Connexive Conditional Logic

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Connexive logics are contra-classical logics that validate the so-called Aristotle's Theses and Boethius' Theses:

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\begin{array}{ll}
\text{(AT)} & {\sim}({\sim}A \to A), \\
\text{(AT)}' & {\sim}(A \to {\sim}A), \\
\text{(BT)} & (A \to B) \to {\sim}(A \to {\sim}B), \\
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- Avoid triviality
- Avoid inconsistency

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Richard Routley (Sylvan), 1970s: Dialectical logics

“Dialectical logics are formally viable, and are coming to be part of the logical scene: they cannot simply be ruled out of court as not ‘logics’. Nor can they be dismissed on the grounds that they fail to discriminate valid arguments since an inconsistency entails everything. For such paradoxes of deducibility any worthwhile dialectical logic would repudiate.” (*Relevance logics and their rivals*, 1982)

A straightforward and conceptually clear road to obtaining systems of connexive logic consists of requiring suitable falsity conditions for implications.

This approach is particularly natural for expansions of first-degree entailment logic, **FDE**, because **FDE** is a basic and simple four-valued logic which clearly separates truth and falsity from each other as two independent semantical dimensions.

The connexive logic **C** from (Wansing 2005) imposes such falsity conditions on the constructive implication in a certain expansion of **FDE**, namely David Nelson's paraconsistent logic **N4**.

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We introduce connexive expansions of **FDE** starting from the perspective of *conditional logic* where the conditional is usually denoted by "$\Box\!\!\rightarrow$".

The point of departure is the logic **CK** introduced by B. Chellas (1975) as a basic system of conditional logic. The semantics in (Chellas 1975) employs what are now called "Chellas frames".

In addition to a non-empty set W of states, a Chellas frame contains a ternary relation $R \subseteq W \times W \times Pow(W)$.

In a model based on a Chellas frame, the relation R can be seen as comprising a collection of binary accessibility relations $R_{[\![A]\!]}$ on W, indexed by the truth set $[\![A]\!]$ of A in that model, for every formula A.

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K. Segerberg (1989) considered general Chellas frames. To obtain correspondences between schematic formulas and conditions on a kind of *frames* instead of models, we consider a suitable modification of Segerberg frames (i.e., a modification of general Chellas frames). A modified Segerberg frame is a triple $\langle W, R, P \rangle$, where $\langle W, R \rangle$ is a Chellas frame and $P \subseteq Pow(W) \times Pow(W)$.

The set P may be seen as a set of allowable (or admissible) extension/anti-extension pairs.

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$$\begin{array}{l} \textbf{FDE} \searrow \\ \qquad\qquad \textbf{CK}_{\textbf{FDE}} \longrightarrow \textbf{CCK} \longrightarrow \textbf{CCK}^{+} \\ \textbf{CK} \nearrow \end{array}$$

The language $\mathfrak{L}$ makes use of a denumerable set PV of propositional variables and the connectives $\sim$ (unary) and $\wedge$, $\vee$, and $\Box\!\!\rightarrow$ (binary).

Definition

A pair $\langle W, R\rangle$ is a Chellas frame (or just a frame) iff

- *W is a non-empty set, intuitively understood as a set of information states, and*
- *$R \subseteq W \times W \times Pow(W)$, where $Pow(W)$ is the power set of W.*

If $\langle W, R\rangle$ is a frame, then $\mathfrak{M} = \langle W, R, v^+, v^-\rangle$ is a model iff v^+ and v^- are valuation functions $v^+\colon \mathrm{PV} \longrightarrow Pow(W)$ and $v^-\colon \mathrm{PV} \longrightarrow Pow(W)$.

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Definition

A model $\mathfrak{M} = \langle W, R, v^+, v^- \rangle$ *is a model for* **CCK** *iff support of truth and support of falsity relations* $\models^+$ *and* $\models^-$ *between* $\mathfrak{M}$, *a state* $w \in W$, *and a formula from* $\mathfrak{L}$ *are inductively defined as follows:*

$$
\begin{array}{lll}
\mathfrak{M}, w \models^+ p & \text{iff} & w \in v^+(p) \text{ for } p \in \mathrm{PV} \\
\mathfrak{M}, w \models^- p & \text{iff} & w \in v^-(p) \text{ for } p \in \mathrm{PV} \\
\mathfrak{M}, w \models^+ {\sim}A & \text{iff} & \mathfrak{M}, w \models^- A \\
\mathfrak{M}, w \models^- {\sim}A & \text{iff} & \mathfrak{M}, w \models^+ A \\
\mathfrak{M}, w \models^+ A \wedge B & \text{iff} & \mathfrak{M}, w \models^+ A \text{ and } \mathfrak{M}, w \models^+ B \\
\mathfrak{M}, w \models^- A \wedge B & \text{iff} & \mathfrak{M}, w \models^- A \text{ or } \mathfrak{M}, w \models^- B \\
\mathfrak{M}, w \models^+ A \vee B & \text{iff} & \mathfrak{M}, w \models^+ A \text{ or } \mathfrak{M}, w \models^+ B \\
\mathfrak{M}, w \models^- A \vee B & \text{iff} & \mathfrak{M}, w \models^- A \text{ and } \mathfrak{M}, w \models^- B \\
\mathfrak{M}, w \models^+ A \mathrel{\Box\!\!\rightarrow} B & \text{iff} & (\forall w' \in W)\ wR_{[\![A]\!]^{\mathfrak{M}}}w' \Rightarrow \mathfrak{M}, w' \models^+ B \\
\mathfrak{M}, w \models^- A \mathrel{\Box\!\!\rightarrow} B & \text{iff} & (\forall w' \in W)\ wR_{[\![A]\!]^{\mathfrak{M}}}w' \Rightarrow \mathfrak{M}, w' \models^- B
\end{array}
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where the set $[\![A]\!]^{\mathfrak{M}}$ *is defined as* $\{w \mid \mathfrak{M}, w \models^+ A\}$. *When the context is clear we sometimes write* $[\![A]\!]$ *instead of* $[\![A]\!]^{\mathfrak{M}}$.

The support of truth and support of falsity conditions for formulas $A \boxright B$ privilege the set $\{w \mid \mathfrak{M}, w \models^{+} A\}$ over the set $\{w \mid \mathfrak{M}, w \models^{-} A\}$, i.e. over $[\![\sim A]\!]$. We may think of $\{w \mid \mathfrak{M}, w \models^{+} A\}$ as the extension of A in $\mathfrak{M}$ and of $\{w \mid \mathfrak{M}, w \models^{-} A\}$ as the anti-extension of A in $\mathfrak{M}$.

