The homotopy theory of polyhedral products associated with flag complexes

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This is joint work with Taras Panov.

Polyhedral products have received considerable attention recently as they unify diverse constructions from several seemingly separate areas of mathematics: toric topology (moment-angle complexes), combinatorics (complements of complex coordinate subspace arrangements), commutative algebra (the Golod property of monomial rings), complex geometry (intersections of quadrics), and geometric group theory (Bestvina-Brady groups). In this talk we investigate the homotopy theory of polyhedral products associated to flag complexes.

Let K be a simplicial complex on the vertex set $[m] = \{1, 2, \ldots, m\}$. For $1 \leq i \leq m$, let (X_i, A_i) be a pair of pointed CW-complexes, where A_i is a pointed subspace of X_i. Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$ be the sequence of pairs. For each simplex $\sigma \in K$, let $(\underline{X}, \underline{A})^\sigma$ be the subspace of $\prod_{i=1}^m X_i$ defined by

$$(\underline{X}, \underline{A})^\sigma = \prod_{i=1}^{m} Y_i \qquad \text{where} \qquad Y_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

The *polyhedral product* determined by $(\underline{X}, \underline{A})$ and K is

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma \subseteq \prod_{i=1}^{m} X_i.$$

A simplicial complex K is *flag* if any set of vertices of K which are pairwise connected by edges spans a simplex.

The *flagification* of K, denoted K^f, is the minimal flag complex on the same set $[m]$ that contains K. We therefore have a simplicial inclusion $K \to K^f$. We prove the following.

Theorem 1 *Let K be a simplicial complex on the vertex set $[m]$, let K^f be the flagification of K, and let L be the simplicial complex given by m disjoint points. Let $(\underline{X}, \underline{A})^L \xrightarrow{g} (\underline{X}, \underline{A})^K \xrightarrow{f} (\underline{X}, \underline{A})^{K^f}$ be the maps of polyhedral products induced by the maps of simplicial complexes $L \longrightarrow K \longrightarrow K^f$. Then the following hold:*

(a) the map Ωf has a right homotopy inverse;

(b) the composite $\Omega f \circ \Omega g$ has a right homotopy inverse.

In particular, consider the special case when each A_i is a point. Write $(\underline{X}, \underline{*})$ for $(\underline{X}, \underline{A})$ and notice that $(\underline{X}, \underline{*})^L = X_1 \vee \cdots \vee X_m$. If K is a flag complex on the vertex set $[m]$ then the simplicial map $L \longrightarrow K$ induces a map

$$f \colon X_1 \vee \cdots \vee X_m = (\underline{X}, \underline{*})^L \longrightarrow (\underline{X}, \underline{*})^K.$$

By Theorem **??**, Ωf has a right homotopy inverse. That is, $\Omega(\underline{X}, \underline{*})^K$ is a retract of $\Omega(X_1 \vee \cdots \vee X_m)$. This informs greatly on the homotopy theory of $\Omega(\underline{X}, \underline{*})^K$ since the homotopy type of $\Omega(X_1 \vee \cdots \vee X_m)$ has been well studied; in particular, when each X_i is a suspension the Hilton-Milnor Theorem gives an explicit homotopy decomposition of the loops on the wedge.