Dense sphere packings: state of the art and algebraic geometry constructions

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How dense can we pack equal spheres in the Euclidean space $\mathbb{R}^N$? The question looks natural and is treated by humanity at least since the end of 16th century. The first four hundred years of research gave us the answers only in dimensions 1, 2, and 3. Quite recently, the answers for $N = 8$ and $N = 24$ — that we always presumed to be true — were proved by an elegant technique using modular forms [1], [2].

If we restrict ourselves to the easier situation when the centers of the spheres form a lattice (an additive subgroup of $\mathbb{R}^N$) the answer is known for N from 1 to 8, and, of course, for $N = 24$. Not too much either ...

We have to ask easier questions. Can we bound the density and how? Which constructions give us packings that, if not being the best, are however dense enough?

Number fields and curves over finite fields provide lovely constructions [3]. To find out their densities we need to know a lot about our algebraic geometry objects. In particular, we study their zeta-functions.

As usual, when we do not know the answer for a given N we try to look at what happens when $N \to \infty$. This time we need to understand the asymptotic behaviour of zeta-functions when the genus tends to ∞, cf. [4], [5], [6], [7].

My dream is a nice theory of limit objects such as projective limits of curves or infinite extensions of $\mathbb{Q}$, as yet we are very far from it.

Another great challenge is to construct lattice sphere packings that are denser that those given by a random construction (so-called Minkowski bound).

References

[1] M. Viazovska, The sphere packing problem in dimension 8, *Annals of Math.* **185** (2017), 991–1015.

[2] J. Oesterlé, Densité maximale des empilements de sphères en dimensions 8 et 24, *Sém. Bourbaki* **69**:1133 (juin 2017).

[3] M. Rosenbloom, M. Tsfasman, Multiplicative lattices in global fields, *Invent. Math.* **101**:1 (1990), 687–696.

[4] M. Tsfasman, Some remarks on the asymptotic number of points, *Coding theory and algebraic geometry*, Springer LN 1518 (1991), 178–192.

[5] M. Tsfasman, S. Vlăduţ, Infinite global fields and the generalized Brauer–Siegel Theorem, *Moscow Math. J.* **2**:2 (2002), 329–402.

[6] M. Tsfasman, Serre's theorem and measures corresponding to abelian varieties over finite fields, *Contemporary Math.*, to appear.

[7] J.-P. Serre, Distribution asymptotique des valeurs propres des endomorphismes de Frobenius, *Sém. Bourbaki* **70**:1146 (mars 2018).