Analogies Between Cubic Surfaces and Cubic Curves

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Cubic curves E: $y^2 = x^3 + ax + b$

- Topology: (complex) torus
- Geometry: One-dimensional abelian variety – good for arithmetic
- Moduli: one-dimensional ($\mathbb{C} \cong \Gamma\backslash\mathcal{H}$)
- Moduli space is quotient of hermitian symmetric space

Cubic surfaces X: e.g., $x^3 + y^3 + z^3 + w^3 = 0$

- Topology: $\mathbb{P}^2$ with six points blown up
- Geometry: A rational surface with 27 lines
- Moduli: four-dimensional $\Gamma\backslash B^4$
- Moduli space is a quotient of hermitian symmetric space
- $X \Rightarrow$ special five-dimensional abelian variety J.
- Good for arithmetic???

Other cubics?

Moduli of Cubic Curves

Smooth cubic curve can be put in the form

$$E : y^2 = x^3 + ax + b$$

Solution set is a torus

$$T = \mathbb{C}/\Lambda, \qquad \Lambda = 1\,\mathbb{Z} + \tau\mathbb{Z}$$

What are the moduli? — i.e., the essential parameters that describe the isomorphism classes of E. One answer:

the ratio $j = \frac{1728a^3}{4a^3+27b^2}$

Another answer: The class of Λ – i.e. the class of τ modulo the action of $SL_2(\mathbb{Z})$.

Moduli space $= \Gamma\backslash\mathcal{H} \cong \mathbb{C}$

Answers are related by modular forms which express j in terms of τ.

Hodge theory

Cohomology of E:

$$H^1(E,\mathbb{C}) = H^{1,0} \oplus H^{0,1},$$

where $H^{1,0}$ is spanned by the holomorphic one-form

$$\omega = dx/y = dx/\sqrt{x^3 + ax + b},$$

and $H^{0,1}$ is spanned by $\bar{\omega}$. Thus $H^{0,1} = \overline{H^{1,0}}$.

The data

$$[\ H^1(E,\mathbb{Z}), H^{1,0}, \text{cup-product}\ ]$$

is a (polarized) Hodge structure of weight one. Polarized?

$$\Im(\tau) > 0 \qquad \Leftrightarrow \qquad \sqrt{-1}\int_E \omega \wedge \bar{\omega} > 0$$

$$E \cong J(E) = (H^{1,0})^*/H_1(E,\mathbb{Z}) \qquad \text{Abel's theorem}$$

Framed Hodge structures

A Hodge structure is framed by a basis for the integral lattice that is standard for the cup product.

Cycles δ and γ where $\delta \cdot \gamma = 1$.

A framing determines an isomorphism

$$\phi : H^1(E, \mathbb{Z}) \longrightarrow \mathbb{Z}^2$$

where

$$m\delta^* + n\gamma^* \mapsto (m, n).$$

A framed Hodge structure for E determines a line in $\mathbb{C}^2$:

$$L(E, \phi) = \phi(H^{1,0}(E))$$

Moduli from Hodge Theory

What is the line $L(E, \phi)$?

$$\omega = \left(\int_\delta \omega\right) \delta^* + \left(\int_\gamma \omega\right) \gamma^*$$

Coefficient integrals are the periods of ω. Define a ratio of periods

$$\tau = \int_\gamma \omega \Big/ \int_\delta \omega$$

Then

$$\mathbb{C}\omega = \mathbb{C}(\delta^* + \tau\gamma^*)$$

So

$$L(E, \phi) = \mathbb{C}(1, \tau) \subset \mathbb{C}^2.$$

Moduli of framed Hodge structures $= \{$slopes $\tau\} = \mathcal{H}$

Moduli of Hodge structures $= \{$slopes mod F.L.T.'s$\} = \Gamma \backslash \mathcal{H}$

Moduli of cubic surfaces

Dimension of space of cubic forms $F(x, y, z, w) = \binom{6}{3} = 20$

Dimension of $GL(4) = 16$

Dimension of moduli space $\mathcal{M} = 4$

Existence of $\mathcal{M}$: GIT   But what does $\mathcal{M}$ look like?

Moduli via Hodge theory?

$$H^2(X) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$$

where

$$H^{2,0} : f\, dz \wedge dw \qquad H^{1,1} : f\ dz \wedge \bar{d}w$$

For cubic surface X: $H^{2,0} = 0.$ $H^2(X) = H^{1,1}(X).$

No moduli in Hodge theory.

Picard's Trick

Let Y be the cubic threefold obtained as a triple branched cover of $\mathbb{P}^3$ branched over X: $u^3 + F(x, y, z, w) = 0$

Hodge structure: $H^3(Y) = H^{2,1} \oplus H^{1,2} \qquad \dim H^{2,1} = 5.$

5-dimensional abelian variety: $J(Y) = H^{2,1}(Y)^*/H_3(Y, \mathbb{Z})$

$J(Y)$ determines Y: Torelli theorem of Clemens-Griffiths.

Mumford's proof: Θ divisor of J has unique isolated cubic singularity p.

Projectivized tangent cone of p is Y.

