MODAL LOGIC MEETS SIMPLICIAL SETS

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Simplicial semantics for modal and superintuitionistic predicate logics was introduced by Dmitry Skvortsov in the early 1990s as a "maximal" Kripke-type semantics. The proofs of basic results on this semantics — soundness and "maximality" theorems (sketchy) and completeness theorem (for a certain class of logics) were given in [1]. The book [2] (Chapter 5) contains a detailed proof of soundness for metaframe semantics, which is a particular case of simplicial semantics*. In this note we state some further results on completeness and incompleteness.

Let us briefly recall the main definitions.

Modal (predicate) formulas are constructed from predicate letters P_k^n (countably many for each arity $n \geq 0$), a countable set of individual variables, classical propositional connectives, quantifiers, and the modal operator $\Box$. Individual constants, function letters and equality are not used.

A *modal predicate logic* is a set of modal predicate formulas containing classical predicate axioms, the axiom of $\mathbf{K}$ and closed under Modus Ponens, Generalization, $\Box$-introduction, and predicate substitutions. $\mathbf{Q\Lambda}$ denotes the smallest predicate logic containing the propositional logic $\mathbf{\Lambda}$.

A *predicate Kripke frame* over a propositional frame $F = (W, R)$ is a pair (F, D), in which $D = (D_u)_{u \in W}$, $D_u \neq \varnothing$, and $D_u \subseteq D_v$ whenever uRv. The notion of validity for Kripke frames is standard and well-known. By soundness theorem, the set of formulas valid in a certain class of frames is always a modal predicate logic. Logics of this form are called *Kripke complete*.

Many modal predicate logics are known to be Kripke incomplete, so other Kripke-type semantics were proposed, in particular, Kripke sheaf semantics, Ghilardi's functor semantics, metaframe semantics, and simplicial semantics — the strongest of them. It is defined as follows.

Let $I_n = \{1, \dots, n\}$, $I_0 = \varnothing$, and let Σ_{mn} be the set of all maps from I_m to I_n (Σ_{0n} consists of a single map $\varnothing_n$, and $\Sigma_{m0} = \varnothing$ for $m > 0$). Also let $\Sigma = \bigcup_{m \geq 0,\, n > 0} \Sigma_{mn}$. There are specific maps:

$\delta_i^n \in \Sigma_{n-1,n}$ sends $1, \dots, n-1$ respectively to $1, \dots, i-1, i+1, \dots, n$;

*The name 'simplicial semantics' was introduced in [2] as an allusion to algebraic topology. 'Simplicial frames' from [2] correspond to 'metaframes' from [1], and 'metaframes' from [2] correspond to 'Cartesian metaframes' from [1].

$\sigma^+ \in \Sigma_{m+1,n+1}$ prolongs $\sigma \in \Sigma_{mn}$ with $\sigma^+(m+1) = n+1$.

A *simplicial frame* based on a propositional Kripke frame F is a tuple $\mathbb{F} = (F, \overrightarrow{D}, \overrightarrow{R}, \pi)$, where $\overrightarrow{D} = (D^n)_{n\geq 0}$ is a family of (non-empty) sets, $\overrightarrow{R} = (R^n)_{n\geq 0}$ is a family of relations $R^n \subseteq D^n \times D^n$, with $F = \left(D^0, R^0\right)$; $\pi = (\pi_\sigma)_{\sigma\in\Sigma}$ is a family of maps $\pi_\sigma : D^n \longrightarrow D^m$ for $\sigma \in \Sigma_{mn}$.

If D^n is the Cartesian power of a set $D\,(= D^1)$, and $\pi_\sigma(a_1, \ldots, a_n) = (a_1, \ldots, a_n) \cdot \sigma := (a_{\sigma(1)}, \ldots, a_{\sigma(m)})$ for $n > 0$, $\mathbb{F}$ is called a *metaframe*.

Remark. *Simplicial sets* are mathematical structures closely related to simplicial frames. By definition [3], a simplicial set consists of non-empty sets $(D^n)_{n\geq 1}$ and maps $\pi_\sigma : D^n \longrightarrow D^m$ corresponding to $\sigma : I_m \longrightarrow I_n$ that are monotonic w.r.t. $\leq$. π_σ should also preserve composition and identity as in sound simplicial frames (cf. Theorem 1 below). So a sound simplicial frame (without level 0) can be regarded as a simplicial set with extra maps π_σ corresponding to permutations $I_n \longrightarrow I_n$ and with extra relations R_n.