If one wants to simultaneously impose conditions on the relation R related to both extensions and anti-extensions, one would have to use a four-place relation $R \subseteq W \times W \times Pow(W) \times Pow(W)$.

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Definition

A triple $\langle W, R, P \rangle$ is a general frame (or Segerberg frame) for **CCK** *iff*

- $\langle W, R \rangle$ *is a frame,*
- $R \subseteq W \times W \times I(P)$, *where* $I(P) = \{X \mid \langle X, Y \rangle \in P\}$ *and*
- $P \subseteq (Pow(W) \times Pow(W))$, *where P satisfies the following conditions:*
  1. *if* $\langle X, Y \rangle \in P$, *then* $\langle Y, X \rangle \in P$
  2. *if* $\langle X, Y \rangle, \langle X', Y' \rangle \in P$, *then* $\langle X \cap X', Y \cup Y' \rangle \in P, \langle X \cup X', Y \cap Y' \rangle \in P$
  3. *if* $\langle X, Y \rangle, \langle X', Y' \rangle \in P$, *then* $\langle \{w \in W \mid \forall w' \in W(wR_X w' \Rightarrow w' \in X')\}, \{w \in W \mid \forall w' \in W(wR_X w' \Rightarrow w' \in Y')\} \in P$.

The set P is a set of pairs of sets of states; intuitively P contains the admissible extension/anti-extension pairs.

Definition

Let $\langle W, R, P\rangle$ be a general frame for **CCK**. The tuple $\langle W, R, P, v^+, v^-\rangle$ is a general model for **CCK** iff $\langle W, R, v^+, v^-\rangle$ is a model and $\langle [\![p]\!], [\![\sim p]\!]\rangle \in P$ for every $p \in \mathrm{PV}$. Support of truth and support of falsity relations $\models^+$ and $\models^-$ are defined as in the case of models for **CCK**.

Lemma

*Let $\langle W, R, P, v^+, v^-\rangle$ be a general model for **CCK**. Then for every $\mathfrak{L}$-formula A, $\langle [\![A]\!], [\![\sim A]\!]\rangle \in P$.*

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Lemma

Let $\langle W, R, P, v^{+}, v^{-}\rangle$ be a general model for **CCK**. *Then for every $\mathfrak{L}$-formula A, $\langle [\![A]\!], [\![{\sim}A]\!]\rangle \in P$.*

Definition

We say that a formula A is valid in a model $\mathfrak{M} = \langle W, R, v^+, v^- \rangle$ or a general model $\mathfrak{M} = \langle W, R, P, v^+, v^- \rangle$ iff $\mathfrak{M}, w \models^+ A$ for all $w \in W$. If A is valid in $\mathfrak{M}$, we write $\mathfrak{M} \models A$. We say that A is valid on a frame $\mathfrak{F} = \langle W, R \rangle$ or a general frame $\mathfrak{F} = \langle W, R, P \rangle$, and write $\mathfrak{F} \models A$, iff $\mathfrak{M} \models A$ for all models $\mathfrak{M} = \langle W, R, v^+, v^- \rangle$, respectively general models $\mathfrak{M} = \langle W, R, P, v^+, v^- \rangle$, based on $\mathfrak{F}$. A formula is valid with respect to a class of (general) models or (general) frames iff it is valid in every (general) model, respectively on every (general) frame from that class.

Definition

We semantically define two basic connexive conditional logics.

The logic **CCK** is the set of all $\mathfrak{L}$-formulas valid with respect to the class of all models for **CCK**.

The logic **CCK**$^+$ is the set of all $\mathfrak{L}$-formulas valid with respect to the class of all models for **CCK** and satisfying the following condition on frames:

$$\mathbb{C}_{A\Box\!\!\rightarrow A} \qquad (\forall X \subseteq W)(\forall w, w' \in W) wR_X w' \Rightarrow w' \in X.$$

If a model, general frame, or general model for **CCK** satisfies $\mathbb{C}_{A\Box\!\!\rightarrow A}$ it will be called a model, general frame, or general model, respectively, for **CCK**$^+$.

Definition

Let $\Gamma \cup \{A\}$ be a set of $\mathfrak{L}$-formulas, and let **L** be a logic. We say that Γ entails A in **L** ($\Gamma \models_{\mathbf{L}} A$) iff for every model $\mathfrak{M}$ for **L** and state w of $\mathfrak{M}$ it holds that if $\mathfrak{M}, w \models^+ B$ for every $B \in \Gamma$, then $\mathfrak{M}, w \models^+ A$.

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Definition

A formula A C-corresponds to a frame condition $\mathbb{C}$ (and vice versa) iff for all frames $\mathfrak{F}$ it holds that $\mathfrak{F}$ satisfies $\mathbb{C}$ iff $\mathfrak{F} \models A$. A formula A S-corresponds to a frame condition $\mathbb{C}$ (and vice versa) iff for all general frames $\mathfrak{F}$ it holds that $\mathfrak{F}$ satisfies $\mathbb{C}$ iff $\mathfrak{F} \models A$.

Definition

Let $A, B_1, \ldots, B_n$ be $\mathfrak{L}$-formulas. A derivability statement $\{B_1, \ldots, B_n\} \vdash A$ C-corresponds to a frame condition $\mathbb{C}$ (and vice versa) iff for all frames $\mathfrak{F}$ it holds that $\mathfrak{F}$ satisfies $\mathbb{C}$ iff (for every model $\mathfrak{M}$ based on $\mathfrak{F}$ and every state w of $\mathfrak{M}$, if $\mathfrak{M}, w \models^+ B_1, \ldots, \mathfrak{M}, w \models^+ B_n$, then $\mathfrak{M}, w \models^+ A$). A statement $\{B_1, \ldots, B_n\} \vdash A$ S-corresponds to a frame condition $\mathbb{C}$ (and vice versa) iff for all general frames $\mathfrak{F}$ it holds that $\mathfrak{F}$ satisfies $\mathbb{C}$ iff (for every general model $\mathfrak{M}$ based on $\mathfrak{F}$ and every state w of $\mathfrak{M}$, if $\mathfrak{M}, w \models^+ B_1, \ldots, \mathfrak{M}, w \models^+ B_n$, then $\mathfrak{M}, w \models^+ A$).

