HIGHER TYPES OF RECURSION AND LOW LEVELS OF DETERMINACY

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We consider how one may lift Kleene's theory of recursion in finite types ([3]) to more expanded notions of recursion. Kleene himself in [4] and [5] sought to show how his previous definitions using an equational calculus could also be grounded in an equivalent formulation using Turing machines (thus perhaps providing backing for the higher type recursive notions). 'Kleene Recursion' (at type 2) has come down to us as a theory of hyperarithmetic sets of reals.

A notion of a more 'generalized-recursion' can roughly speaking be obtained by replacing Turing machines in Kleene's [4] and [5], by so called *infinite time Turing machines* (ittm's) [2]. The characteristics of such conceptual devices have been investigated in [7] and they can be shown to compute codes for an initial segment of the constructible universe ([1]). This higher type involvement of ittm's is an interesting construction in its own right, and deserves, we believe, further investigation. Here it is shown that there are applications of ittm-theory to classical descriptive set theory. For, we can already give an exact characterisation of complete ittm-semi-decidable sets formed relative to a particular type 2 functional. (This results in a definition similar to one already used by [6].)

This comes through a theorem connected with low level determinacy. For the 'Kleene recursion' above, there were already connections with open determinacy: Player I in an open, or Σ^0_1, game on Cantor or Baire space has a hyperarithmetic, hence 'Kleene recursive', strategy. A listing of the open games won by Player I formed a complete 'Kleene semi-decidable-in-oJ' (for *ordinary jump*) set of integers.

We show the equivalence between the existence of winning strategies for $G_{\delta\sigma}$ (or Σ^0_3) games in Cantor or Baire space, and the existence of functions generalized ittm-recursive in a certain higher type-2 functional eJ' (for *extended jump*). This allows us to lift in a natural fashion the Kleenean results to this level: the list of Σ^0_3 games won by Player I is now a complete generalized semi-decidable-in-eJ set of integers. (See [8].)

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