Proving Undecidability

of Certain Affine Geometries

for Graduate Students in Computer Science

in the 21st Century

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Based on: J.A. Makowsky (2017, arXiv:1712.07474)
Can one design a geometry engine?
On the (un)decidability of affine Euclidean geometries

Outline

- The origins of this talk
- Hommage to P. Bernays
- Decidability of Analytic Geometry
- Undecidability of theories of fields
- Undecidability in Synthetic Geometry

This talk has a long gestation

- In my course (Technion CS 236714) Topics in automated Theorem Proving I usually devote about one third on theorem proving in geometry. There I survey decidability and undecidability results.

- In 2016 R. Kahle invited me to give a talk on the occasion of the inauguration of a plaquette honoring P. Bernays at the house in Göttingen where P. Bernays used to live.

- At the same time I saw a call for papers on automated theorem proving for a special issue of Annals of Mathematics in Artificial Intelligence.

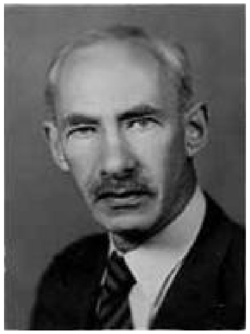
- A preliminary version is posted:

  Can one design a geometry engine?
  On the (un)decidability of affine Euclidean geometries
  https://arxiv.org/abs/1712.07474

  The revised version will follow these days.

Paul Bernays (1888-1977)

in Göttingen from 1917-1934

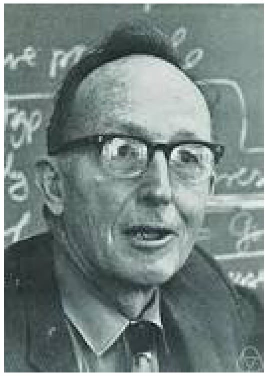


- P. Bernays' influence on Computer Science
- P. Bernays and Geometry

Lieber Herr Bernays.........

- Paul Bernays founded the **Monday Logic Seminar**, at ETH Zürich in 1939, together with F. Gonseth and G. Polya.
- Later it was run jointly with E. Specker and H.Läuchli, and after H.Läuchli's premature death, by E. Specker till 2002.
- I met Paul Bernays first in the **Monday Logic Seminar** in **1967**.
- He introduced me, on my request, to G. Kreisel, which became a decisive event for my further career.
- I became very friendly with P. Bernays till his death in 1977.
- P. Bernays was a guest of honor at my PhD party in 1974

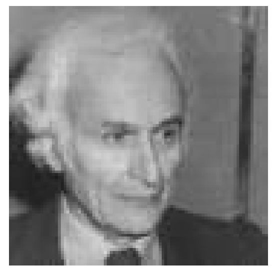
Paul Bernays and Computer Science, I



Doctoral students in Göttingen

- Haskell Curry (1900–1982) PhD 1930
  Combinatory Logic, Programming languages
- Gerhard Gentzen (1909–1945) PhD 1933
  Proof Theory, Proof theoretic Semantics
- Saunders Mac Lane (1909–2005) PhD 1934
  Category Theory

Paul Bernays and Computer Science, II



Doctoral students at ETH Zürich

- Julius Richard Büchi (1924-1984), PhD 1950
  Finite Automata, Descriptive Complexity

- Corrado Böhm (1923–), PhD 1951
  Programming languages, Structured programming, λ-calculus

- Erwin Engeler (1930–) PhD 1958
  First Professor of Logic and Computer Science at ETH Zürich, 1972-1997

Paul Bernays and Computer Science, III

Collaborators and postdoctoral visitors in Göttingen

1914–1924 Moses Ilyich Schönfinkel (1889–1942)
Founder of Combinatory Logic

1929 László Kalmár (1905–1976)
First Professor of Logic and Computer Science in Hungary

1933 Rózsa Péter (1905-1977)
Founder of Recursion Theory as a discipline

Ernst Specker (1920–2011)

- Ernst Specker got his habilitation from **P. Bernays** in 1951 for his work in set theory.
- E. Specker was at ETH Zürich from 1950 on,a and became Full Professor of Logic in 1955.
- E. Specker and V. Strassen had an influential seminar from 1973–1988 on algorithmic problems.

FESTSCHRIFT
ZUR FEIER DER ENTHÜLLUNG DES GAUSS-WEBER-DENKMALS IN GÖTTINGEN. HERAUSGEGEBEN VON DEM FEST-COMITEE. I. THEIL.

GRUNDLAGEN DER GEOMETRIE

VON

DR. DAVID HILBERT,
O. PROFESSOR AN DER UNIVERSITÄT GÖTTINGEN.

MIT 50 IN DEN TEXT GEDRUCKTEN FIGUREN.