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In expansions of **CK**, with classical logic as its base logic, the formula $\sim A \wedge A \mathrel{\Box\!\!\rightarrow} \sim A$, for example, corresponds to the frame condition $(\forall X \subseteq W)(\forall w, w' \in W)(xR_{\overline{X} \cap X}\, w' \Rightarrow w' \in \overline{X})$.

In the semantics for **CCK**, the evaluation of a propositional variable $\sim p$ is independent of the evaluation of p, and it is not clear how to capture $\sim A \wedge A \mathrel{\Box\!\!\rightarrow} \sim A$ by a condition that does not make use of the binary predicate P.

The following general frame condition, however, corresponds to $\sim A \wedge A \mathrel{\Box\!\!\rightarrow} \sim A$:

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Similarly, $\sim A \wedge B \,\Box\!\!\rightarrow \sim A$ corresponds to

$$(\forall X, Y, X', Y' \subseteq W)(\forall w, w' \in W)(\langle X, Y\rangle \in P \Rightarrow (\langle X', Y'\rangle \in P \Rightarrow (wR_{Y \cap X'} w' \Rightarrow w' \in Y))).$$

Remark

The four-place accessibility relation introduced earlier brings us back to frame correspondence instead of general frame correspondence. The formulas $\sim A \wedge A \,\Box\!\!\rightarrow \sim A$ *and* $\sim A \wedge B \,\Box\!\!\rightarrow \sim A$*, for example, correspond to*

$$(\forall X, Y \subseteq W)(\forall w, w' \in W)(wR_{\langle Y \cap X, X \cup Y\rangle} w' \Rightarrow w' \in Y)$$

and

$$(\forall X, Y, X', Y' \subseteq W)(\forall w, w' \in W)(\langle X', Y'\rangle \in R \Rightarrow (wR_{\langle Y \cap X', X \cup Y'\rangle} w' \Rightarrow w' \in Y)),$$

respectively. The latter condition is non-trivial; all set variables involved do occur at argument places of 'R'.

Similarly, $\sim A \wedge B \boxright \sim A$ corresponds to

$$(\forall X, Y, X', Y' \subseteq W)(\forall w, w' \in W)(\langle X, Y\rangle \in P \Rightarrow (\langle X', Y'\rangle \in P \Rightarrow (wR_{Y\cap X'}w' \Rightarrow w' \in Y))).$$

Remark

The four-place accessibility relation introduced earlier brings us back to frame correspondence instead of general frame correspondence. The formulas $\sim A \wedge A \boxright \sim A$ and $\sim A \wedge B \boxright \sim A$, for example, correspond to

$$(\forall X, Y \subseteq W)(\forall w, w' \in W)(wR_{\langle Y\cap X, X\cup Y\rangle}w' \Rightarrow w' \in Y)$$

and

$$(\forall X, Y, X', Y' \subseteq W)(\forall w, w' \in W)(\langle X', Y'\rangle \in R \Rightarrow (wR_{\langle Y\cap X', X\cup Y'\rangle}w' \Rightarrow w' \in Y)),$$

respectively. The latter condition is non-trivial; all set variables involved do occur at argument places of 'R'.

We extend the tableau calculus for **FDE** presented in (Priest 2008) by tableau rules for $\boxminus\!\!\rightarrow$ that suitably modify the rules for $\boxminus\!\!\rightarrow$ there, in order to obtain a tableau calculus for **CCK** in the language $\mathfrak{L}$. We will assume some familiarity with the tableau method as applied by Priest.

In tableaux for **CCK** tableau nodes are of the form $A, +i$, or $A, -i$, or $ir_A j$, where A is an $\mathfrak{L}$-formula, i and j are natural numbers representing information states, $+$ indicates support of truth ($\models^+$), $-$ indicates failure of support of truth ($\not\models^+$), and r_A represents the accessibility relation $R_{[\![A]\!]}$ in the countermodel one tries to construct in unfolding a tableau.

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Tableaux for a single conclusion derivability statement $\Delta \vdash B$ start with nodes of the form $A, +0$ for every premise A from the finite premise set Δ and a node of the form $B, -0$. Then tableau rules are applied (if that is possible) to tableau nodes leading to a more complex tableau.

A branch of the tableau closes iff it contains a pair of nodes $A, +i$ and $A, -i$. The tableau closes iff all of its branches close. If a tableau (tableau branch) is not closed, it is called open. A tableau branch is said to be complete iff no more rules can be applied to expand it. A tableau is said to be complete iff each of its branches is complete. Closed branches are marked by '$\times$'.

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The tableau rules for the connectives of **FDE** can be stated as follows:

$$
\begin{array}{c} A \wedge B, +i \\ \downarrow \\ A, +i \\ B, +i \end{array}
\qquad
\begin{array}{c} A \wedge B, -i \\ \swarrow \quad \searrow \\ A, -i \quad B, -i \end{array}
\qquad
\begin{array}{c} A \vee B, +i \\ \swarrow \quad \searrow \\ A, +i \quad B, +i \end{array}
\qquad
\begin{array}{c} A \vee B, -i \\ \downarrow \\ A, -i \\ B, -i \end{array}
$$

$$
\begin{array}{c} {\sim}{\sim}A, \underline{+}i \\ \downarrow \\ A, \underline{+}i \end{array}
\qquad
\begin{array}{c} {\sim}(A \wedge B), \underline{+}i \\ \downarrow \\ {\sim}A \vee {\sim}B, \underline{+}i \end{array}
\qquad
\begin{array}{c} {\sim}(A \vee B), \underline{+}i \\ \downarrow \\ {\sim}A \wedge {\sim}B, \underline{+}i \end{array}
$$

where he symbol $\underline{+}$ is to be read uniformly either as $+$ or as $-$.

The tableau rules for the conditional $\Box\!\!\rightarrow$ in **CCK** then are as follows:

$$
\begin{array}{c}
A \Box\!\!\rightarrow B, +i \\
i r_A j \\
\downarrow \\
B, +j
\end{array}
\qquad
\begin{array}{c}
{\sim}(A \Box\!\!\rightarrow B), \pm i \\
\downarrow \\
A \Box\!\!\rightarrow {\sim}B, \pm i
\end{array}
\qquad
\begin{array}{c}
A \Box\!\!\rightarrow B, -i \\
\downarrow \\
i r_A j \\
B, -j
\end{array}
$$

The left rule is applied whenever a line $i r_A j$ occurs on the branch; the right rule requires the introduction of a new natural number j not already occurring in the tableau.

