On face numbers of flag simplicial complexes

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Let $\mathcal{K}$ be an n-dimensional simplicial complex. Denote by f_i the number of i-dimensional simplices of $\mathcal{K}$. Characterization of possible f-vectors $(f_0, \ldots, f_n)$ of various classes of simplicial complexes is a classical problem of enumerative combinatorics.

Among the most well-known results in this direction are: (1) the Kruskal-Katona theorem describing all possible f-vectors of general simplicial complexes; (2) Analogue of the Kruskal-Katona theorem for Cohen-Macaulay simplicial complexes; (3) The upper bound theorem due to McMullen, which gives necessary conditions for a tuple of integers to be the f-vector of a triangulation of an n-dimensional sphere; (4) g-Theorem, characterizing the f-vectors of simplicial polytopes.

The proofs of these results led to numerous constructions, associating certain algebraic and topological objects to combinatorial objects (simplicial complexes, triangulations of spheres, polytopes, etc). These constructions allow to employ methods of homological algebra, algebraic geometry and algebraic topology in purely combinatorial problems.

Following a similar path, we derive a series of inequalities on the f-vectors of flag simplicial complexes. Our talk is built upon the results of Denham, Suciu [1] and Panov, Ray [2], where the authors relate the Poincaré series of a face ring of a flag simplicial complex to the Poincaré series of a free graded algebra. The main result can be formulated as follows.

Theorem 1 ([3, Thm. 1.1]) *Let $\mathcal{K}$ be a flag simplicial complex with f-vector $(f_0, \dots, f_n)$. Then for any $N \geq 1$ we have*

$$(-1)^N \sum_{d|N} \mu(N/d)(-1)^d p_d(\underline{\alpha}) \geq 0, \qquad (1)$$

where p_d is d-th Newton polynomial expressed in elementary symmetric polynomials $\underline{\alpha} = (\alpha_1, \alpha_2, \dots)$ with

$$\alpha_n := \sum_{i=0}^{n-1} f_i \binom{n-1}{i},$$

$\mu(n)$ is the Möbius function

$$\mu(n) = \begin{cases} (-1)^k, & \text{if } n \text{ is a product of } k \text{ distinct prime factors;} \\ 0, & \text{otherwise,} \end{cases}$$

and the summation is taken over all positive divisors of N.

References

[1] G. Denham and A. I. Suciu, *Moment-angle complexes, monomial ideals and Massey products*, Pure Appl. Math. Q., 3 (2007), pp. 25–60.

[2] T. E. Panov and N. Ray, *Categorical aspects of toric topology*, in Toric topology, vol. 460 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2008, pp. 293–322.

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