LEIPZIG,
VERLAG VON B. G. TEUBNER.
1899.

Paul Bernays and Geometry

- D. Hilbert's **Grundlagen der Geometrie** appeared first in **1899**.
- P. Bernays was involved in preparing the lectures for Hilbert since **1917**.
- From its 5th German edition (**1922**) collaboration with P. Bernays is acknowledged.
- P. Bernays began editing revised editions in **1956** (8th edition).
- P. Bernays' preface to this 10th ed. is dated Feb., **1968**.

History:

From Euclid to Hilbert-(Bernays)

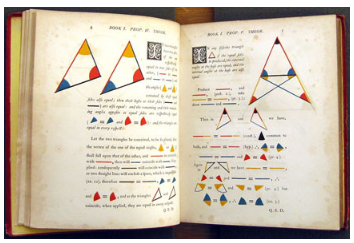
and beyond.......(Tarski, Wu)

Back to outline

The Classics, I

Euclides: *Elements of Geometry*

The most influential mathematical text ever written.
Latin versions: Peletier, 1557; F. Commandino, 1572; C. Clavius, 1574.
Italian version: F. Commandino, 1575
French version: F. Peyrard, 1804
English versions: Simson, 1756; Playfair 1795; Heath, 1926

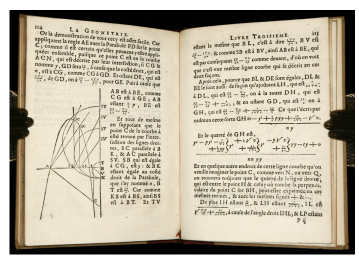


(wikipedia) **Euclides of Alexandria**, fl. 300 BC, was a Greek mathematician, often referred to as the "Father of Geometry". He was active in Alexandria during the reign of Ptolemy I (323283 BC). His Elements is one of the most influential works in the history of mathematics, serving as the main textbook for teaching mathematics (especially geometry) from the time of its publication until the late 19th or early 20th century.

The Classics, II

René Descartes 31 March 1596 11 February 1650
Latinized: Renatus Cartesius; adjectival form: "Cartesian"; was a French philosopher, mathematician, and writer who spent most of his life in the Dutch Republic. He has been dubbed The Father of Modern Philosophy, and much subsequent Western philosophy is a response to his writings, (...) Descartes' influence in mathematics is equally apparent; the Cartesian coordinate system allowing reference to a point in space as a set of numbers, and allowing algebraic equations to be expressed as geometric shapes in a two-dimensional coordinate system (and conversely, shapes to be described as equations) was named after him. He is credited as the father of analytical geometry, the bridge between algebra and geometry, crucial to the discovery of infinitesimal calculus and analysis


LA
GEOMETRIE.
LIVRE PREMIER.



(wikipedia) Discours sur la méthode,
with an appendix *La Géométrie* 1637 and 1664.

The Classics, III

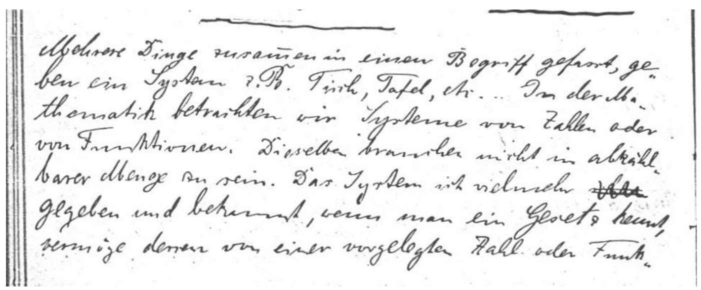
Euclides Danicus: Georg Mohr (1640-1697), published in 1672

(wikipedia): Jorgen Mohr (Latinised Georg(ius) Mohr) (April 1, 1640 January 26, 1697) was a Danish mathematician. He traveled in the Netherlands, France, and England. Mohr was born in Copenhagen. His only original contribution to geometry was the proof that any geometric construction which can be done with compass and straightedge can also be done with compasses alone, a result now known as the MohrMascheroni theorem. He published his proof in the book Euclides Danicus, Amsterdam, 1672.

The Classics, IV

Hilbert: *Grundlagen der Geometrie*, 1899 ff.

