be linear functionals. Moreover, assume that

$$0 \notin \operatorname{Conv}\{\xi_1, \dots, \xi_m\}.$$

Then there exist elements $\eta_1, \dots, \eta_m \in X$ such that

$$0_m \notin \operatorname{Conv}\left(\begin{pmatrix} \langle \xi_1, \eta_1 \rangle \\ \dots \\ \langle \xi_1, \eta_m \rangle \end{pmatrix}, \dots, \begin{pmatrix} \langle \xi_m, \eta_1 \rangle \\ \dots \\ \langle \xi_m, \eta_m \rangle \end{pmatrix} \right),$$

where $\langle \xi_i, \eta_j \rangle$ is the action of the linear functional $\xi_i \in X^*$ on the element $\eta_j \in X$.

Proof of the Kuhn-Tucker Theorem in a Banach space

Let

$$0_{X^*} \notin \mathrm{Conv}\,\{f'(x^*), g_i'(x^*), i = 1, \dots, m\}.$$

Then for

$$\xi_1 = f'(x^*), \xi_2 = g_1'(x^*), \dots, \xi_{m+1} = g_m{}'(x^*)$$

by Lemma $\exists\, \eta_1, \dots, \eta_{m+1} \in X$

$$0_{m+1} \notin \mathrm{Conv}\left(\begin{pmatrix} \langle f'(x^*), \eta_1 \rangle \\ \dots \\ \langle f'(x^*), \eta_{m+1} \rangle \end{pmatrix}, \dots, \begin{pmatrix} \langle g_m{}'(x^*), \eta_1 \rangle \\ \dots \\ \langle g_m{}'(x^*), \eta_{m+1} \rangle \end{pmatrix}\right) \triangleq \widetilde{K}.$$

$$\exists p = (p_1, \dots, p_{m+1})^\top, \qquad p = \mathrm{Pr}_{\widetilde{K}}\, 0_{m+1}$$

$$\langle f'(x^*), -p_1\eta_1 - \dots - p_{m+1}\eta_{m+1}\rangle < 0$$

$$\langle g_i'(x^*), -p_1\eta_1 - \dots - p_{m+1}\eta_{m+1}\rangle < 0$$

$$i = 1, \dots, m$$

$$\overline{p} = p_1\eta_1 + \dots + p_{m+1}\eta_{m+1}$$

$$\Rightarrow f(x^* - \alpha\overline{p}) < f(x^*), \qquad g_i(x^* - \alpha\overline{p}) < 0,$$

$$i = 1, \dots, m, \quad \alpha > 0. \ \square$$