Newton polygon method and solvability of equations by quadratures

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Consider a homogeneous linear differential equation

$$y^n + a_1(t)y^{n-1} + \cdots + a_n(t)y = 0 \tag{1}$$

whose coefficients a_i belong to a differential field K.

Theorem 1 *The equation* (1) *can be solved by quadratures over K if and only if the following conditions hold: 1) the equation* (1) *has a solution $y_1 = \int f(t)dt$ where f is an algebraic function over K, 2) the linear differential equation of order $(n-1)$ obtained from* (1) *by the reduction of order using the solution y_1 is solvable by quadratures over the differential field $K(y_1)$.*

The standard proof (E. Picard and E. Vessiot, 1910) of Theorem 1 uses the differential Galois theory and is rather involved. In the talk I will discuss an elementary proof of Theorem 1 based on old arguments suggested by J. Liouville, J. Ritt and M. Rosenlicht.

J. Liouville in 1839 proved Theorem 1 for $n = 2$. J. Ritt in 1948 simplified his proof [1]. He used expansion of solutions (as functions of a parameter) into converging Puiseux series. J. Ritt studied algebraic properties of such series using the Newton polygon method.

M. Rosenlicht in 1973 proved [2] the following theorem.

Theorem 2 *Let n be a positive integer, and let f be a polynomial in several variables with coefficients in a differential field K and of total degree less than n. Then if the differential equation*

$$y^{(n)} = f(y, y', y'', \dots) \tag{2}$$

has a solution representable by quadratures over K, it has a solution algebraic over K.

A homogeneous linear differential equation (1) of second order can be reduced to the nonlinear Riccati equation

$$u' + a_1(t)u + a_2(t) + u^2 = 0 \tag{3}$$

which is a particular case of (2) for $n = 2$. To prove Theorem 1 for $n = 2$ Liouville and Ritt proved first Theorem 2 for the Riccati equation (3). To prove Theorem 1 in general case M. Rosenlicht proved first Theorem 2 for a generalized Riccati equation of order $n - 1$. The reduction of Theorem 1 to Theorem 2 for the generalized Riccati equation is straightforward. But Rosenlich's proof of Theorem 2 is rather involved. It is applicable to abstract differential fields of characteristic zero and makes use of the valuation theory.

In the talk I will discuss a proof of Theorem 2 which does not rely on the valuation theory. It generalizes Ritt's arguments (makes use of the Puiseux expansion and Newton polyhedron method) and provides an elementary proof of the classical Theorem 1.

References

[1] J. F. Ritt, *Integration in finite terms*, Columbia Univ. Press, New York, 1948.

[2] M. Rosenlicht, An analogue of l'Hospital's rule, *Proc. Amer. Math. Soc.* 37 (1973), **37**:2 (1973), 369–373.

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