David Hilbert (later editions with P.Bernays),
English version by Leo Unger, 1971

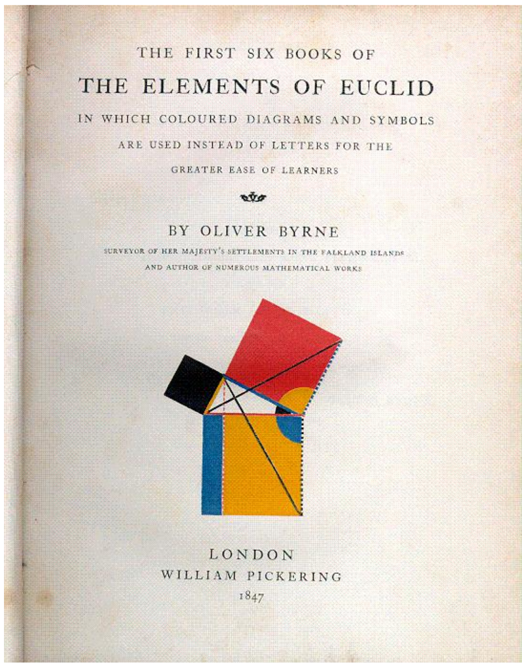


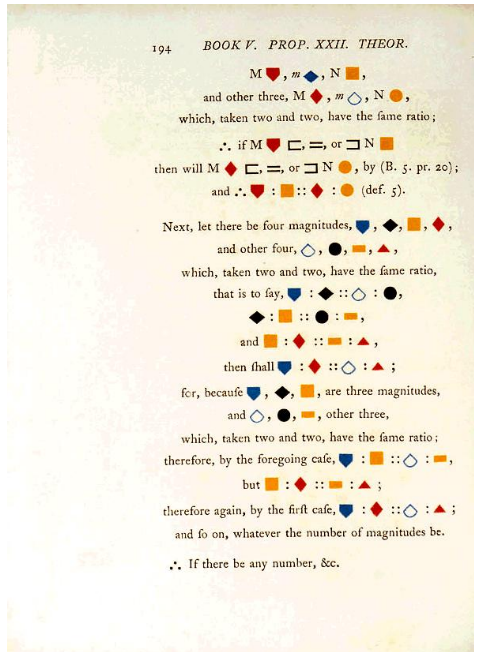
Mehrere Dinge zusammen in einen Begriff gefasst, geben ein System z.B. Tisch, Tafel, etc... In der Mathematik betrachten wir Systeme von Zahlen oder von Functionen. Dieselben brauchen nicht in abzählbarer Menge zu sein. Das System ist vielmehr gegeben und bekannt, wenn man ein Gesetz kennt, vermöge dessen von einer vorgelegten Zahl oder Funkt.

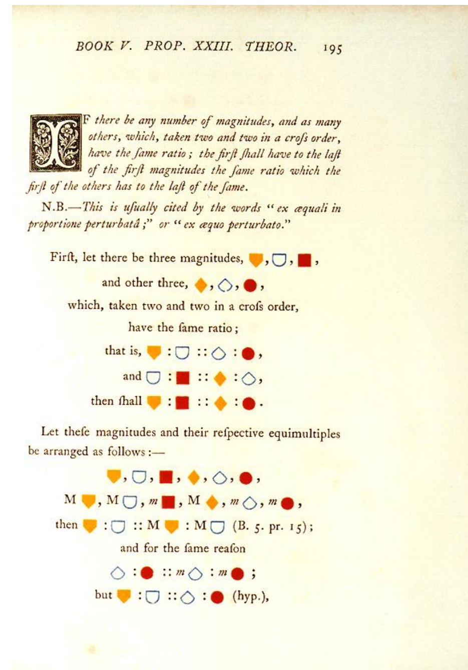
Hilbert's Geometry Axioms

The Oliver Byrne edition of Euclid, 1847

A masterpiece of visualization


THE FIRST SIX BOOKS OF
THE ELEMENTS OF EUCLID
IN WHICH COLOURED DIAGRAMS AND SYMBOLS
ARE USED INSTEAD OF LETTERS FOR THE
GREATER EASE OF LEARNERS
BY OLIVER BYRNE
SURVEYOR OF HER MAJESTY'S SETTLEMENTS IN THE FALKLAND ISLANDS
AND AUTHOR OF NUMEROUS MATHEMATICAL WORKS
LONDON
WILLIAM PICKERING
1847


194 BOOK V. PROP. XXII. THEOR.


BOOK V. PROP. XXIII. THEOR. 195

Available online:

https://www.math.ubc.caa∼/cass/euclid/byrne.html

Apology

- There are no new technical results in this talk.

- I just report on what I learned when I reviewed the question while preparing a course first in 2003, and later till 2015.

- But I would like to draw attention to **M. Ziegler's results** and bred their significance for the question.

  They have been **widely overlooked**, due to the fact that they were published in German in a Swiss-French periodical in 1982 (and presented in 1980 at the occasion of E. Specker's 60th birthday..

- I also offer a comprehensive view, both Algebra-Geometrical and Algorithmic.

