Polytopal realizations of cluster, subword and accordion complexes and representation theory

Mikhail A. Gorsky *(Bielefeld University)*,
mgorsky@math.uni-bielefeld.de

Associahedra are a family of polytopes appearing in different branches of mathematics. The were first introduced by D. Tamari and rediscovered by J. Stasheff in the early 1960s in the context of the associativity. The vertices of the $n-$dimensional

associahedron bijectively correspond to the triangulations of the $(n+3)-$dimensional regular polygon, and the edges correspond to flips. This is one of many combinatorial descriptions of the structure of the associahedron. Later, there were introduced a lot of generalizations of this family of polytopes. I will discuss three of them, their different geometric realizations and their relation to the representation theory.

In early 2000s, S. Fomin and A. Zelevinsky introduced the notion of *cluster algebras* in order to study dual canonical bases in double Bruhat cells and the phenomenon of total positivity. A cluster algebra is defined by a *quiver*, or an oriented graph, without loops or 2-cycles.It has a set of generators, called *cluster variables*, that are grouped in overlapping subsets of a fixed cardinality, called *clusters*. Relations correspond to the operation of mutation between clusters different only in one cluster variable. One may thus study abstract simplicial complexes whose vertices correspond to cluster variables, maximal simplices correspond to clusters, and edges correspond to mutations. These are called *cluster complexes.* Fomin showed that for quivers of finite Dynkin type, these complexes are polytopal, and in type A_n the dual polytope is the $n-$dimensional associahedron. More generally, the polytopes dual to cluster complexes are called *generalized associahedra.* For any initial cluster, one may construct 2 different geometric realizations of a generalized associahedron. Their dual fans are the $d-$fan and the $g-$fan of the algebra, encoding all its algebraic structure. They also encode the so-called *wall and chamber structure* of the path algebra of the corresponing quiver. The toric variety associated to the $g-$fan is the toric degeneration of the corresponding cluster variety.

Let W be a finite Coxeter group, $S = \{s_1, \dots, s_n\}$ be a set of simple reflections generating W. Consider a word $\mathbf{Q} := \mathbf{Q}_1 \dots \mathbf{Q}_m$ in the alphabet of simple reflections ($\mathbf{Q}_i \in S$ $\forall\, i = 1, \dots, m$) and an element π of the group W. *The subword complex* $\Delta(\mathbf{Q}; \pi)$ is a pure simplicial complex on the set of vertices $\{\mathbf{Q}_1, \dots, \mathbf{Q}_m\}$ corresponding to the letters (more precisely, to their positions) in the word Q. A set of vertices yields a simplex if the complement in Q to the corresponding subword contains a reduced expression of π. The maximal simplices correspond to the complements of reduced expressions of π in the word $\mathbf{Q}$. Subword complexes were introduced by A. Knutson and E. Miller in the article [5]. They showed in [6] that $\Delta(\mathbf{Q}; \pi)$ is spherical if and only if the Demazure product of the word Q equals π; otherwise, $\Delta(\mathbf{Q}; \pi)$ is a triangulated ball. For spherical subword complexes, there arise natural questions of the existence, of the combinatorial description and of geometric realizations of their polar dual polytopes. In the group W there exists the unique longest element denoted by w_o. We will consider subword complexes of the form

$\Delta(\mathbf{c}\,\mathbf{w}_o; w_o)$, where $\mathbf{c}$ is a reduced expression of a Coxeter element, $\mathbf{w}_o$ is an arbitrary reduced expression of w_o. Such complexes admit a realization by *brick polytopes* of V.Pilaud–C. Stump [7] that we will denote by $\mathbf{B}(\mathbf{c}\,\mathbf{w}_o; w_o)$. C. Ceballos, J.-P. Labbé and C. Stump [1] proved that the complexes $\Delta(\mathbf{c}\,\mathbf{w}_o(\mathbf{c}); w_o)$, where $\mathbf{w}_o(\mathbf{c})$ is the so-called $\mathbf{c}$-*sorting word* for w_o, are the generalized cluster complexes of type W. Therefore, the polytopes $\mathbf{B}(\mathbf{c}\,\mathbf{w}_o(\mathbf{c}); w_o)$ realize the $c-$associahedra of type W. The choice of a Coxeter element c is equivalent to the choice of a quiver Q being an orientation of the Coxeter diagram of the group W. The dual fan to the brick polytope realizing the generalized associahedron is the $g-$fan of the corresponding cluster algebra.

The choice of an arbitrary reduced expression $\mathbf{w}_o$ of the element w_o is equivalent to the choice of a Dyer total order on the set of positive roots of the corresponding root system Φ, which in turn is equivalent to the choice maximal green sequence of mutations of the quiver Q of a (non-necessarily linear) stability condition on the category of representations of Q over some ground field. Thus, one can introduce the notion of $(\mathbf{c}, \mathbf{w}_o)$-*stable* positive roots forming the set $\mathrm{Stab}(\mathbf{c}, \mathbf{w}_o)$, resp. stable representations. In [2] (see also [3] for more details), i prove the following theorem.

