18. *Lawler E.L.* Efficient implementation of dynamic programming algorithms for sequencing problems: CWI. Technical Reports. Stichting Mathematish Centrum. Mathematische Besliskunde. 1979. BW 106/79. P. 1–16.

On the stationarity conditions in an optimal control problem for a trajectory with smooth boundary contact on a single interval*

Andrei Dmitruk, Ivan Samylovskiy

Lomonosov Moscow State University, Moscow, Russia

dmitruk@member.ams.org, ivan.samylovskiy@cosmos.msu.ru

Our aim is to apply the technique of the reduction of an optimal control problem demonstrated in [1, 2, 4] to a problem with state constraints and to obtain a full system of stationarity conditions, including the nonnegativity of the density of measure (the state constraint multiplier) and the signs of the jumps of the measure at junction points, by two-stage variation approach including the reduction from the pure state-constrained problem to a mixed control-state constrained one.

In this work we generalize results obtained in [6, 7] to a problem with state constraint of order 2 (i.e., with smooth contact with state boundary) of the following base class:

$$\begin{cases} \dot z = f(z,y,x,u), \qquad \varphi_s(u(t)) \le 0, \\ \dot y = x, \qquad y(t) \ge 0, \\ \dot x = g(z,y,x,u), \\ J_A \to \min, \quad \text{where} \quad J_A = J(z(0), z(T), y(0), y(T), x(0), x(T)). \end{cases}$$

Here, $z \in \mathbb{R}^n$, $y \in \mathbb{R}^1$ and $x \in \mathbb{R}^1$ are state variables, $u \in \mathbb{R}^m$ is a control, the functions $z(\cdot)$, $y(\cdot)$ and $x(\cdot)$ are absolutely continuous, $u(\cdot)$ is measurable and bounded. We will assume that the functions f, g and φ of dimensions n, 1 and $\mathrm{d}(\varphi)$, respectively, are defined and continuous (moreover, f and g are Lipschitz together with their first and second derivatives

*Supported by the Russian Foundation for Basic Research (project nos. 16-01-00585 and 18-31-00091).

w.r.t. z, y, x, u) on an open subset $\mathcal{Q} \subset \mathbb{R}^{n+1+m}$ together with their first-order partial derivatives w.r.t. z, x, u. (The function $\varphi(u)$ can be formally considered as a function of the variables z, x, u.)

We consider a process $w^0 = (z^0, y^0, x^0, u^0)$ such that the trajectory $y^0(t)$ touches the state boundary only on a segment $[t_1^0, t_2^0]$, where $0 < t_1^0 < t_2^0 < T$. In addition, we suppose the control u^0 to be continuous on Δ_1, Δ_3 and Lipschitz continuous on Δ_2 with its derivative (for convenience, we assume that the function u^0 at the time moments t_1^0, t_2^0 has both left and right values); moreover, $\varphi_s(u^0(t)) < 0$ on Δ_2 for all s, and the equalities

$$\dot{y}^0(t_1^0 - 0) = x^0(t_1^0 - 0) = 0, \qquad \dot{y}^0(t_2^0 + 0) = x^0(t_2^0 + 0) = 0$$

and the strict inequalities

$$\dot{x}^0(t_1^0 - 0) > 0, \qquad \dot{x}^0(t_2^0 + 0) > 0$$

hold at the moments t_1^0, t_2^0, which means that reaching the state boundary and leaving it occur with zero first and nonzero second time derivatives of $y(t)$.

We also suppose that $g'_u(z^0(t), y^0(t), x^0(t), u^0(t)) \neq 0$ on the boundary arc Δ_2, i.e., that the state constraint is of order 2, and the gradients $\varphi'_s(u^0(t))$, $s \in I(u^0(t))$, are positive independent for all $t \in \Delta_1 \cup \Delta_3$ (i.e., their nontrivial linear combination with non-negative coefficients cannot vanish). Here $I(u) = \{s \colon \varphi_s(u) = 0\}$ is the set of active indices.

Reducing the state constraint $y(t) \geq 0$ to the triple of terminal constraint $y(t_1) \geq 0$, terminal constraint $x(t_1) = 0$ and mixed control-state constraint $\dot{x} = 0$ on Δ_2 and replicating variables according to [3], we come to the following optimal control problem on the time interval $\tau \in [0, 1]$:

$$\text{minimize} \quad J_B := J\left(r_1(0),\, r_3(1),\, \xi_1(0),\, \xi_3(1)\, \eta_1(0),\, \eta_3(1)\right)$$

under the following constraints:

$$\frac{dr_i}{d\tau} = \rho_i f(r_i, \xi_i, \eta_i, v_i), \quad r_1(1) = r_2(0), \quad r_2(1) = r_3(0),$$

$$\frac{d\xi_i}{d\tau} = \rho_i \eta_i, \quad \xi_1(1) - \xi_2(0) = 0, \quad \xi_2(1) - \xi_3(0) = 0, \quad \xi_2(0) \geq 0,$$

$$\frac{d\eta_i}{d\tau} = \rho_i g(r_i, \xi_i, \eta_i, v_i), \quad \eta_1(1) - \eta_2(0) = 0, \quad \eta_2(1) - \eta_3(0) = 0, \quad \eta_2(0) = 0,$$

$$\frac{dt_i}{d\tau} = \rho_i, \quad t_1(0) = 0, \quad t_1(1) = t_2(0), \quad t_2(1) = t_3(0), \quad t_3(1) = T,$$

$$g(r_2, \xi_2, \eta_2, v_2) \equiv 0, \qquad \varphi(v_1(\tau)) \le 0, \qquad \varphi(v_3(\tau)) \le 0.$$

By formulating necessary stationarity conditions [3] in problem B and rewriting them in terms of the original problem, we perform the first stage of our two-stage variation method. To obtain the conditions of the nonnegativity of the state constraint multiplier and jumps of the adjoint variable at the junction points t_1^0 and t_2^0, we perform the second stage of the variation method, which includes variations $\bar{x}(t)$ concentrated on Δ_2.