Automated Theorem Proving

for High School Geometry

- Herbert Gelernter, 1929 – 2015

- Empirical Explorations of the Geometry Theorem Machine
  H. Gelernter, J.R. Hansen and D.W. Loveland,
  IBM Report 1960

Decidability

Alfred Tarski (1901–1983)

and Wu Wenjun (Wen-Tsün Wu) (1919–2017)

A. Tarski   Wu Wenjun

- W. Schwabhäuser, W. Szmielev and A. Tarski,
  Metamathematische Methoden in der Geometry, Springer 1983

- Wen-Tsün Wu,
  Mechanical Theorem Proving in Geometries: Basic Principles,
  Springer 1994, First Chinese edition 1984

Tarski's Theorem of 1931 and 1951

Elementary Euclidean Geometry is decidable.

This needs clarifications:

- What is elementary Geometry?
- What exactly is decidable?

The theory of fields, 1949-1960

J. Robinson, 1949 The first order consequences of the field axioms are not recursive.

J. Robinson, 1949 The complete theory of the field of rational numbers is not r.e.

A. Tarski, 1931 or 1951 The theory of algebraic closed fields of characteristic 0 is complete and decidable, and admits elimination of quantifiers.

A. Tarski, 1931 or 1951 The theory of ordered real closed fields is complete and decidable, and admits elimination of quantifiers.

Tarski's conjecture:
If T is a finite subtheory of RCF or ACF_0, then T is undecidable.

The theory of fields 1960-1990

G.E. Collins, 1975
Quantifier elimination for RCF can be done in doubly exponential time.

M. Ziegler, 1982 Tarski's conjecture is proved.
If T is a finite subtheory of RCF or ACF_0, then T is undecidable.

A. Macintyre, K. McKenna, L. van der Dries, 1983
If T is a theory of fields consistent with RCF or ACF_0 which admits elimination of quantifiers, then T is logically equivalent to RCF, resp. ACF_0.

J.H. Davenport and J. Heintz, 1988 Quantifier elimination for RCF requires doubly exponential time.

D. Grigoriev and N. Vorobjov, 1987 Quantifier elimination for RCF for existential formulas can be performed in simple exponential time.
It is open whether this can be done in polynomial time.
This one of the millennium problems.

Analytic geometry over a(n) (ordered) field, I

Geometry can be interpreted using quantifier-free formulas in models of RCF and ACF_0:

- In a model of RCF we get a model of Euclidean geometry with betweenness (Hilbert planes, Euclidean planes, etc).
- In a model of ACF_0 we get a model of Wu's orthogonal geometry without betweenness.

Conclusions:

- It is decidable whether a statement formulated in Hilbert's language of geometry is true in the geometry interpreted in a model of RCF.
- It is decidable whether a statement formulated in Wu's language of geometry is true in the geometry interpreted in a model of ACF_0.

Verification of Geometric Constructions
High School Geometry

In text book problems in Geometry we are a given a construction of points $P_1, P_2, \ldots, P_n$ and lines $l_1, l_2, \ldots, l_m$ using **ruler and compass**. The theorem then asserts or forbids that a subset of points either meet, are colinear or co-circular, a subset of lines either meet, are parallel or perpendicular, or a subset of pairs of points are pairwise equidistant.

Translating this into the language of (ordered) fields we get a formula of the form

$$\forall \bar{x} \left(\left(\bigwedge_{i \in I} f_i(\bar{x}) = 0 \wedge \bigwedge_{j \in J} h_j(\bar{x}) \neq 0 \right) \rightarrow g(\bar{x}) = 0 \right)$$

Here the f_i, h_j, g are polynomials of degree 2. In particular, the statement is of the form $\forall \bar{x} \Phi(\bar{x})$, with Φ quantifier free.

This remains true if we allow also constructions with **marked ruler** which allows us to trisect angles.

The universal theory of Affine Geometry

Theorem: (Schur)
For any model of Affine Geometry Π the ring $\mathfrak{F}_\Pi$ is a commutative field of characteristic 0.
Conversely, for a commutative field of characteristic 0, $\mathfrak{F}$, the Geometry $\Pi_\mathfrak{F}$ is a model of Affine Geometry.

Theorem: Let T be a set of τ_{Wu}-sentences ($\tau_{Hilbert}$-sentences) and let ϕ be a universal τ_{Wu}-sentence ($\tau_{Hilbert}$-sentence).

(i) If every model of T is a field of characteristic 0, then

$$T \vdash \phi \text{ iff } ACF_0 \vdash \phi.$$

(ii) If every model of T is an ordered field, then

$$T \vdash \phi \text{ iff } RCF \vdash \phi.$$

In particular, in both cases the universal theory of the Geometry derived from T is decidable.

Analytic geometry over a(n) (ordered) field, II

Using the facts that

- Universal formulas are preserved under substructures, and
- universal formulas of geometry for the interpreted geometry correspond to universal formulas in the field,

we get more:

Proposition:

- Let T be a finite set axioms in Hilbert's language of geometry true in the interpretation over a model of RCF, and let ϕ a universal formula in the same language. Then it is decidable whether ϕ follows from T.
- The analogous statement is true for Wu's geometry and ACG_0.