We need one further rule, ($\mathit{reg}\ \Box\!\!\rightarrow$):

$$
\begin{array}{c}
A \mathrel{\Box\!\!\rightarrow} C, +i \\
B \mathrel{\Box\!\!\rightarrow} C, -i \\
\swarrow \qquad \searrow \\
\begin{array}{cc}
A, +k & A, -j \\
B, -k & B, +j
\end{array}
\end{array}
$$

where k and j are indices new to the tableau.

The above set of tableau rules for the connectives from $\mathfrak{L}$ constitutes a tableau calculus T**CCK** for **CCK**.

We need one further rule, (*reg* $\Box\!\!\rightarrow$):

$$\begin{array}{c} A \Box\!\!\rightarrow C, +i \\ B \Box\!\!\rightarrow C, -i \\ \swarrow \quad \searrow \\ \begin{array}{cc} A, +k & A, -j \\ B, -k & B, +j \end{array} \end{array}$$

where k and j are indices new to the tableau.

The above set of tableau rules for the connectives from $\mathfrak{L}$ constitutes a tableau calculus *T***CCK** for **CCK**.

The rule ($\mathit{reg}\ \Box\!\!\rightarrow$) is needed because the Left Logical Equivalence (LLE) rule, saying that if A and B are provably equivalent, then $B \Box\!\!\rightarrow C$ can be derived from $A \Box\!\!\rightarrow C$, is validated by our semantics.

As an example of a tableau proof, we present a proof of a very simple instance of LLE. We show that $(p \wedge q) \Box\!\!\rightarrow r \vdash_{T\mathbf{CCK}_{\mathbf{FDE}}} (q \wedge p) \Box\!\!\rightarrow r$:

$$(p \wedge q) \Box\!\!\rightarrow r, +0$$
$$(q \wedge p) \Box\!\!\rightarrow r, -0$$

Left branch ($\swarrow$):

$$(p \wedge q), +0$$
$$(q \wedge p), -0$$
$$p, +0$$
$$q, +0$$

which splits into $q, -0$ ($\times$) and $p, -0$ ($\times$).

Right branch ($\searrow$):

$$(p \wedge q), -0$$
$$(q \wedge p), +0$$
$$q, +0$$
$$p, +0$$

which splits into $p, -0$ ($\times$) and $q, -0$ ($\times$).

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As an example of a tableau proof, we present a proof of a very simple instance of LLE. We show that $(p \wedge q) \Box\!\!\rightarrow r \vdash_{T\mathbf{CCK_{FDE}}} (q \wedge p) \Box\!\!\rightarrow r$:

$$
\begin{array}{c}
(p \wedge q) \Box\!\!\rightarrow r, +0 \\
(q \wedge p) \Box\!\!\rightarrow r, -0 \\
\swarrow \qquad\qquad \searrow \\
\begin{array}{cc}
\begin{array}{c}
(p \wedge q), +0 \\
(q \wedge p), -0 \\
p, +0 \\
q, +0 \\
\swarrow \quad \searrow \\
\begin{array}{cc} q, -0 & p, -0 \\ \times & \times \end{array}
\end{array}
&
\begin{array}{c}
(p \wedge q), -0 \\
(q \wedge p), +0 \\
q, +0 \\
p, +0 \\
\swarrow \quad \searrow \\
\begin{array}{cc} p, -0 & q, -0 \\ \times & \times \end{array}
\end{array}
\end{array}
\end{array}
$$

To obtain the tableau calculus $T\mathbf{CCK}^{+}$ for $\mathbf{CCK}^{+}$, we add the following tableau rule to $T\mathbf{CCK}$:

$$R_{A\Box\!\!\rightarrow A} \qquad \begin{array}{c} ir_A j \\ \downarrow \\ A, +j \end{array}$$

Definition

Let **L** *be a logic and let* $T\mathbf{L}$ *be a tableau calculus for* **L**. *If* $\Delta = \{B_1, \ldots, B_n, A\}$ *is a finite set of* $\mathfrak{L}$*-formulas, then* A *is derivable from* Δ *in* $T\mathbf{L}$ *(in symbols:* $\Delta \vdash_{T\mathbf{L}} A$*), iff there exists a closed and complete tableau for* $B_1, +0$, …, $B_n, +0$, $A, -0$ *in* $T\mathbf{L}$.

Theorem

Let $\Delta \cup \{A\}$ be a finite set of $\mathfrak{L}$-formulas. Then $\Delta \models_{\mathbf{CCK}} A$ iff $\Delta \vdash_{T\mathbf{CCK}} A$.

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To prove the latter theorem, we need general frames.

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To prove the latter theorem, we need general frames.

Definition

Let br be a complete open tableau branch. Then the structure $\langle W_{br}, R_{br}, v^{+}_{br}, v^{-}_{br}\rangle = \mathfrak{M}_{br}$ induced by br is defined as follows:

1. $W_{br} := \{w_j \mid j \text{ occurs on } br\}$,
2. $w_j R_{brX} w_k$ iff there is an A with $X = [\![A]\!]$ and $jr_A k$ as well as some expression of the form $(B \mathrel{\Box\!\!\rightarrow} C), \pm j$ or $(B \mathrel{\Diamond\!\!\rightarrow} C), \pm j$ occurs on br (for $X \subseteq W_{br}$, $w_j, w_k \in W_{br}$),
3. $w_j \in v^{+}_{br}(p)$ iff $p, +j$ occurs on br,
4. $w_j \in v^{-}_{br}(p)$ iff $\sim p, +j$ occurs on br.

One can show that $\mathfrak{M}_{br}$ is a model (induced by br).

Definition

Let $\mathfrak{M}_{br} = \langle W_{br}, R_{br}, v^{+}_{br}, v^{-}_{br}\rangle$ be a model for **CCK**. Then the structure $\mathfrak{M}_{br} = \langle W_{br}, R_{br}, P_{br}, v^{+}_{br}, v^{-}_{br}\rangle$ is defined by requiring that P_{br} is the *smallest* subset of $(Pow(W) \times Pow(W))$ such that $\langle [\![q]\!], [\![{\sim}q]\!]\rangle \in P_{br}$ for every $q \in \mathrm{PV}$ and such that P_{br} satisfies the following conditions:

1. if $\langle X, Y\rangle \in P_{br}$, then $\langle Y, X\rangle \in P_{br}$
2. if $\langle X, Y\rangle, \langle X', Y'\rangle \in P_{br}$, then $\langle X \cap X', Y \cup Y'\rangle \in P_{br}, \langle X \cup X', Y \cap Y'\rangle \in P_{br}$
3. if $\langle X, Y\rangle, \langle X', Y'\rangle \in P_{br}$, then $\langle\{w \in W \mid \forall w' \in W(wR_X w' \Rightarrow w' \in X')\}, \{w \in W \mid \forall w' \in W(wR_X w' \Rightarrow w' \in Y')\} \in P_{br}$.