We prove that for a process $w^0 = (z^0, y^0, x^0, u^0)$ providing the extended weak minimum [5] in problem A and for a solution $\bar{w} = (\bar{z}, \bar{y}, \bar{x}, \bar{u})$ of the corresponding differential equation in variations we get

$$J'(w^0)\,\bar{w} = -\Delta\psi_x(t_1^0)\,\bar{x}(t_1^0) - \Delta\psi_x(t_2^0)\,\bar{x}(t_2^0) - \Delta\psi_y(t_1^0)\,\bar{y}(t_1^0) \\ - \Delta\psi_y(t_2^0)\,\bar{y}(t_2^0) + \int_{\Delta_2} (-\ddot{m})\,\bar{y}\,dt \ge 0$$

for any W^2_∞-function $\bar{y}(t) = \varkappa(t) > 0$. Considering special $\varkappa$ "concentrated" inside Δ_2^0 or in the neighbourhood of $t_{1,2}^0$, we get $-\ddot{m}(t) \ge 0$ on Δ_2 (i.e., $m(t)$ is concave on Δ_2), $\psi_y(t_i) = 0$ and $\psi_x(t_i) \le 0$, where $-\ddot{m}$ is the state constraint multiplier.

Moreover, we formulate necessary conditions in the original problem using the measure

$$\mu(t) = \begin{cases} \alpha_1(t_1^0 - t) + \dot{m}(t_1^0 + 0)t - 2m(t_1^0 + 0) + \beta_{10} & \text{on } \Delta_1, \\ [2pt] - m(t) + \dot{m}(t_1^0 + 0)t & \text{on } \Delta_2, \\ [2pt]\dot{m}(t_1^0 + 0)t - 2m(t_2^0 - 0) & \text{on } \Delta_3 \end{cases}$$

and write the jumps of the adjoint variables in a "symmetrical" form

$$\Delta\psi_x(t_i^0) = -\Delta\mu(t_i^0) = 0, \qquad \Delta\psi_y(t_i^0) = -\Delta\mu(t_i^0) \le 0$$

and nonnegativity of a state constraint multiplier in the form $\ddot{\mu}(t) \ge 0$.

References

1. *Denbow C.H.* A generalized form of the problem of Bolza // Contributions to the calculus of variations, 1933–1937. Univ. Chicago Press, 1937. P. 180–215.
2. *Warga J.* The reduction of certain control problems to an "ordinary differential" type // SIAM Rev. 1968. V. 10, N 2. P. 219–222.
3. *Milyutin A.A., Osmolovskii N.P.* Calculus of variations and optimal control. Am. Math. Soc., 1998.

4. *Dmitruk A.V., Kaganovich. A.A.* Maximum principle for optimal control problems with intermediate constraints // Comput. Math. Model. 2011. V. 22, N 2. P. 180–215.
5. *Dmitruk A. V., Osmolovskii N.P.* Necessary conditions for a weak minimum in optimal control problems with integral equations on a variable time interval // Discrete Contin. Dyn. Syst. 2015. V. 35, N 9. P. 4323–4343.
6. *Dmitruk A.V., Samylovskiy. I.A.* On two approaches to necessary conditions for an extended weak minimum in optimal control problems with state constraints // 2016 Int. Conf. "Stability and Oscillations Of Nonlinear Control Systems" (Pyatnitskiy's Conference). IEEE, 2016. P. 1–5.
7. *Dmitruk A.V., Samylovskiy. I.A.* On the relation between two approaches to necessary optimality conditions in problems with state constraints // J. Optim. Theory Appl. 2017. V. 173, N 2. P. 391–420.

О некоторых задачах позиционного граничного управления для волнового уравнения (On some positional boundary control problems for the wave equation)*

А. А. Дряженков (A. A. Dryazhenkov), М. М. Потапов (M. M. Potapov)

Московский государственный университет им. М.В. Ломоносова, Москва, Россия

andrja@yandex.ru, mmpotapovrus@gmail.com

Рассматривается следующая пространственно-одномерная динамическая система с граничным управлением:

$$y_{tt}(t,x) = y_{xx}(t,x) - q(x)y(t,x), \qquad 0 < t < T, \quad 0 < x < l,$$

$$-y_x + \sigma_0 y\,|_{x=0} = u(t), \qquad y_x + \sigma_1 y\,|_{x=l} = 0, \qquad 0 < t < T,$$

$$y\,|_{t=0} = v^0(x), \qquad y_t\,|_{t=0} = v^1(x), \qquad 0 < x < l.$$

*Исследование выполнено за счет гранта Российского научного фонда (проект 14-11-00539).