Elementary geometry is decidable

So what is elementary geometry?

- Elementary = high-school geometry
- Elementary = First order logic
- Elementary = the first order theory of the reals (Euclid, Hilbert, Tarski)
- Elementary = the first order theory of the complex numbers (orthogonal geometry, Wu)

Open problems, I

- What is the complexity of deciding the truth of universal sentences true in RCF, resp. ACF_0?
- Is the Proposition of the previous slide also true for $\forall\exists$-sentences?

Undecidability

Undecidability of various axiomatizations of synthetic geometry, I

Let $\mathcal{F}$ be a field and $\Pi(\mathcal{F}))$ the model of geometry with betweenness interpreted in $\mathcal{F}$. Conversely, let Π a Pappian plane with no finite lines, and let $\mathcal{F}(\Pi)$ be its field of segment arithmetic.

In 1909, F. Schur showed:

Theorem:

(i) $\mathcal{F}$ is a field of characteristic 0 iff $\Pi(\mathcal{F}))$ is a Pappian plane with no finite lines.

(ii) Π is a Pappian plane with no finite lines iff $\mathcal{F}(\Pi))$ is a field of characteristic 0.

This can also be found in E. Artin's book Geometric Algebra from 1957.

Undecidability of various axiomatizations of synthetic geometry, II

Using the undecidability of the theory of
fields (also of characteristic 0)
one concludes in 1949, without providing all the details,
that the first order consequences of Pappian planes (whatever they are) is not recursive (but is r.e.).

However, one would need more:

(i) The fields $\mathcal{F}$ and $\mathcal{F}(\Pi(\mathcal{F}))$ are isomorphic.

(ii) The Pappian planes Π and $\Pi(\mathcal{F}(\Pi))$ are isomorphic as incidence structures.

(iii) Both interpretations, geometry in a field, and field in a geometry, are first order definable.

Finding a written version of the undecidability proof suitable for today's graduate students in CS or AI

- A. Tarski announced some undecidability results in 1949 at the 11th meeting of the Association of Symbolic Logic.

- In W. Hodges monograph *Model Theory* proving such an undecidability results is given as Exercise 10 of Section 5.4.

- A complete proof is buried in the German book by W. Schwabhäuser (based on notes by A. Tarski and W. Szmielew) from 1983.

- Various undecidability results are almost completely proved in the *Handbook of spatial logic* of 2007, chapter by P. Balbiani, V. Goranko, R. Kellerman and D. Vakarelov: *Logical theories for fragments of elementary geometry*.

None of these references are helpful for the graduate students I have in mind.

Incidence geometries

This is the simplest family of geometries:

- We have two sorts: Points and lines.
- We have only one binary relation $\in$ between points and lines. $p \in \ell$ says that the point p is incident with the line ℓ.
- We have three incidence axioms I-1, I-2, I-3.

Incidence axioms

(I-1): For any two distinct points A, B there is a unique line l with $A \in l$ and $B \in l$.

(I-2): Every line contains at least two distinct points.

(I-3): There exists three distinct points A, B, C such that no line l contains all of them.

They can be formulated in First Order Logic FOL using the incidence relation only.

Other axioms for incidence only, I

There are more axioms which can be formulated with the incidence relation only:

Parallel axiom: We define: $Par(l_1, l_2)$ or $l_1 \parallel l_2$ if l_1 and l_2 have no point in common.

(ParAx): For each point A and each line l there is at most one line l' with $l \parallel l'$ and $A \in l'$.

$Par(l_1, l_2)$ can be formulated in FOL using the incidence relation only, hence also the Parallel Axiom.

Pappus' axiom:

(Pappus): Given two lines l, l' and points $A, B, C \in l$ and $A', B', C' \in l'$ such that $AC' \parallel A'C$ and $BC' \parallel B'C$. Then also $AB' \parallel A'B$.

Other axioms for incidence only, II

Axioms of Desargues and of infinity:

(Inf): Given distinct A, B, C and l with $A \in l, B, C \notin l$ we define $A_1 = Par(AB, C) \times l$, and inductively, $A_{n+1} = Par(A_n B, C) \times l$. Then all the A_i are distinct.

(De-1): If AA', BB', CC' intersect in one point or are all parallel, and $AB \parallel A'B'$ and $AC \parallel A'C'$ then $BC \parallel B'C'$.

(De-2): If $AB \parallel A'B'$, $AC \parallel A'C'$ and $BC \parallel B'C'$ then AA', BB', CC' are all parallel.