The structure $\mathfrak{M}_{br}$ is a general model for **CCK**; we call it the general model induced by *br*.

Lemma

*(Completeness lemma) Suppose that br is a complete open tableau branch of a tableau in T***CCK***, and let* $\langle W_{br}, R_{br}, P_{br}, v^{+}_{br}, v^{-}_{br}\rangle = \mathfrak{M}_{br}$ *be the general model induced by br. If* $\mathfrak{M}_{br}$ *is a general model for* **CCK***, then*

- *If* $A, +i$ *occurs on br, then* $\mathfrak{M}_{br}, w_i \models^{+} A$;
- *If* $A, -i$ *occurs on br, then* $\mathfrak{M}_{br}, w_i \not\models^{+} A$;
- *If* $\sim A, +i$ *occurs on br, then* $\mathfrak{M}_{br}, w_i \models^{-} A$;
- *If* $\sim A, -i$ *occurs on br, then* $\mathfrak{M}_{br}, w_i \not\models^{-} A$.

To prove completeness for $\mathbf{CCK}^+$, it is enough to show that the frame of the induced general model, $\mathfrak{M}_{br}$, satisfies $\mathbb{C}_{A \square\!\!\rightarrow A}$: $(\forall X \subseteq W_{br})(\forall w_i, w_j \in W_{br})\, w_i R_{brX} w_j \Rightarrow w_j \in X$. By an earlier observation, we know that for every A, $\langle [\![A]\!], [\![{\sim}A]\!] \rangle \in P_{br}$, and by the definition of the general model $\mathfrak{M}_{br}$, it follows that $Pow(W_{br}) = \{[\![A]\!]^{\mathfrak{M}_{br}} \mid A \text{ is an } \mathfrak{L}\text{-formula}\}$. Thus, it is enough to show that for every $\mathfrak{L}$-formula A and every w_i, $w_j \in W_{br}$, $w_i R_{br[\![A]\!]^{\mathfrak{M}_{br}}} w_j$ implies $w_j \in [\![A]\!]^{\mathfrak{M}_{br}}$. Suppose, w_i, $w_j \in W_{br}$ and $w_i R_{br[\![A]\!]^{\mathfrak{M}_{br}}} w_j$. Then $ir_A j$ occurs on br and by completeness of br and rule $R_{A \square\!\!\rightarrow A}$, $A, +j$ occurs on br. Therefore, by the previous Lemma, $\mathfrak{M}_{br}, w_j \models^+ A$, i.e., $w_j \in [\![A]\!]^{\mathfrak{M}_{br}}$.

The system **CCK** validates Boethius' theses in rule form:

$$(A \mathrel{\Box\!\!\rightarrow} B) \vdash {\sim}(A \mathrel{\Box\!\!\rightarrow} {\sim}B), \; (A \mathrel{\Box\!\!\rightarrow} {\sim}B) \vdash {\sim}(A \mathrel{\Box\!\!\rightarrow} B).$$

as shown by the following tableau proofs in *T***CCK**:

$$\begin{array}{c} A \mathrel{\Box\!\!\rightarrow} B, +0 \\ {\sim}(A \mathrel{\Box\!\!\rightarrow} {\sim}B), -0 \\ \downarrow \\ A \mathrel{\Box\!\!\rightarrow} {\sim\sim}B, -0 \\ \downarrow \\ 0 r_A 1 \\ {\sim\sim}B, -1 \\ \downarrow \\ B, -1 \\ \downarrow \\ B, +1 \\ \times \end{array}
\qquad
\begin{array}{c} A \mathrel{\Box\!\!\rightarrow} {\sim}B, +0 \\ {\sim}(A \mathrel{\Box\!\!\rightarrow} B), -0 \\ \downarrow \\ A \mathrel{\Box\!\!\rightarrow} {\sim}B, -0 \\ \times \end{array}$$

In T**CCK**$^{+}$, Aristotle's and Boethius' theses are provable by making use of rule $R_{A \boxright A}$,e.g.:

$$
\begin{array}{c}
\sim(A \boxright \sim A), -0 \\
\downarrow \\
(A \boxright \sim\sim A), -0 \\
\downarrow \\
0 r_A 1 \\
\sim\sim A, -1 \\
\downarrow \\
A, -1 \\
\downarrow \\
A, +1 \\
\times
\end{array}
\qquad
\begin{array}{c}
(A \boxright B) \boxright \sim(A \boxright \sim B), -0 \\
\downarrow \\
0 r_{A \boxright B} 1 \\
\sim(A \boxright \sim B), -1 \\
\downarrow \\
(A \boxright \sim\sim B), -1 \\
\downarrow \\
1 r_A 2 \\
\sim\sim B, -2 \\
\downarrow \\
B, -2 \\
\downarrow \\
A \boxright B, +1 \\
\downarrow \\
B, +2 \\
\times
\end{array}
$$

- Abelard's First Principle: $\sim((A \rightarrow B) \wedge (A \rightarrow \sim B))$, and
- Aristotle's Second Thesis: $\sim((A \rightarrow B) \wedge (\sim A \rightarrow B))$

fail to be valid for the conditional $\Box\!\!\rightarrow$.