A *valuation* in a simplicial frame $\mathbb{F}$ is a function ξ sending every predicate letter P^n_k to a subset $\xi(P^n_k) \subseteq D^n$. An *assignment* of length n in $\mathbb{F}$ is a pair $(\mathbf{x}, \mathbf{a})$, where $\mathbf{a} \in D^n$, $\mathbf{x}$ is a list of different variables of length n. For a formula A, an assignment $(\mathbf{x}, \mathbf{a})$ involving all its parameters and a model $M = (\mathbb{F}, \xi)$ the *truth relation* $M, \mathbf{a}/\mathbf{x} \vDash A$ is defined by induction, in particular

- $M, \mathbf{a}/\mathbf{x} \vDash P^m_k(\mathbf{x}\cdot\sigma)$ iff $\pi_\sigma \mathbf{a} \in \xi(P^m_k)$ (for $\sigma \in \Sigma_{mn}$);
- $M, \mathbf{a}/\mathbf{x} \vDash \Box B$ iff $\forall \mathbf{b} \in R^n(\mathbf{a})\ M, \mathbf{b}/\mathbf{x} \vDash B$;
- $M, \mathbf{a}/\mathbf{x} \vDash \exists y B$ iff $\exists \mathbf{c} \in D^{n+1} \left(\pi_{\delta^{n+1}_{n+1}} \mathbf{c} = \mathbf{a}\ \&\ M, \mathbf{c}/\mathbf{x}y \vDash B\right)$, where y does not occur in $\mathbf{x}$;
- $M, \mathbf{a}/\mathbf{x} \vDash \exists x_i B$ iff $M, \pi_{\delta^n_i} \mathbf{a}/(\mathbf{x}\cdot\delta^n_i) \vDash \exists x_i B$.

A formula is called *valid* in a simplicial frame if it is true under every valuation and variable assignment (for its parameters); a formula is *strongly valid* if all its substitution instances are valid.

Theorem 1 ([1]). *Let $\mathbb{F} = (F, \overrightarrow{D}, \overrightarrow{R}, \pi)$ be a simplicial frame such that:*

- *$\pi_{\varnothing_1}$ is surjective;*
- *every π_σ for $\sigma \in \Sigma_{mn}$ is a p-morphism from (D^n, R^n) to (D^m, R^m), i.e., $\pi_\sigma(R^n(\mathbf{a})) = R^m(\pi_\sigma(\mathbf{a}))$ for every $\mathbf{a}$;*
- *π preserves composition and sends identity maps to identity maps;*
- *if $\pi_{\delta^{m+1}_{m+1}}(\mathbf{b}) = \pi_\sigma(\mathbf{a})$, $\sigma \in \Sigma_{mn}$, then there exists $\mathbf{c} \in D^{n+1}$ such that $\pi_{\sigma^+}(\mathbf{c}) = \mathbf{b}$, $\pi_{\delta^{n+1}_{n+1}}(\mathbf{c}) = \mathbf{a}$.*

Then the set of formulas strongly valid in $\mathbb{F}$ is a modal predicate logic.

A simplicial frame satisfying these conditions is called *sound*. Note that metaframes always satisfy the fourth condition.

A modal logic of some class of sound simplicial frames (respectively, metaframes) is called *simplicially complete* (respectively, *metaframe complete*).

Theorem 2 ([1]). *If $\mathbf{\Lambda}$ is a canonical (d-persistent) propositional modal logic, then $\mathbf{Q\Lambda}$ is simplicially complete.*

Now consider the propositional modal logics

$$\begin{aligned}\mathbf{D4.1} &:= \mathbf{K} + \Box p \to \Box\Box p + \Diamond\top + \Box\Diamond p \to \Diamond\Box p,\\ \mathbf{S4.1} &:= \mathbf{K} + \Box p \to \Box\Box p + \Box p \to p + \Box\Diamond p \to \Diamond\Box p,\\ \mathbf{SL4} &:= \mathbf{K} + \Box p \leftrightarrow \Diamond p + \Box p \to \Box\Box p.\end{aligned}$$

Theorem 3. *Let $\mathbf{\Lambda}$ be a propositional modal logic between* **D4.1** *and* **SL4**. *Then $\mathbf{Q\Lambda}$ is metaframe incomplete.*

The crucial formula for the proof is

$$\Box\Diamond\forall x\forall y(\Box\Diamond P(x,y) \to \exists x'\exists y'(P(x',y') \wedge \Diamond P(x,y'))).$$

It is strongly valid in sound metaframes strongly validating **QD4.1**. However, it is not provable in **QSL4**, because it can be refuted in a simplicial frame strongly validating **QSL4**.

Corollary 4. *The logics* **QD4.1**, **QS4.1**, **QSL4** *are simplicially complete, but metaframe incomplete.*

In fact, completeness follows from Theorem 2 and incompleteness from Theorem 3.

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References

[1] D. P. Skvortsov and V. B. Shehtman. Maximal Kripke-type semantics for modal and superintuitionistic predicate logics. *Annals of Pure and Applied Logic* 63(1):69–101, 1993. DOI: 10.1016/0168-0072(93)90210-5

[2] D. M. Gabbay, V. Shehtman and D. Skvortsov. *Quantification in Nonclassical Logic. Volume 1*, Studies in Logic and the Foundations of Mathematics 153. Elsevier, 2009.

[3] S. I. Gelfand and Yu. I. Manin. *Methods of Homological Algebra*. Springer, 2003. DOI: 10.1007/978-3-662-12492-5

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