The axiom of infinity is not first order definable but consists of an infinite set of first order formulas with infinitely many new constant symbols for the points A_i, and the incidence relation. The two Desargues axioms are first order definable using the incidence relation only.

The undecidability of some incidence geometries

W. Rautenberg in his Diploma thesis (1960) announced (published as a sketch in 1961) a proof of the following:

THEOREM: The set of first order consequences of I-1, I-2, I-3 is not recursive (but r.e.)

The proof is incomplete but the **basic idea** can be completed.

In his PhD thesis 1962, he generalized the theorem to **projective incidence geometry** which consists of three projective versions of the incidence axioms and the axiom of Desargue.

The basic proof ideas

- We use various versions of interpretability is described in A. Tarski, A. Mostowski, R. Robinson from 1953 [TMR].

  However, the version of interpretability needed is **not spelled out** in [TMR] and is used wrongly in Rautenberg.

- We also use J. Robinson's undecidability of the first order theory of (infinite) fields from 1949 [JR].

- We have to use the fact that models of incidence geometry which satisfy the axiom of Pappus can be coordinatized.

  Again, in Rautenberg, the coordinatization is **claimed**, but **essential details are left out and/or overlooked**.

- Finally, we use that undecidability is preserved under subtheories when omitting finitely many axioms:

  If a theory T' is undecidable, and T' is obtained from a theory T over the same vocabulary by adding a finite set of axioms, then T is undecidable.

  Note that the converse is not true.

(Almost) complete proofs

- An **almost complete** proof is buried in the German monograph from 1983 by W. Schwabhaeuser based on notes by A. Tarski (1901/1983) and W. Szmielev (1918-1976).

  The interpretability argument still overlooks a crucial condition, which remains unverified in the coordinatization argument.

- The same oversight occurs in the English chapter in the Handbook of Spatial Logics, 2007, by P. Balbiani, V. Gorenko, R. Kellerman and D. Vakarelov: *Logical theories for fragments of elementary geometry*.

Translation schemes

Let τ be a vocabulary consisting of one binary relation symbol R, σ be a vocabulary consisting of one ternary relation symbol S. We want to interpret a σ structure on k-tuples of elements of a τ-structure.

- A $\tau - \sigma$-**translation scheme** $\Phi = (\phi, \phi_S)$ consists of a formula $\phi(\bar{x})$ with k free variables and a formula ϕ_S with $3k$ free variables. Φ is quantifier-free if all its translation formulas are quantifier-free.

- Let $\mathfrak{A} = \langle A, R^A \rangle$ be a τ-structure.
  We define a σ-structure $\Phi^*(\mathfrak{A}) = \langle B, S^B \rangle$ as follows:

  The universe is given by $B = \{\bar{a} \in A^k : \mathfrak{A} \models \phi(\bar{a}\}$
  and
  $S^B = \{\bar{b} \in A^{k \times 3} : \mathfrak{A} \models \phi_S(\bar{b}\}$ Φ^* is called a **transduction**.

- Let θ be a σ-formula. We define a τ-formula $\Phi^\sharp(\theta)$ inductively by substituting occurrences of $S(\bar{b}$ by their definition via ϕ_S where the free variables are suitable named. $\Phi^\sharp$ is called a **translation**.

The fundamental property of translation schemes

The fundamental property of

translation schemes,
transductions and
translations

is the following:

Let Φ be a $\tau - \sigma$-translation scheme, and θ be a σ-formulas.

$$\mathfrak{A} \models \Phi^{\sharp}(\theta) \text{ iff } \Phi^{*}(\mathfrak{A}) \models \theta$$

If θ has free variables, the assignment have to be chosen accordingly.

Furthermore, if Φ is quantifier-free, and θ is a universal formula, $\Phi^{\sharp}(\theta)$ is also universal.

Transfer of (un)decidability

Let Φ be a $\tau - \sigma$-translation scheme.

(i) Let $\mathfrak{A}$ be a τ-structure. If the complete first order theory T_0 of $\mathfrak{A}$ is decidable, so is the complete first order theory T_1 of $\Phi^*(\mathfrak{A})$.

(ii) There is a τ-structure $\mathfrak{A}$ such that the complete first order theory T_1 of $\Phi^*(\mathfrak{A})$ is decidable, but the complete first order theory T_0 of $\mathfrak{A}$ is undecidable.

(iii) If however, $\Phi^\sharp$ is **onto**, i.e., for every $\phi \in$ FOL(τ) there is a formula $\theta \in$ FOL(σ) with $\Phi^\sharp(\theta) = \phi$, then the converse of (i) also holds.

(iv) Let $T \subseteq$ FOLτ be a decidable theory and $T' \subseteq$ FOL(σ) and Φ^* be such that $\Phi^*|_{Mod(T)} : Mod(T) \to Mod(T')$ be onto. Then T' is decidable.