Also: $J(Y(X))$ determines X.

Hodge structure of cyclic threefold Y

Let $\sigma(u) = \omega u$, where $\omega = \exp(2\pi i/3)$.

The transformation σ decomposes H^3 into eigenspaces:

$$H^3(Y) = H^3(Y)_\omega \oplus H^3(Y)_{\bar{\omega}}.$$

Each is five-dimensional.

The decomposition is compatible with the Hodge structure, e.g.,

$$H^3(Y)_{\bar{\omega}} = H^{2,1}(Y)_{\bar{\omega}} \oplus H^{1,2}(Y)_{\bar{\omega}}$$

where $\dim H^{2,1}(Y)_{\bar{\omega}} = 1$ and $\dim H^{1,2}(Y)_{\bar{\omega}} = 4$.

Let $\phi : H^3(Y)_{\bar{\omega}} \to \mathbb{C}^5$ be a suitable framing. Define a line

$$L(Y, \phi) = \phi(H^{2,1}(Y)_{\bar{\omega}}) \subset \mathbb{C}^5$$

This determines a point in the Grassmannian of lines in $\mathbb{C}^5 = \mathbb{P}^4$.

The line $L(Y, \phi)$

Consider affine equation for Y: $u^3 + F(x, y, z, 1) = 0$

Set

$$\omega = \frac{dx \wedge dy \wedge dz}{u^{4/3}}$$

The form ω generates $H^{2,1}(Y)_{\bar{\omega}}$. Moreover,

$$\sqrt{-1} \int_Y \omega \wedge \bar{\omega} < 0$$

Now let $\gamma_0, \gamma_1, \ldots, \gamma_4$ be homology classes which determine a framing.

One may normalize these so that $\gamma_0^2 = -1$, $\gamma_i^2 = 1$ for $i > 0$.

This is because the cup product form $\sqrt{-1}(x, \bar{y})$ is negative on $H^{2,1}(Y)_{\bar{\omega}}$ and positive on $H^{1,2}(Y)_{\bar{\omega}}$.

The period vector

Normalize ω so that its integral over γ_0 is 1. Set

$$\tau_i = \int_{\gamma_i} \omega$$

Then $L(Y, \phi)$ is the line generated by

$$(1, \tau_1, \tau_2, \tau_3, \tau_4)$$

The relation $i \int_Y \omega \wedge \bar{\omega} < 0$ yields

$$|\tau_1|^2 + |\tau_2|^2 + |\tau_3|^2 + |\tau_4|^2 < 1$$

That is, the vector $\tau = (\tau_1, \tau_2, \tau_3, \tau_4)$ lies in the unit ball B^4.

Conclusion: Framed Hodge structures of cyclic cubic threefolds are parametrized by the unit ball B^4.

Framings

The homology of Y is a module for the ring of Eisenstein integers $\mathcal{E} = \mathbb{Z}[\omega]$: ω acts via σ.

A framing is a basis of $H_3(Y, \mathbb{Z})$ as a hermitian $\mathcal{E}$ module that diagonalizes the hermitian form as $(-1, +1, +1, +1, +1)$.

$$2h(x, y) = ((\sigma - \sigma^{-1})x \cdot y) - (\omega - \omega^{-1})(x \cdot y)$$

The group which acts on the framings is $\Gamma = SU(1, 4, \mathcal{E})$.

Conclusion:

$$\{ \text{ Hodge structures} + \text{symmetry } \sigma \} = \Gamma \backslash B^4$$

Constructed: a period map

$$f : \mathcal{M} \to \Gamma \backslash B^4$$

Analysis of period map

Injectivity? Yes: Torelli theorem.

Surjectivity? No: Cubic surfaces with a node correspond to a totally geodesic hyperplane $\delta^\perp$ in B^4. These are not in the image of f.

Let $\mathcal{H} = \bigcup_\delta \delta^\perp$. Then

$$f : \mathcal{M} \to \Gamma \backslash (B^4 - \mathcal{H})$$

is surjective.

Moreover, if $\mathcal{M}_s$ denotes the moduli of stable cubic surfaces, then

$$f : \mathcal{M}_s \to \Gamma \backslash B^4$$

is an isomorphism.

Corollaries

Let PC_0 be the projective space of smooth cubic forms.

What is $\pi_1(P\mathcal{C}_0)$?

Monodromy representation:

$$\rho : \pi_1(P\mathcal{C}_0) \to Aut(H^2(X))$$

has finite image

Second monodromy representation:

$$\rho' : \pi_1(P\mathcal{C}_0) \to PAut(H^3(Y))$$

has infinite image

The second monodromy representation also has (virtually) infinite kernel, essentially $\pi_1(B^4 - \mathcal{H})$.

Speculations

X = cubic surface / $\mathbb{F}_p$. Has zeta function — from trace of Frobenius map.

But only finitely many zeta functions — because only finitely many conjugacy classes in $W(E_6)$.

But roughly p^4 points in moduli space of cubics over $\mathbb{F}_p$.

So # points in moduli space $>>$ # zeta functions.

Better (complete) invariant — $J(X)$?