Although the conditional $\Box\!\!\rightarrow$ in **CCK**$^+$ is reflexive, it is still a very weak conditional. It does not, e.g., validate Modus Ponens. If we add to **CCK** the following tableau rule:

$$R_{MP} \qquad \begin{array}{c} A, +i \\ \downarrow \\ ir_A i \end{array}$$

we can prove the derivability statement $\{A, A \Box\!\!\rightarrow B\} \vdash B$:

$$\begin{array}{c} A, +0 \\ A \Box\!\!\rightarrow B, +0 \\ B, -0 \\ \downarrow \\ 0r_A 0 \\ \downarrow \\ B, +0 \\ \times \end{array}$$

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The statement $\{A, A \boxright B\} \vdash B$ C-corresponds to

$$\mathbb{C}_{MP} \quad (\forall X \subseteq W)(\forall w \in W) w \in X \Rightarrow wR_X w.$$

The logic defined as the set of all $\mathfrak{L}$-formulas valid in the class of all Chellas models satisfying $\mathbb{C}_{MP}$ validates Modus Ponens. But even if we assume $\mathbb{C}_{A \boxright A}$ and $\mathbb{C}_{MP}$ and add the rules $R_{A \boxright A}$ and R_{MP} to T**CCK**, $\boxright$ still is weaker than intuitionistic implication. The following formula, e.g., is not validated.

$$A \boxright (B \boxright A).$$

Like the connexive logic **C**, the system **CCK**$^+$ is a non-trivial but inconsistent logic. Both $(A \wedge {\sim}A) \boxright A$ and ${\sim}((A \wedge {\sim}A) \boxright A)$ are provable in T**CCK**$^+$.

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Extension by a constructive conditional

The logic **FDE** lacks a conditional that satisfies Modus Ponens. This defect is overcome in Nelson's four-valued constructive logic **N4**, which results from **FDE** by adding a constructive implication. The system **N4** is both paracomplete and paraconsistent.

Recently, Andreas Kapsner and Hitoshi Omori (2017) suggested to add a connexive conditional, $\Box\!\!\rightarrow$ (which they write as $\sqsupset$), to Nelson's three-valued logic **N3**, which is paracomplete but not paraconsistent.

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If the modification of the support of falsity-conditions for the constructive implication, $\rightarrow$, that leads from **N4** to the connexive logic **C** is applied to **N3**, the result is the trivial system in the language of **N3**.

In order to avoid triviality, Kapsner and Omori impose a consistency constraint on both the support of truth and the support of falsity conditions for $\Box\!\!\rightarrow$.

The following definition presents their Lewis-Nelson models in a way that facilitates comparison with the Chellas models for **CCK**$^{+}$.

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The following definition presents their Lewis-Nelson models in a way that facilitates comparison with the Chellas models for $\mathbf{CCK}^+$.

Definition

A Lewis-Nelson model is a structure $\mathfrak{M} = \langle W, \leq, \{R_A \mid A \text{ is a formula}\}, v^+, v^- \rangle$*, where* W *is a non-empty set (of states),* $\leq$ *is a partial order on* W*,* $\{R_A \mid A \text{ is a formula}\}$ *is a collection of binary relations on* W*, and* v^+ *and* v^- *are valuation functions* $v^+ \colon \mathrm{PV} \longrightarrow Pow(W)$ *and* $v^- \colon \mathrm{PV} \longrightarrow Pow(W)$*. For all* $p \in \mathrm{PV}$ *and for all* $w, w' \in W$*, (i)* $v^+(p) \cap v^-(p) = \varnothing$*, and (ii) if* $w \in v^*(p)$ *and* $w \leq w'$*, then* $w' \in v^*(p)$*, for* $* \in \{+, -\}$*, and the relations* R_A *satisfy the following conditions, where* $f_A(w) := \{w' \in W \mid wR_A w'\}$*:*

(1) $f_A(w) \subseteq [\![A]\!]$ *(i.e.,* $(\forall w, w' \in W) wR_A w' \Rightarrow w' \in [\![A]\!]$*).*

(2) *If* $w \in [\![A]\!]$*, then* $w \in f_A(w)$ *(i.e.,* $(\forall w \in W) w \in [\![A]\!] \Rightarrow wR_A w$*).*

(3) *If* $[\![A]\!] \neq \varnothing$*, then* $f_A(w) \neq \varnothing$*.*

(4) *If* $f_A(w) \subseteq [\![B]\!]$ *and* $f_B(w) \subseteq [\![A]\!]$*, then* $f_A(w) = f_B(w)$*.*

(5) *If* $f_A(w) \cap [\![B]\!] = \varnothing$*, then* $f_{A \wedge B}(w) \subseteq f_B(w)$*.*

(6) *If* $w \in [\![A]\!]$ *and* $w' \in f_A(w)$*, then* $w = w'$*.*

Definition (Definition continued)

The support of truth and support of falsity conditions coincide with those from the Definition of Chellas models, except that

$\mathfrak{M}, w \models^+ A \to B$	iff	for all $w' \in W$ such that $w \leq w'$ it holds that $\mathfrak{M}, w \not\models^+ A$ or $\mathfrak{M}, w \models^+ B$
$\mathfrak{M}, w \models^- A \to B$	iff	$\mathfrak{M}, w \models^+ A$ and $\mathfrak{M}, w \models^- B$
$\mathfrak{M}, w \models^+ A \mathrel{\Box\!\!\rightarrow} B$	iff	for some $w' \in W$, $wR_A w'$ and for all $w' \in W$ such that $wR_A w'$ it holds that $\mathfrak{M}, w' \models^+ B$
$\mathfrak{M}, w \models^- A \mathrel{\Box\!\!\rightarrow} B$	iff	for some $w' \in W$, $wR_A w'$ and for all $w' \in W$ such that $wR_A w'$ it holds that $\mathfrak{M}, w' \models^- B$.

If $\mathfrak{M} = \langle W, \leq, \{R_A \mid A \text{ is a formula}\}, v^+, v^- \rangle$ is a Lewis-Nelson model, then $\mathfrak{F} = \langle W, \leq, \{R_A \mid A \text{ is a formula}\} \rangle$ is said to be a Lewis-Nelson frame, and $\mathfrak{M}$ is said to be based on $\mathfrak{F}$.

Definition (Definition continued)

The support of truth and support of falsity conditions coincide with those from the Definition of Chellas models, except that

$\mathfrak{M}, w \models^+ A \to B$ iff for all $w' \in W$ such that $w \leq w'$ it holds that $\mathfrak{M}, w \not\models^+ A$ or $\mathfrak{M}, w \models^+ B$

$\mathfrak{M}, w \models^- A \to B$ iff $\mathfrak{M}, w \models^+ A$ and $\mathfrak{M}, w \models^- B$

$\mathfrak{M}, w \models^+ A \,\Box\!\!\rightarrow B$ iff for some $w' \in W$, $wR_A w'$ and for all $w' \in W$ such that $wR_A w'$ it holds that $\mathfrak{M}, w' \models^+ B$

$\mathfrak{M}, w \models^- A \,\Box\!\!\rightarrow B$ iff for some $w' \in W$, $wR_A w'$ and for all $w' \in W$ such that $wR_A w'$ it holds that $\mathfrak{M}, w' \models^- B$.