Interpretability, transductions and translations

- In the monograph by A. Tarski, A. Mostowski and R.M. Robinson from 1953 the method is defined and used.

- In Everet Beth's *The Foundation of Mathematics* of 1959, Chapter 21 the method is used to show various undecidability results, but not of geometry.

- In M. Rabin's 1965 paper the first complete account of the methods is given.

  Rabin liked to tell the story that Tarski abandoned editing a second edition of the above monograph because of this paper.

- The method is described correctly in the monographs by D. Monk, H.D.Ebbinghaus J. Flum and W. Thomas, and W.Hodges. It is often used imprecisely in papers appearing before 1965, and papers in Computer Science till today.

Coordinatization via planar ternary rings

First we look $\tau_{\in}$-structures, i.e., at models of the incidence relation alone.

Let Π be plane satisfying I-1, I-2, and I-3 with distinguished lines ℓ, m, d and points $O = (0,0)$ and $I = (1,0)$.

There are first order translation schemes RR_{ptr} and RF_{field} such that

(i) $RR^*_{ptr}(\Pi)$ is a planar ternary ring.

(ii) Π is a (infinite) Pappian plane iff $RF^*_{field}(\Pi)$ is a field (of characteristic 0).

To prove undecidability, the properties of the translation scheme RR_{field} above **are not enough**.

We still have to show that RR^*_{field} is **onto** as a transduction from Pappus planes to fields.

RF^*_{field} is onto‘

There is a first order translation scheme $PP_\in$ such that for every field $\mathcal{F}$ $PP^*_\in(\mathcal{F}_\Pi)$ is a Pappus plane.

$PP_\in$ is just the usual definition of geometry inside a field using Cartesian coordinates.

(i) If Π is a Pappus plane there is a field $\mathcal{F}_\Pi$ such that $PP^*_\in(\mathcal{F}_\Pi)$ is isomorphic to Π.

(ii) If additionally Π satisfies (Inf), $\mathcal{F}_\Pi$ is a field of characteristic 0.

In fact, $\mathcal{F}_\Pi$ can be chosen to be $RF^*_{field}(\Pi)$ from previous slide.

Conclusion:

RF^*_{field} is onto as a transduction from Pappus planes to fields.

Finally, Rautenberg's claim proved.

We now use:

(i) (J. Robinson, 1949)
The first order theory of (infinite) fields is undecidable.

(ii) The transduction RF^{*}_{field} is onto from (infinite) Pappian planes to fields (of characteristic 0).

(iii) The precise use of translation schemes.

(iv) The preservation of undecidability.

This gives:

- The first order theory of Pappian planes is undecidable.
- The first odder theory of incidence geometry is undecidable.

Models of geometry using incidence, betweenness, equidistance and equiangularity ($\tau_{Hilbert}$).

Hilbert Plane:
Axioms I-1, ..., I-3,
B-1, ... , B-4,
C-1, ..., C-6.

Euclidean Plane:
Hilbert Plane with
Parallel Axiom and Axiom E.

Hilbert planes and Euclidean planes

In Hilbert planes and Euclidean planes the coordinatization gives fields satisfying an additional finite set of field axioms.

These are the Pythagorean fields and the Euclidean ordered fields.

A field has the Pythagorean Property if square roots of sums of squares exist, i.e.

$$\forall z(\exists x, y(z = x^2 + y^2) \to \exists u(u^2 = z)$$

An ordered fields has the Euclidean Property if every positive element has a square root.

The proof sketched so far does not give undecidability for the first order theory of Hilbert planes and Euclidean planes.

Ziegler's Theorem

Theorem:(M. Ziegler, 1982)

Let T be a finite theory consistent with the theory of algebraically closed fields of characteristic 0 or with the theory of (real closed) fields, then T is undecidable.

Rautenberg and Hauschild - a Cold War Tale

1973: W. Rautenberg and K. Hauschild in East Berlin announce their result, that the theory of Pythagorean fields is undecidable.

1973 Rautenberg leaves East Berlin in an adventurous and illegal way to the West and visits Berkeley. Taking merit for the result he becomes Professor in West Berlin.

1974: The result is published in Fundamenta Mathematicae without Rautenberg's name in the paper (but it does appear on the top of even numbered pages).

1977: K. Hauschild publishes an Addendum to the paper in Fundamenta Mathematicae.

1979: H. Ficht in his M.Sc. thesis written under A. Prestel finds an irreparable mistake in the proof. M. Ziegler is a co-examiner.

1980: Martin Ziegler presents his alternative and more general proof.

Many undecidable geometries

Using Ziegler's Theorem we can prove undecidability of many geometrical theories.

In particular also

- Wu's orthogonal geometry
- Huzita's Origami geometry.

Conclusions

- Quantified first order properties of
  the **Real (Euclidean) Plane** are decidable.

