и выполняется условие максимума

$$u_*(t)\psi^1(t) \overset{\text{п.в.}}{=} \max_{u\in[0,1]}\{u\psi^1(t)\}.$$

Можно показать, что особые режимы в задаче (P) отсутствуют. Данное обстоятельство позволяет свести решение задачи (P) к исследованию краевой задачи принципа максимума.

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An existence theorem for infinite-horizon optimal control problems and its application to a model of optimal exploitation of a renewable resource

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Consider the following problem (P):

$$J(x(\cdot),u(\cdot)) = \int_0^\infty f^0(t,x(t),u(t))\,dt \to \max,$$

$$\dot{x}(t) = f(t,x(t),u(t)), \qquad x(0) = x_0,$$

$$u(t) \in U.$$

Here $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $t \geq 0$, U is a nonempty closed (not necessary bounded) set in $\mathbb{R}^m$, and $x_0 \in G$ where G is an open convex set in $\mathbb{R}^n$. The class of *admissible controls* consists of all $u(\cdot) \in L^\infty_{\mathrm{loc}}([0,\infty),\mathbb{R}^m)$ such that $u(t) \in U$ for all $t \geq 0$. It is assumed that for any $u(\cdot)$ the corresponding *admissible trajectory* $x(\cdot)$ exists on $[0,\infty)$ in G and the function $t \mapsto f^0(t,x(t),u(t))$ is locally integrable on $[0,\infty)$. An admissible pair $(x_*(\cdot),u_*(\cdot))$ is *optimal* in problem (P) if the integral functional $J(x(\cdot),u(\cdot))$ converges and for any other admissible pair $(x(\cdot),u(\cdot))$ the following inequality holds:

$$J(x_*(\cdot),u_*(\cdot)) \geq \limsup_{T\to\infty} \int_0^T f^0(t,x(t),u(t))\,dt.$$

Assume that the following conditions take place:

(A1) *For a.e. $t \in [0,\infty)$ the partial derivatives $f_x(t,x,u)$ and $f^0_x(t,x,u)$ exist for all $(x,u) \in G \times U$. The functions $f(\cdot,\cdot,\cdot)$, $f^0(\cdot,\cdot,\cdot)$, $f_x(\cdot,\cdot,\cdot)$ and $f^0_x(\cdot,\cdot,\cdot)$ are measurable in t for all $(x,u) \in G\times U$, continuous in (x,u) for a.e. $t \in [0,\infty)$ and locally bounded.*

(A2) *For an arbitrary admissible pair $(x_*(\cdot),u_*(\cdot))$ there exist a $\beta > 0$ and an integrable function $\lambda\colon [0,\infty) \mapsto \mathbb{R}^1$ such that for any $\zeta \in G$ with $\|\zeta - x_0\| < \beta$, there is a solution $x(\zeta;\cdot)$ to the Cauchy problem*

$$\dot{x}(t) = f(t,x(t),u_*(t)), \qquad x(0) = \zeta,$$

which is defined on $[0,\infty)$, lies in G, and

$$\max_{x\in[x(\zeta;t),x_*(t)]} \big|\big\langle f^0_x(t,x,u_*(t)), x(\zeta;t) - x_*(t)\big\rangle\big| \overset{\text{a.e.}}{\leq} \|\zeta - x_0\|\lambda(t).$$

(A3) *For any $M > 0$ there is a compact set $U_M \subset U$ such that $\{u \in U\colon \|u\| \leq M\} \subset U_M$ and for a.e. $t \geq 0$ for all $x \in G$ the set*

$$Q_M(t,x) = \big\{(z^0,z) \in \mathbb{R}^{n+1}\colon\ z^0 \leq f^0(t,x,u),\ z = f(t,x,u),\ u \in U_M\big\}$$

is convex.

(A4) *There is a positive function $\omega\colon [0,\infty) \mapsto \mathbb{R}^1$, $\omega(t) \to +0$ as $t \to \infty$, such that $\int_T^{T'} f^0(t,x(t),u(t))\,dt \leq \omega(T)$, $0 \leq T \leq T'$, for any admissible pair $(x(\cdot),u(\cdot))$.*

For an arbitrary $(x(\cdot),u(\cdot))$ denote by $Z(\cdot)$ the normalized fundamental matrix solution of the linear system $\dot{z}(t) = -[f_x(t,x(t),u(t))]^* z(t)$ and put

$$\psi_T(t) = Z(t)\int_t^T Z^{-1}(s) f^0_x(s,x(s),u(s))\,ds, \qquad 0 \leq t \leq T, \quad T > 0.$$

The following existence result does not assume any uniform boundedness of admissible controls (see [1] for details).

