# Toric topology of balanced simplicial complex

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Stanley introduced in [4] an important class of simplicial complexes that arise often in combinatorics, topology and algebra so called balanced simplicial complexes. A $n$-dimensional simplicial complex $K$ is called balanced if its set of vertices can be splitted into $n$ disjoint subsets such that there is no two vertices spanning the edge of $K$ and belonging to the same subset. One of central questions in combinatorics is description of all integer vectors that may appear as the face vectors of convex polytopes. The most celebrated result in this problematic is the famous $g$-theorem.

In the paper [2] Klee and Novik conjectured a stronger bound concerning the face numbers of balanced simplicial $n$-polytopes. The conjecture known as the Balanced Lower Bound Theorem is proved in [1, Theorem 1.3] ("if" part) and in [2, Theorem 5.8] ("only if" part).

**Theorem 1 (Balanced Lower Bound Theorem)** *Let $P$ be a balanced simplicial $n$-polytope. Then*

$$\frac{h_0(P)}{\binom{n}{0}} \leq \frac{h_1(P)}{\binom{n}{1}} \leq \cdots \leq \frac{h_{\lfloor \frac{n}{2} \rfloor}}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}. \tag{1}$$

*The equality $\frac{h_{i-1}(P)}{\binom{n}{i-1}} = \frac{h_i(P)}{\binom{n}{i}}$ for some $i \leq \frac{n}{2}$ if and only if $P$ has the balanced $(i-1)$-stacked property.*

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The proof of the Balanced Generalized Lower Bound Theorem given by Juhnke-Kubitzke and Murai in [1] is based on the existence of so called linear system of parameters and a Lefschetz element of the Stanley-Reisner ring of simplicial polytopes. Corollaries have insightful topological meanings for canonical quasitoric manifolds which we explain in this contribution.

# References


[1] M. Juhnke-Kubitzke and S. Murai, *Balanced Generalized Lower Bound Inequality for Simplicial Polytopes*, Preprint, https://arxiv.org/abs/1503.06430

[2] S. Klee and I. Novik, *Lower Bound Theorems and a Generalized Lower Bound Conjecture for balanced simplicial complexes*, Mathematika, **62** (2016), 441-477.

[3] S. Murai and E. Nevo, *On the generalized lower bound for simplicial polytopes and spheres*, Acta Math. **210** (2013), 185-202.

[4] R. Stanley, *Balanced Cohen-Macaulay complexes*, Trans. Amer. Math. Soc. **249** (1979), no. 1, 139-157.