Between Physics and Model Theory

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The story so far

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B.Zilber, *A class of quantum Zariski geometries* In: **Model Theory with applications to algebra and analysis**, I and II, CUP, Cambridge. 2008

———, *Perfect infinities and finite approximation* In: **Infinity and Truth**. IMS Lecture Notes Series, V.25, 2014

J.A.Cruz and B.Zilber, *The geometric semantics of algebraic quantum mechanics* Phil. Trans. R. Soc. A 2015 373 20140245

B.Zilber, *The semantics of the canonical commutation relations* arxiv.org/abs/1604.07745

———-, *Structural approximation and quadratic quantum mechanics* In preparation

Problems with limits

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The passage from discrete (**finite**) to continuous.

Cf. with Hilbert 6.

Sub-problem. “Very large” finite structure

The relevant formalism - **pseudo-finite** structures.

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Definition. An L-structure $\mathbf{M}$ is a limit of a family $\{\mathrm{N}_i : i \in I\}$ of L-structures along an ultrafilter $\mathcal{D}$ if there is a surjective L-homomorphism

$$\lim : \mathrm{N} \twoheadrightarrow \mathbf{M} \text{ for } \mathrm{N} = \prod_{i \in I} \mathrm{N}_i / \mathcal{D}.$$

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$$\lim_{\mathcal{D}} : \mathrm{N}_i \to \mathbf{M}.$$

Determining a limit

Note.
1.Homomorphisms preserve positively definable relations, "equations".

2. An example of lim is the *standard part* map.

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is satisfied iff **M** is *compact*, i.e.

- the intersection of any filter of closed subsets of $\mathbf{M}^n$ is non-empty;
- the image of a closed $S \subset \mathbf{M}^{n+1}$ under projection $\mathbf{M}^{n+1} \to \mathbf{M}^n$ is closed. I.e. $\exists$-positive definable sets are closed (positive quantifier elimination).

Positive model theory

(Compare also with *positive model theory* by Ben-Yaacov, Poizat and others)

Positive model theory

A pair of structures with a morphism

$$\mathrm{N} \twoheadrightarrow \mathbf{M}$$

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We want that **M** does not depend on the choice of N in $\mathcal{N}$.

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In general, N is not unique for a given $\mathbf{M}$; there is a whole class $\mathcal{N}$ of L-structures N with

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We look in the cases when $\mathcal{N}$ has the amalgamation property with respect to $\twoheadrightarrow$ (instead of $\hookleftarrow$) and use "upside-down" Fraïssé construction to build **the model completion** $\mathcal{N}^*$ **of** $\mathcal{N}$, in the sense of positive model theory**).**

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Cf. similar notions and results by Irwin and Solecki (2005) and earlier construction by Cherlin, van den Dries and Macintyre (1981)

Approximation of fields and rings

Any locally compact field (such as $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}_p$) can be *compactified* by adding ∞.
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Theorem.
1. The compactified field $\bar{\mathbb{C}}$ (or any other ACF) is approximable by finite fields.

2. The rings $\mathbb{Z}_p$ are approximable by finite rings $\mathbb{Z}/p^n$.

3. The compact subring $\hat{\mathbb{Z}} = \prod \mathbb{Z}_p$ of adeles is approximable by finite rings $\mathbb{Z}/N$, for N "highly divisible".

4. The compactified field $\bar{\mathbb{R}}$ is NOT approximable by finite fields or finite rings. Neither are any other locally compact fields.

How to approximate $\mathbb{R}$?

Weak ring structure on $\bar{\mathbb{R}}$

The “weak ring structure” $\mathbb{R}_{\text{weak}}$ is a two-sorted structure of the form

$$((\bar{\mathbb{R}}; +, \leq, r\cdot, ...)_{r\in\mathbb{R}} \longrightarrow (\mathbb{S}, \cdot, ...))$$

where $\mathbb{S} \subset \mathbb{C}$ is the group of the unit circle in the complexes, $\{r\cdot, ...\}_{r\in\mathbb{R}}$ the family of linear operators $x \mapsto r \cdot x$ on $\bar{\mathbb{R}}$ and $\longrightarrow$ is the bilinear map

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Note. The weak structure on $\bar{\mathbb{R}}$ allows to recover the usual ring structure on $\mathbb{R}$, if one uses all the power of first-order logic. But, *the ring structure is not recoverable by positive formulas*.

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Further relations and functions on $\mathbb{R}_{\text{weak}}$ are of the form

$$z \in r \cdot \mathbb{S}, \quad z = \int_{\mathbb{R}} \mathrm{e}^{iF(x,y)} dx, \quad z = r_f \cdot \mathrm{e}^{ig(y)}$$

where $F(x, y) \in \mathbb{R}[x, y]$, $g(y) \in \mathbb{R}[y]$

Weak ring structure on $\mathbb{Z}/N$

This is given as

$$(\mathbb{Z}/N; +, \leq, nx = my, \ldots)_{\frac{m}{n}=r} \longrightarrow (\mathbb{Z}/N; +, \ldots)$$

where, $\leq$ is the cyclic order and $\longrightarrow$ is

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Note that

$$(\mathbb{Z}/N; +) \cong \mathbb{C}_N \subset \mathbb{S} \subset \mathbb{C}$$

multiplicative group of roots of unity of order N.