Theorem 1. *Assume that there is an admissible pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ such that $J(\bar{x}(\cdot), \bar{u}(\cdot)) > -\infty$. Assume also that there are a continuous nonnegative function $M\colon [0,\infty) \mapsto \mathbb{R}^1$ and a positive function $\delta\colon [0,\infty) \mapsto \mathbb{R}^1$, $\lim_{t\to\infty} \delta(t)/t = 0$, such that for any admissible pair $(x(\cdot), u(\cdot))$ that satisfies on a set $\mathfrak{M} \subset [0,\infty)$, $\operatorname{meas}\mathfrak{M} > 0$, for all $t \in \mathfrak{M}$ the inequality $\|u(t)\| > M(t)$, for a.e. $t \in \mathfrak{M}$ for all $T \geq t + \delta(T)$ we have*

$$\sup_{u\in U\colon \|u\|\leq M(t)} \mathcal{H}(t, x(t), u, \psi_T(t)) - \mathcal{H}(t, x(t), u(t), \psi_T(t)) > 0.$$

Then there is an optimal admissible control $u_*(\cdot)$ in problem (P) and for a.e. $t \geq 0$ the following estimate is true:

$$\|u_*(t)\| \leq M(t). \tag{1}$$

Moreover, if for a.e. $t \in \mathfrak{M}$ we have

$$\inf_{\substack{T>0\colon\\ t\leq T-\delta(T)}} \left\{ \sup_{u\in U\colon \|u\|\leq M(t)} \mathcal{H}(t, x(t), u, \psi_T(t)) - \mathcal{H}(t, x(t), u(t), \psi_T(t)) \right\} > 0,$$

then estimate (1) is true for any optimal admissible control $u_*(\cdot)$ in (P).

The following problem $(P1)$ is a model of optimal exploitation of a renewable resource:

$$J(S(\cdot), u(\cdot)) = \int_0^\infty e^{-\rho t}[\ln S(t) + \ln u(t)]\, dt \to \max,$$

$$\dot{S}(t) = rS(t)\left(1 - \frac{S(t)}{K}\right) - u(t)S(t), \qquad S(0) = S_0 > 0,$$

$$u(t) \in (0,\infty).$$

Here the class of admissible controls consists of all $u(\cdot) \in L^\infty_{\text{loc}}([0,\infty), \mathbb{R}^1)$ such that $u(t) \in (0,\infty)$ for all $t \geq 0$. Obviously, for any admissible trajectory $S(\cdot)$ we have $S(t) \in G = (0,\infty)$.

Using the Bernoulli transformation $x(t) = 1/S(t)$, $t \geq 0$, one can prove that $(P1)$ is equivalent to the following problem $(P2)$ (see [2, 3]):

$$J(x(\cdot), u(\cdot)) = \int_0^\infty e^{-\rho t}[\ln u(t) - \ln x(t)]\, dt \to \max,$$

$$\dot{x}(t) = (u(t) - r)x(t) + a, \qquad x(0) = x_0 = \frac{1}{S_0},$$

$$u(t) \in [\rho, \infty).$$

Here $a = r/K$ and the set of admissible controls consists of all locally bounded measurable functions $u\colon [0,\infty) \mapsto [\rho,\infty)$.

The application of Theorem 1 to problem $(P2)$ implies the following result.

Theorem 2. *There is an optimal admissible control* $u_*(\cdot)$ *in problem* $(P2)$ (*and hence in* $(P1)$). *Moreover, for any optimal admissible pair* $(x_*(\cdot), u_*(\cdot))$ *the following inequality takes place*:

$$u_*(t) \overset{\text{a.e.}}{\leq} \left(1 + \frac{1}{Kx_*(t)}\right)(r + \rho), \qquad t \geq 0.$$

Notice that the Hamiltonian of problem $(P2)$ is not concave. This fact considerably complexifies the application of standard sufficient optimality conditions of Arrow's type to problem $(P2)$. However, Theorem 1 justifies the application of an appropriate version of the Pontryagin maximum principle for infinite-horizon problems (see [4]) to problem $(P2)$ (see [2, 3] for more details).

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