If $\mathfrak{M} = \langle W, \leq, \{R_A \mid A \text{ is a formula}\}, v^+, v^- \rangle$ is a Lewis-Nelson model, then $\mathfrak{F} = \langle W, \leq, \{R_A \mid A \text{ is a formula}\} \rangle$ is said to be a Lewis-Nelson frame, and $\mathfrak{M}$ is said to be based on $\mathfrak{F}$.

Due to condition (i), the semantics gives rise to a system that fails to be paraconsistent.

Since condition $\mathbb{C}_{A \Box\!\!\rightarrow A}$ is assumed, the constraint that for some $w \in W$, $wR_{[\![A]\!]^{\mathfrak{M}}}w$, restricts the support of truth and the support of falsity conditions of $A \Box\!\!\rightarrow B$ to consistent antecedents A.

Validity is defined as support of truth at any state of any Lewis-Nelson model, and the entailment relation, $\models$, between set of formulas and single formulas is defined as preservation of support of truth: $\Delta \models A$ iff for all models $\mathfrak{M} = \langle W, \leq, \{R_A \mid A \text{ is a formula}\}, v^+, v^- \rangle$ and all $w \in W$, it holds that $\mathfrak{M}, w \models^+ A$ if $\mathfrak{M}, w \models^+ B$ for all $B \in \Delta$.

Since Lewis-Nelson models $\langle W, \leq, \{R_A \mid A \text{ is a formula}\}, v^+, v^- \rangle$ use *binary relations* R_A *on* W, the semantics does not allow for a purely structural correspondence theory based on *frames* (or general frames).

Kapsner and Omori's semantically defined system has a number of noteworthy properties:

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Validity is defined as support of truth at any state of any Lewis-Nelson model, and the entailment relation, $\models$, between set of formulas and single formulas is defined as preservation of support of truth: $\Delta \models A$ iff for all models $\mathfrak{M} = \langle W, \leq, \{R_A \mid A \text{ is a formula}\}, v^+, v^- \rangle$ and all $w \in W$, it holds that $\mathfrak{M}, w \models^+ A$ if $\mathfrak{M}, w \models^+ B$ for all $B \in \Delta$.

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Kapsner and Omori's semantically defined system has a number of noteworthy properties:

- The conditional $\Box\!\!\rightarrow$ is not reflexive: $(A \wedge {\sim}A) \Box\!\!\rightarrow (A \wedge {\sim}A)$ is not valid.
- Simplification fails for $\Box\!\!\rightarrow$ as neither $(A \wedge {\sim}A) \Box\!\!\rightarrow A$ nor $(A \wedge {\sim}A) \Box\!\!\rightarrow {\sim}A$ is valid.
- Actually, for no formula B, $(A \wedge {\sim}A) \Box\!\!\rightarrow B$ is valid; *contradictione nihil implicat*.
- For no formula B, $((A \wedge {\sim}A) \Box\!\!\rightarrow (A \wedge {\sim}A)) \Box\!\!\rightarrow B$ is valid.
- Weakening fails for $\Box\!\!\rightarrow$ as $p \Box\!\!\rightarrow p$ is valid, but $(p \wedge {\sim}p) \Box\!\!\rightarrow p$ is not.
- The logic is not closed under substitution because $p \Box\!\!\rightarrow p$ is valid, but $(p \wedge {\sim}p) \Box\!\!\rightarrow (p \wedge {\sim}p)$ is not.

In addition to the lack of a purely structural correspondence theory, the semantics in terms of Lewis-Nelson models may be seen to have at least two other drawbacks.

Kapsner and Omori motivate adding a connexive conditional to **N3** instead of **N4** by remarking that "the move to the **N4**-based logic works well technically, but philosophically is a doubtful one". This may be clearly criticized.

There is no convincing reason to prefer truth value gaps over truth value gluts, and to prefer paracompleteness over paraconsistency when it comes to information processing, and it is not without reason that **FDE**, Belnap and Dunn's useful four-valued logic, and Nelson's **N4** treat verification and falsification on a par.

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On the contrary, the four-valued semantics is well-motivated and natural not only from the point of view of information processing but also from the point of view of proof-theoretic semantics.

Moreover, the property *contradictione nihil implicat* echoes the idea of negation as cancellation. If the cancellation model of negation is meant to justify *contradictione nihil implicat*, this is a very problematic justificatory base.

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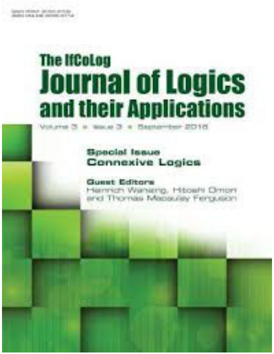
McCall, Storrs, 2012, “A History of Connexivity”, in D.M. Gabbay et al. (eds.), *Handbook of the History of Logic. Volume 11. Logic: A History of its Central Concepts*, Amsterdam: Elsevier, 415–449.

Connexive Logics, *IfCoLog Journal of Logics and their Applications* 3 (2016), Heinrich Wansing, Hitoshi Omori and Thomas M. Ferguson (eds).



We shall now expand the connexive conditional logics **CCK** and **CCK**$^+$ and introduce an additional constructive conditional in these systems.

The resulting constructive connexive logics are named "**CCCK**" and "**CCCK**$^+$", respectively, and are formulated in the expanded language $\mathfrak{L}_\rightarrow$.

The use of a binary relation, $\leq$, for interpreting the constructive implication, and of relations R_X for every set of states X, requires a decision on how to extend the persistence (alias heredity, alias monotonicity) requirement from **N4**, i.e., $(\forall w, w' \in W)$ if $w \in v^*(p)$ and $w \leq w'$, then $w' \in v^*(p)$, for $* \in \{+, -\}$, to all $\mathfrak{L}_\rightarrow$-formulas.

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Various options for guaranteeing persistence in intuitionistic modal logics are discussed and compared with each other in (Simpson 1994). Here we use conditions employed in (Došen 1984, 1985).