Theorem 1 (i) *The vertices of* $\Delta(\mathbf{c}\,\mathbf{w}_o; w_o)$ *and, equivalently, the facets of* $\mathbf{B}(\mathbf{c}\,\mathbf{w}_o; w_o)$ *are in a one-to-one correspondence with the simple negative and the* $(\mathbf{c}, \mathbf{w}_o)$*-stable positive roots in the system* Φ.

(ii) *Let expressions* $\mathbf{w}_o$, $\mathbf{w}'_o$ *be such that* $\mathrm{Stab}(\mathbf{c}, \mathbf{w}_o) \subset \mathrm{Stab}(\mathbf{c}, \mathbf{w}'_o)$. *Then the complex* $\Delta(\mathbf{c}\,\mathbf{w}'_o; w_o)$ *can be obtained from the complex* $\Delta(\mathbf{c}\,\mathbf{w}_o; w_o)$ *by a sequence of edge subdivi-*

sions. Similarly, a ceratin geometric realization of the polytope $\mathbf{B}(\mathbf{c}\,\mathbf{w}_o'; w_o)$ *can be obtained from the polytope realizing* $\mathbf{B}(\mathbf{c}\,\mathbf{w}_o; w_o)$ *by a sequence of truncations of faces of codimension* 2*. In particular, for any expression,* $\mathbf{B}(\mathbf{c}\,\mathbf{w}_o; w_o)$ *is combinatorially equivalent to a* $2-$*truncated cube.*

The polytopes for words related by an elementary braid move of order m either coincide, or are related by a truncation of one face of codimension 2 $(m-2)$ times. This comes from the fact such an operation might change the set of stable positive roots only in one root subsystem of rank 2. We call the polytopes $\mathbf{B}(\mathbf{c}\,\mathbf{w}_o; w_o)$ *stability associahedra*. Theorem 1 implies that all the stability associahedra and, in particular, all generalized associahedra are $2-$truncated cubes. The dual fans of this realization of generalized associahedra are the $d-$fans of cluster algebras. In a joint work in progress with Vincent Pilaud and Salvatore Stella, we work on the definition algebras whose structure is naturally encoded by these $d-$fans and by the $g-$fans given by the brick polytopes.

Theorem 1 provides a partial order on the set of reduced expressions of w_o given by the inclusion of sets of stable positive roots. In type A_n with the linear orientation, the resulting poset is isomorphic with the poset of triangulations of the cyclic polytope of dimension 3 with the 2nd (higher) Tamari-Stasheff order. In spirit of Reading's Cambrian lattices, we call it *the* 2*nd higher Cambrian order* of type (W, c). In [4] i show that in terms of complexes, or corresponding fans, Theorem 1 can be generalized to any acyclic quiver Q, and we get a semi-lattice of maximal green sequences. For the corresponding complexes, the order is given by the order of edge subdivisions. The choice of a maximal green sequence is equivalent to the choice of a (non-necessarily linear) stability condition with finitely many stable objects.

Another generalization of associahedra is given by polytopes dual to so-called *accordion complexes*. Their vertices correspond to certain dissections of regular polytopes, instead of triangulations. Recently, their combinatorics and geometric realizations were linked to the wall and chamber structure of gentle algebras. I will overview the progress in this area.

Theorem 1 provides a partial order on the set of maximal green sequences given by the inclusion of sets of stable positive roots. In type A_n with the linear orientation, the resulting poset is isomorphic with the poset of triangulations of the cyclic polytope of dimension 3 with the 2nd (higher) Tamari-Stasheff order. In spirit of Reading's Cambrian lattices, we call it *the 2nd higher Cambrian order* of type (W, c). In [4] i show that in terms of complexes, or corresponding fans, Theorem 1 can be generalized to any acyclic quiver Q, and we get a semi-lattice of maximal green sequences. For the corresponding complexes, the order is given by the order of edge subdivisions. The choice of a maximal green sequence is equivalent to the choice of a (non-necessarily linear) stability condition with finitely many stable objects.

References

[1] C. Ceballos, J.-P. Labbé, and C. Stump. *Subword complexes, cluster complexes, and generalized multi-associahedra*, preprint, arXiv:1108.1776, 2011.

[2] M. Gorsky. *Subword complexes and 2-truncated cubes*, Russian Mathematical Surveys, 69:3 (2014), 572–574.

[3] M. Gorsky. *Geometry and combinatorics of subword complexes and their dual polytopes*, Ph. D. thesis (in Russian). Available at http://www.mi.ras.ru/dis/ref14/gorskii/gorskii_dis.pdf

[4] M. Gorsky. *Stability associahedra, maximal green sequences and second Cambrian orders*, *in preparation.*

[5] A. Knutson and E. Miller. *Subword complexes in Coxeter groups*. Adv. Math., 184(1):161-176, 2004.

[6] A. Knutson and E. Miller. *Groebner geometry of Schubert polynomials*. Ann. of Math. (2), 161(3):1245-1318, 2005.

[7] V. Pilaud and C. Stump. *Brick polytopes of spherical subword complexes: A new approach to generalized associahedra*. Advances in Mathematics, 276 (2015), 1 – 61.