THE CONCEPT OF PROOF

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The concept of proof is one of the most fundamental building blocks of mathematics. The Hilbertian revolution at the beginning of the 20th century is based on an atomic notion of proof which is the foundation of the axiomatic method:

"A proof is a finite sequence of formulas $A_1, \ldots, A_n$ such that each A_i is instance of an axiom or follows by direct application of a rule from $A_{i_1}, \ldots, A_{i_k}$ with all $i_j < i$".

No scientific revolution is however total, but there is a trend to disregard all alternatives to the successful method. In this lecture we discuss more global notions of proof, where subproofs are not necessarily proofs themselves. Examples are among others:

1. protoproofs in the sense of Euler's famous solution to the Basel problem, which uses analogical reasoning and where additional external justifications are necessary;
2. circular notions of proof, where the concept of proof itself incooperates induction. The most significant example is Pierre de Fermat's Methode de Descente, for a modern setting cf. [1];
3. sound proofs based on locally unsound rules cf. [2];
4. proofs based on abstract proof descriptions prominent e.g. in Bourbaki, where only the choice of a suitable result makes a verification possible cf. [3].

We discuss the benefits of these alternative concepts and the possibility that innovative concepts of proof adapted to the problems in question might lead to strong mathematical results and constitute a novel area of Proof Theory.

References

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