Definition

A constructive frame is a structure $\langle W, R, \leq \rangle$, where $\langle W, R \rangle$ is a Chellas frame and

1. $\leq$ is a reflexive and transitive binary relation on W,
2. $(\forall X \subseteq W)\,(\leq \circ R_X) \subseteq (R_X \circ \leq)$,
3. $(\forall X \subseteq W)\,(\leq^{-1} \circ R_X) \subseteq (R_X \circ \leq^{-1})$.

where '$\circ$' stands for the concatenation of binary relations.
If $\langle W, R, \leq \rangle$ is a constructive frame, then $\mathfrak{M} = \langle W, R, \leq, v^+, v^- \rangle$ is a constructive model iff v^+ and v^- are valuation functions $v^+\colon \mathrm{PV} \longrightarrow Pow(W)$ and $v^-\colon \mathrm{PV} \longrightarrow Pow(W)$ such that if $w \in v^*(p)$ and $w \leq w'$, then $w' \in v^*(p)$, for $* \in \{+, -\}$.

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Definition

A constructive model $\mathfrak{M} = \langle W, R, \leq, v^+, v^- \rangle$ is a model for **CCCK** iff support of truth and support of falsity relations $\models^+$ and $\models^-$ between $\mathfrak{M}$, states $w \in W$, and formulas from $\mathfrak{L}_\rightarrow$ are inductively defined as in the case of **CCK**, and using Nelson's clauses for the constructive implication, i.e.:

$$\mathfrak{M}, w \models^+ A \rightarrow B \quad \text{iff} \quad \text{for all } w' \in W \text{ such that } w \leq w' \text{ it holds that } \mathfrak{M}, w' \not\models^+ A \text{ or } \mathfrak{M}, w' \models^+ B$$

$$\mathfrak{M}, w \models^- A \rightarrow B \quad \text{iff} \quad \mathfrak{M}, w \models^+ A \text{ and } \mathfrak{M}, w \models^- B.$$

Lemma

(Persistence) Let $\mathfrak{M} = \langle W, R, \leq, v^+, v^- \rangle$ be a model for **CCCK***. Then for every $w, w' \in W$ and every $\mathfrak{L}_{\rightarrow}$-formula A it holds that if $w \leq w'$ and $\mathfrak{M}, w \models^* A$, then $\mathfrak{M}, w' \models^* A$, for $* \in \{+, -\}$.*

We next define general frames and general models for **CCCK**.

Definition

A quadruple $\langle W, R, \leq, P\rangle$ is a general frame for **CCCK** iff $\langle W, R, \leq\rangle$ is a constructive frame, $\langle W, R, P\rangle$ is a general frame for **CCK** and P in addition satisfies the following condition:

5. if $\langle X, Y\rangle, \langle X', Y'\rangle \in P$, then $\langle\{w \in W \mid \forall w' \in W(w \leq w' \Rightarrow (w' \notin X \text{ or } w' \in X'))\}, \{w \in W \mid w \in X \wedge w \in Y'\}\rangle \in P$.

Definition

Let $\langle W, R, \leq, P\rangle$ be a general frame for **CCCK**. The tuple $\langle W, R, \leq, P, v^+, v^-\rangle$ is a general model for **CCCK** iff $\langle W, R, \leq, v^+, v^-\rangle$ is a constructive model and $\langle [\![p]\!], [\![\sim p]\!]\rangle \in P$ for every $p \in \mathrm{PV}$. Support of truth and support of falsity relations $\models^+$ and $\models^-$ are defined as in the case of models for **CCCK**.

Lemma

Let $\langle W, R, \leq, P, v^+, v^-\rangle$ be a general model for **CCCK**. *Then for every $\mathfrak{L}_{\rightarrow}$-formula A, $\langle [\![A]\!], [\![\sim A]\!]\rangle \in P$.*

Eventually, we define our basic constructive conditional logics.

Definition

The logic **CCCK** (**CCCK**$^+$) is the set of all $\mathfrak{L}_{\rightarrow}$-formulas valid with respect to the class of all models for **CCCK** (all models for **CCCK** that satisfy $\mathbb{C}_{A\Box\!\!\rightarrow A}$). If a model, general frame, or general model for **CCCK** satisfies $\mathbb{C}_{A\Box\!\!\rightarrow A}$ it will be called a model, general frame, or general model, respectively, for **CCK**$^+$.

We define tableaux calculi *T***CCCK** and *T***CCCK**$^+$ for **CCCK** and **CCCK**$^+$ that generalize the tableaux calculi for **CCK** and **CCK**$^+$.

Theorem

Let $\Delta \cup \{A\}$ be a finite set of $\mathfrak{L}$-formulas. Then $\Delta \models_{\mathbf{CCCK}} A$ iff $\Delta \vdash_{T\mathbf{CCCK}} A$.

Theorem

$\Delta \models_{\mathbf{CCCK}^+} A$ *iff* $\Delta \vdash_{T\mathbf{CCCK}^+} A$.

It is well known that the syllogistic contains inferences that are not classically valid under the standard translation into predicate logic. One of the most prominent examples is the inference from 'Every *P* is *Q*' to 'Some *P*s are *Q*s':

$$\forall x(P(x) \rightarrow Q(x)) \vdash \exists x(P(x) \wedge Q(x)) \tag{1}$$

Normally, we do not quantify over the empty set. If we assume that the interpretation of *P* is empty, there is hardly any reason to assume that every *P* is *Q*, but if the interpretation of *P* is non-empty, (1) is a valid inference.

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Inference (1) cannot be consistently added as a rule to a proof system for classical predicate logic, as is obvious from the following instance of (1):

$$\forall x((P(x) \wedge {\sim}P(x)) \rightarrow Q(x)) \vdash \exists x((P(x) \wedge {\sim}P(x)) \wedge Q(x)) \quad (2)$$

The premise of (2) is classically valid, whereas the conclusion is classically unsatisfiable. Now, in classical logic, inference (1) is interchangeable with

$$\forall x(P(x) \rightarrow Q(x)) \vdash \exists x{\sim}(P(x) \rightarrow {\sim}Q(x)). \quad (3)$$

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Storrs McCall (1967) pointed out that in a system of connexive logic (3) is a valid inference. This is especially perspicuous in the quantified connexive logic QC introduced in (Wansing 2005), because there

$$\sim(A \to B) \leftrightarrow (A \to \sim B) \tag{4}$$

is an axiom.

One might therefore suggest to translate statements of the form 'Some *P*s are *Q*s' not as $\exists x(P(x) \wedge Q(x))$ but as $\exists x \sim(P(x) \to \sim Q(x))$, which in the system QC is equivalent to $\exists x(P(x) \to Q(x))$.

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