ON CONSTRUCTIVE VERSIONS OF INDEPENDENCE-FRIENDLY LOGIC

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This talk presents the recent results of the joint work of the author, S. O. Speranski and I. Yu. Shevchenko, which are partly presented in [9] and partly are in progress.

The independence-friendly first-order logic (IF-FOL) suggested in [4] has generated numerous dependence and independence logics — whose specific operators can be easily defined in terms of so-called *teams* (see [5, 6]). Recall that a *team* is a family of assignments of elements of the domain of a first-order structure to individual variables, or a family of valuations of propositional variables in the set of truth values; usually it is assumed that all members of a team have the same domain. The logic IF-FOL is an extension of first-order logic (FOL) by means of *independent quantifiers* of the form $\exists x \backslash X$ where $\{x\} \cup X$ is a finite set of individual variables. The validity of a formula $\exists x \backslash X\, \varphi$ in a structure $\mathfrak{M}$ on a team T means that the formula φ is valid in $\mathfrak{M}$ on a team $T' = \{s\,(x/f\,(s)) \mid s \in T\}$* where f is a function from T to the domain of $\mathfrak{M}$ such that $f\,(s) = f\,(s')$ whenever $s\,(y) = s'\,(y)$ for all $y \in \mathrm{dom}\,(s) \setminus X$ — in this way, the value of s on x is independent of its values on the variables in X. Equivalently, IF-FOL can be easily interpreted using skolemisations, so as Skolem terms for occurences of $\exists x \backslash X$ do not contain variables from X.

The logic IF-FOL admits a game theoretical interpretation too. To obtain a game theoretical semantics (GTS) for IF-FOL, we have to pass from standard games used to interpret formulas in FOL to games with imperfect information. Hintikka [3, Chapter 6] motivates the game theoretical approach to interpreting IF-FOL as follows:

> The approach presented in this book has a strong spiritual kinship with constructivistic ideas. This kinship can be illustrated in a variety of ways. One of the basic ideas of constructivists like Michael Dummett [1, 2] is that meaning has to be mediated by teachable, learnable, and practicable human activities. This is precisely the job which semantical games do in game-theoretical semantics.

*Here $s\,(x/a)$ denotes the assignment with domain $\mathrm{dom}\,(s) \cup \{x\}$ such that $s\,(x/a)\,(x) = a$, and $s\,(x/a)\,(y) = s\,(y)$ for $y \neq x$.

In fact this statement made by Hintikka motivated us to compare GTS for FOL and IF-FOL with one standard constructive semantics, namely with the modification of realizability semantics suggested by D. Nelson [8]. To be more precise, let $\sigma_{\mathbb{N}}$ and $\mathfrak{N}$ be the signature of Peano arithmetic and its standard model, i.e.

$$\sigma_{\mathbb{N}} \ := \ \{0, \mathsf{s}, +, \times, =\} \quad \text{and} \quad \mathfrak{N} \ := \ \langle \mathbb{N}; 0^{\mathbb{N}}, \mathsf{s}^{\mathbb{N}}, +^{\mathbb{N}}, \times^{\mathbb{N}}, =^{\mathbb{N}} \rangle.$$

For any $e \in \mathbb{N}$, assignment s in $\mathfrak{N}$ and first-order $\sigma_{\mathbb{N}}$-formula ϕ with $FV(\phi) \subseteq \mathrm{dom}(s)$, D. Nelson [8] inductively defines

$$e \, \text{Ⓟ} \, s, \phi \quad \text{and} \quad e \, \text{Ⓝ} \, s, \phi.$$

If $e \, \text{Ⓟ} \, s, \phi$ (respectively $e \, \text{Ⓝ} \, s, \phi$), then the number e is called *a positive* (*negative*) *realization for ϕ under s*. Roughly speaking, each positive (negative) realization of ϕ under s encodes an effective verification (respectively falsification) procedure for ϕ in $\mathfrak{N}$ under s. Negation can be viewed as a kind of switch between verification and falsification procedures in Nelson's semantics, which is similar to how it behaves in GTS, where the players switch their roles when they see $\neg$. This observation explains our choice of constructive semantics for comparing with GTS. On this way we obtain the following results.

i. First, omitting the requirements of constructivity in the definition of realizations, we define, for any pair s, ϕ with $FV(\phi) \subseteq \mathrm{dom}(s)$, two families of set theoretical objects $S^{+}(s, \phi)$ and $S^{-}(s, \phi)$. In GTS for FOL, two strategies of the same player are called equivalent if the sets of histories played according to these strategies coincide. It turns out that there is a natural one-to-one correspondence between elements of $S^{+}(s, \phi)$, where ϕ is implication-free, and winning strategies for Eloise (the initial verifier in GTS) up to the equivalence just defined. Similarly, there is a natural bijection between $S^{+}(s, \phi)$ and the equivalence classes of winning strategies for Abelard (who is the initial falsifier). By distinguishing effective objects in $S^{+}(s, \phi)$ and $S^{-}(s, \phi)$ and codifying them by natural numbers we get back to positive and negative Nelson's realizations for ψ under s. In this sense Nelson's realizability restricted to the implication-free first-order formulas can be viewed as an effective version of GTS for FOL.

ii. Next we propose a realizability interpretation for IF-FOL. More precisely, for any $e \in \mathbb{N}$, team T of assignments in $\mathfrak{N}$ and IF-FOL-$\sigma_{\mathbb{N}}$-formula ϕ with $FV(\phi) \subseteq \mathrm{dom}(T)$, we inductively define

$$e \, \boxed{\mathtt{P}} \, T, \phi \quad \text{and} \quad e \, \boxed{\mathtt{N}} \, T, \phi.$$

We show that the resulting realizability semantics is related to GTS for IF-FOL in exactly the same way as Nelson's restricted realizability to GTS for FOL.

iii. Finally, we show that the team realizability interpretation for IF-FOL appropriately generalises Nelson's restricted realizability interpretation for the implication-free first-order formulas. In fact, we establish that for 'effective' teams and implication-free first-order formulas, team realizations can be identified with computable sequences of Nelson's realizations.

In conclusion we shall discuss another approach to 'effectivizing' IF-FOL, which is based on the notion of effective strategy (defined as a computable function from sequences of actions to actions). A sketch of this approach can be found already in [3, Chapter 6]. We shall also discuss the equivalence of this approach and the one described above (which is based on the possibility of codifying elements of $S^{+}(s, \phi)$ and $S^{-}(s, \phi)$ by natural numbers). Lastly, we shall describe a version of IF-FOL with implication and discuss the possibility of defining a kind of independent implication. Nelson's realizability gives us a hint for such a definition.

References

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