- Quantified first order properties true in all
  Affine Planes
  (Hilbert, Euclidean, Orthogonal and Metric Wu Planes, Origami geometry)
  are undecidable.

- Universal statements true in
  all Geometries above are decidable.

  What about $\forall\exists$ statements?

Origami Geometry

Axiomatizing Origami Geometry, I

Humiaki Huzita (Fujita Fumiaki, 1924 26 March 2005)

was a Japanese-Italian mathematician and origami artist.

He was born in Japan, emigrating to Italy to study nuclear physics at the University of Padua. He is best known for formulating the first six HuzitaHatori axioms, which described the mathematics of paper folding to solve geometric construction problems.

Axiomatizing Origami Geometry, II

The axioms were first discovered by

Jacques Justin in 1986.

Résolution par le pliage de l'équation du troisième degré et applications géométriques
LOuvert - Journal de lAPMEP dAlsace et de lIREM de Strasbourg (in French). 42: 919

Axioms 1 through 6 were rediscovered by Japanese-Italian mathematician Humiaki Huzita and reported at the First International Conference on Origami in Education and Therapy in 1991.

Axioms 1 though 5 were rediscovered by Auckly and Cleveland in 1995.

Axiom 7 was rediscovered by Koshiro Hatori in 2001; Robert J. Lang also found axiom 7.

We follow in the sequel
Logical and algebraic view of Huzita's origami axioms
with applications to computational origami,
F. Ghourabi and T. Ida and H. Takahashi and M. Marin and A. Kasem,
Proceedings of the 2007 ACM symposium on Applied computing, 767-772 (2009).

Axiomatizing Origami Geometry, III

The resulting geometries of origami are **stronger**

than the geometries of ruler and compass.

Origami geometry allows one to construct

- angle trisection,
- doubling of the cube.
- solutions to third-degree equations.

Origami constructions, I

A line which is obtained by folding the paper is called a *fold*.

The original axioms and there expression as first order formulas in the vocabulary $\tau_{origami}$ are as follows:

(H-1): Given two points P_1 and P_2, there is a unique fold (line) that passes through both of them.

$$\forall P_1, P_2 \exists^{=1} l (P_1 \in l \wedge P_2 \in l)$$

(H-2): Given two points P_1 and P_2, there is a unique fold (line) that places P_1 onto P_2.

$$\forall P_1, P_2 \exists^{=1} l SymLine(P_1, l, P_2)$$

(H-3): Given two lines l_1 and l_2, there is a fold (line) that places l_1 onto l_2.

$$\forall l_1, l_2 \exists k \forall P \, (P \in k \rightarrow Peq(l_1, P, l_2))$$

Origami constructions, II

(H-4): Given a point P and a line l_1, there is a unique fold (line) perpendicular to l_1 that passes through point P.

$$\forall P, l \exists^{=1} k \forall P (P \in k \wedge Or(l, k))$$

(H-5): Given two points P_1 and P_2 and a line l_1, there is a fold (line) that places P_1 onto l_1 and passes through P_2.

$$\forall P_1, P_2 l_1 \exists l_2 \forall P (P_2 \in l_2 \wedge \exists P_2 (SymLine(P_1, l_2, P_2) \wedge P_2 \in l_1))$$

(H-6): Given two points P_1 and P_2 and two lines l_1 and l_2, there is a fold (line) that places P_1 onto l_1 and P_2 onto l_2.

$$\forall P_1, P_2 l_1, l_2 \exists l_3 \, ((\exists Q_1 SymLine(P_1, l_3, Q_1) \wedge Q_1 \in l_1) \wedge (\exists Q_2 SymLine(P_2, l_3, Q_2) \wedge Q_2 \in$$

(H-7): Given one point P and two lines l_1 and l_2, there is a fold (line) that places P onto l_1 and is perpendicular to l_2.

$$\forall P, l_2, l_2 \exists l_3 \, (Or(l_2, l_3) \wedge (\exists Q SymLine(P, l_3, Q) \wedge Q \in l_1))$$

Origami planes

Affine Origami plane: Let τ with $\tau_{origami} \subseteq \tau$ be a vocabulary of geometry. A τ-structure Π is an *affine Origami plane* if it satisfies (I-1, I-2, I-3), the axiom of infinity (Inf), (ParAx) and the Huzita-Hatori axioms (H-1) - (H-7).
We denote the set of these axioms by $T_{a-origami}$

Proposition: The relations $SymLine$ and Peq are first order definable using Eq and Or with existential formulas over $\tau_{f-field}$: Hence the axioms (H-1)-(H-7) are first order definable in FOL(τ_{wu}).

Theorem: The first order theory of Affine Origami planes is undecidable.

Thank you

- for the invitation, and
- for your interest and attention.