Weak ring structure on $\mathbb{Z}/N$

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$$\int_{\mathbb{R}} e^{2\pi iF(x,y)} dx \rightsquigarrow \sum_{0 \le n < A(N)} e^{\frac{2\pi iF(n,m)}{N}} \Delta n$$

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A crucial property: $n << N \to n|N$, follows from pos-model completeness.

Structures over $\mathbb{R}_{\mathrm{w}}$

Theorem. The finite approximation of $\bar{\mathbb{R}}_{\mathrm{w}}$ can be extended to a finite approximation of the mathematical structure of quantum mechanics with quadratic Hamiltonians:

Structures over $\mathbb{R}_w$

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the symplectic space on $\mathbb{R}^{2n}$ *with Hermitian line bundle with connection.*

The metaplectic group $\mathrm{Mp}(n)$ *of symplectomorphisms acts on the space and on the line bundle.*

Its 1-dim subgroups describe the evolution of respective "particles".

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Calculations over finite models can be passed via **lim** to the continuous model of quantum mechanics.

Example of Calculation. Time evolution operator for the quantum harmonic oscillator.

$$K^t := e^{-i\frac{\mathrm{P}^2+\omega^2\mathrm{Q}^2}{\hbar}t}, \quad t \in \mathbb{R}.$$

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The matrix element on row x_1 and column x_2 (*kernel of the Feynman propagator*) is calculated as

$$\langle x_1 | K^t x_2 \rangle = \sqrt{\frac{\omega}{2\pi i\hbar \sin t}} \exp i\omega \frac{(x_1^2 + x_2^2)\cos t - 2x_1x_2}{2\hbar \sin \omega t}.$$

Physicists calculations of path integral

$$I_L = \int_{\mathbb{R}} \dots \int_{\mathbb{R}} e^{i \sum a_{kj} x_k x_j} dx_1 \dots dx_L, \quad K^t = \lim_{L \to \infty} I_L$$

where L defines $\Delta t = \frac{t}{L}$ for which the evolution from t_k to $t_k + \Delta t$ is evaluated. I_L calculates the evolution from t_0 to $t_0 + t$.

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where L defines $\Delta t = \frac{t}{L}$ for which the evolution from t_k to $t_k + \Delta t$ is evaluated. I_L calculates the evolution from t_0 to $t_0 + t$. For the Harmonic oscillator,

$$|I_L| = \sqrt{L} \prod_{n=1}^{L} \sqrt{\left(\frac{\pi^2 n^2}{t^2} - \omega^2\right)^{-1}}$$

does not converge to a finite number, so has to be renormalised (treated as a "generalised limit") in order to get a right answer.

Structural approximation calculations of path integral

$$I_\kappa = \sum_{x_1=-\eta/2}^{\eta/2} \dots \sum_{x_\kappa=-\eta/2}^{\eta/2} e^{i\sum a_{kj}x_k x_j} \Delta x_1 \dots \Delta x_\kappa, \quad K^t = \lim I_\kappa$$

$0 << \kappa << \eta$. $\Delta x = c\frac{1}{\sqrt{\eta}}$, $\Delta t = \frac{t}{\kappa}$.
The control over the pseudo-finite parameters allows to evaluate K^t correctly, no need of renormalisation.

Higher order oscillating integrals

The current aim is to reinterpret the classical *stationary phase formula* in terms of pseudo-finite model

$$\int_{\mathbb{R}} e^{\frac{iF(x)}{h}} dx = e^{\frac{iF(c)}{h}} h^{\frac{1}{2}} I(h), \text{ where } \lim_{h \to +0} I(h) = e^{\frac{\pi i}{4}} \sqrt{2\pi} \frac{1}{\sqrt{|F''(c)|}}$$

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assuming the c the unique critical point, $F'(c) = 0$, $F''(c) \neq 0$. According to the general rule our interpretation of this integral, for $\deg F = d$, $\frac{1}{2\pi h} = H \in \mathbb{Z}$ is

$$\int_{\mathbb{R}} \mathrm{e}^{\frac{iF(x)}{h}} dx := \lim \frac{1}{M} \frac{M^{d-2}}{Hd} \sum_{0 \le n < M^d} \exp(2\pi i H F(\frac{n}{M}))$$

Theorem. For some $J(H, F)$ (sum of roots of unity)

$$\int_{\mathbb{R}} \mathrm{e}^{\frac{iF(x)}{h}} dx = J(H, F).$$