The cohomology rings of Hessenberg varieties and Schubert polynomials

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Let n be a positive integer. The **(full) flag variety** $\mathcal{F}\ell(\mathbb{C}^n)$ in $\mathbb{C}^n$ is the collection of nested linear subspaces $V_\bullet := (V_1 \subset V_2 \subset \ldots \subset V_n = \mathbb{C}^n)$ where each V_i is an i-dimensional subspace in $\mathbb{C}^n$. Considering a linear map $X : \mathbb{C}^n \to \mathbb{C}^n$ and a weakly increasing function $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ satisfying $h(j) \geq j$ for $j = 1, \ldots, n$, called a **Hessenberg function**, a **Hessenberg variety** is defined by

$$\mathrm{Hess}(X, h) := \{V_\bullet \in \mathcal{F}\ell(\mathbb{C}^n) \mid XV_i \subseteq V_{h(i)} \text{ for } i = 1, \ldots, n\}.$$

Here we concentrate on Hessenberg varieties $\mathrm{Hess}(N, h)$ when $X = N$ a nilpotent matrix whose Jordan form consists of exactly one Jordan block. We define a polynomial

$$f_{i,j} := \sum_{k=1}^{j} \Big(\prod_{\ell=j+1}^{i} (x_k - x_\ell) \Big) x_k \tag{1}$$

for $1 \leq j \leq i \leq n$. Here, we take by convention $\prod_{\ell=j+1}^{i}(x_k - x_\ell) = 1$ whenever $i = j$. From the result of [1], the following isomorphism as $\mathbb{Q}$-algebras holds

$$H^*(\mathrm{Hess}(N, h); \mathbb{Q}) \cong \mathbb{Q}[x_1, \ldots, x_n]/(f_{h(1),1}, f_{h(2),2}, \ldots, f_{h(n),n}).$$

Moreover, there is a surprising connection that this presentation can be obtained from a hyperplane arrangement ([2]). The main theorem is as follows.

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Theorem 1 ([3]) *Let i, j be positive integers with $1 \leq j < i \leq n$. Then the polynomial $f_{i-1,j}$ in (1) can be written as an alternating sum of certain Schubert polynomials $\mathfrak{S}_{w_k^{(i,j)}}$:*

$$f_{i-1,j} = \sum_{k=1}^{i-j} (-1)^{k-1} \mathfrak{S}_{w_k^{(i,j)}} \tag{2}$$

where $w_k^{(i,j)}$ $(1 \leq k \leq i-j)$ is a permutation on n letters $\{1, 2, \ldots, n\}$ defined by $(s_{i-k}s_{i-k-1} \ldots s_j)(s_{i-k+1}s_{i-k+2} \ldots s_{i-1})$ using the transpositions s_r of r and $r+1$. Here, we take by convention $(s_{i-k+1}s_{i-k+2} \ldots s_{i-1}) = \mathrm{id}$ whenever $k = 1$.

We can interpret the equality (2) in Theorem 1 from a geometric viewpoint under the circumstances of having a codimension one Hessenberg variety $\mathrm{Hess}(N, h')$ in the original Hessenberg variety $\mathrm{Hess}(N, h)$.

References

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