On optimal solutions in a problem with two-dimensional bounded control*

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We consider an optimal control problem that is affine in two-dimensional control. The origin is a singular trajectory in this problem. We study the structure of optimal solutions in a neighborhood of the origin. We use the resolution of singularity via blow up and the invariant manifold theorems to find a family of optimal solutions.

Consider the following optimal control problem:

$$\int_0^\infty \langle x(t), x(t)\rangle\, dt \to \min, \tag{1}$$

$$\dot{x} = y, \qquad \dot{y} = Kx + u, \tag{2}$$

$$x(0) = x^0, \qquad y(0) = y^0, \tag{3}$$

$$\|u(t)\| \le 1. \tag{4}$$

Here $x, y, u \in \mathbb{R}^2$, K is a 2×2 diagonal matrix, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ are the scalar product and the standard Euclidean norm on $\mathbb{R}^2$. If (x^0, y^0) are sufficiently close to the origin, then there exists a unique solution to (1)–(4). Under additional assumptions on the matrix K, optimal solutions exist for all (x^0, y^0).

We use the Pontryagin maximum principle. It can be shown that the problem is regular; that is, we can define the Hamiltonian as

$$H(x, y, \phi, \psi, u) = -\frac{1}{2}\langle x, x\rangle + \langle y, \phi\rangle + \langle Kx, \psi\rangle + \langle u, \psi\rangle$$

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where ϕ and ψ are adjoint functions. The Hamiltonian system of the Pontryagin maximum principle has the form

$$\begin{aligned} \dot{\phi} &= -\frac{\partial H}{\partial x} = x - K\psi, \qquad \dot{\psi} = -\frac{\partial H}{\partial y} = -\phi, \\ \dot{x} &= \frac{\partial H}{\partial \phi} = y, \qquad \dot{y} = \frac{\partial H}{\partial \psi} = Kx + \hat{u} \end{aligned} \tag{5}$$

where the optimal control $\hat{u}(t)$ is determined by the maximum condition

$$\begin{aligned} H(x(t), \phi(t), \psi(t), \hat{u}(t)) &= \max_{\|u(t)\| \le 1} H(x(t), \phi(t), \psi(t), u) \\ &= -\frac{1}{2}\langle x, x\rangle + \langle y, \phi\rangle + \langle Kx, \psi\rangle + \max_{\|u(t)\| \le 1} \langle u, \psi\rangle. \end{aligned} \tag{6}$$

Because of that, we get $\hat{u}(t) = \psi(t)/\|\psi(t)\|$ if $\psi(t) \neq 0$. If $\psi = 0$, then any admissible control meets (6). Denote $z_1 = \psi$, $z_2 = -\phi$, $z_3 = -x$, $z_4 = -y$. We can rewrite system (5) as follows:

$$\begin{aligned} &\dot{z}_1 = z_2, \qquad \dot{z}_2 = z_3 + Kz_1, \\ &\dot{z}_3 = z_4, \qquad \dot{z}_4 = -\hat{u} + Kz_3, \qquad \hat{u} = \frac{z_1}{\|z_1\|}. \end{aligned} \tag{7}$$

Put $z = (z_1, z_2, z_{3,} z_4) \in \mathbb{R}^8$. A solution $z(t)$ of (7) is said to be *singular* on an interval (t_1, t_2) if $z_1(t) = 0$ for all $t \in (t_1, t_2)$. For (1)–(4), $z(t) = 0$ is a unique singular solution. It was proved [1] that optimal solutions, starting from a small enough neigbourhood of the origin, reach zero in finite time T which depends on (x^0, y^0). Moreover, the optimal control $\hat{u}(t)$ does not have a limit at $t \to T-0$. It was shown in [2] that if the initial data x^0 and y^0 are colinear, then the solutions $x(t)$ and $y(t)$ are colinear for every t. In this case the behavior of the optimal solutions of problem (1)–(4) in the neigbourhood of the origin is similar to the optimal synthesis of the Fuller problem with a scalar control. More precisely, the optimal control $\hat{u}(t)$ has an infinite number of switchings on a finite time interval, i.e., $\hat{u}(t)$ is a chattering control.

In the following, we use the complex notation for vectors in $\mathbb{R}^2$:

$$Re^{i\varphi} = (R\cos\varphi, R\sin\varphi).$$

In [2, 3] for system (7) with $K = 0$ a family of optimal solutions $\hat{z}(t) = (\hat{z}_1(t), \hat{z}_2(t), \hat{z}_3(t), \hat{z}_4(t))$, $0 \le t < T$, was found:

$$\begin{aligned} \hat{z}_m(t) &= -BA_{m-1}(T-t)^{5-m} e^{i\alpha \ln|T-t|}, \qquad m = \overline{1,4}, \\ \hat{u}(t) &= -Be^{i\alpha \ln|T-t|}, \end{aligned} \tag{8}$$

where $B \in \mathcal{SO}(2)$, $i^2 = -1$, $\alpha = \pm\sqrt{5}$, $A_0 = 1/126$, $A_{j+1} = -A_j(4-j+i\alpha)$, $j = 0, 1, 2$.

Note that the trajectories (8) hit the origin in a finite time T. Moreover, the optimal control $\hat{u}(t)$ performs an infinite number of rotations along the circle S^1. If $K = 0$ then system (7) is homogeneous with respect to the action of the Fuller group. In [2, 3] this property was used to find the logarithmic spirals (8). If $K \neq 0$ then (7) does not possess this property. However, we will show that in this case there are similar optimal logarithmic spirals.

Theorem. *In a sufficiently small neighborhood of the origin there exist the following solutions to* (7):

$$z_m^*(t) = C_m\,(T_* - t)^{5-m} e^{i\alpha \ln|T_* - t|}(1 + o(1)), \qquad m = \overline{1,4},$$

$$u^*(t) = C_0\, e^{i\alpha \ln|T_* - t|}(1 + o(1)), \qquad t \to T_* - 0,$$

and all its possible rotations. Here $0 < T_* < \infty$ *is a time at which* $z^*(t)$ *hits the origin* (*hitting time*), $C_m \in \mathbb{C}$, $m = \overline{0,4}$*. The constants* T_* *and* C_m *depend on* $z^*(0)$.

Corollary. *There exist optimal solutions to problem* (1)–(4) *of the following form*:

$$x^*(t) = -C_3\,(T_* - t)^2 e^{i\alpha \ln|T_* - t|}(1 + o(1)),$$

$$y^*(t) = -C_4\,(T_* - t)\, e^{i\alpha \ln|T_* - t|}(1 + o(1)),$$

$$u^*(t) = C_0\, e^{i\alpha \ln|T_* - t|}(1 + o(1)), \qquad t \to T_* - 0,$$

To prove the theorem, we use the procedure of resolution of singularity for the Hamiltonian system (7). We use the same scheme as in [2, 4] and a similar change of coordinates. Doing this turns (7) into a system that is a small perturbation of the corresponding system with $K = 0$. In this case in the new coordinates the logarithmical spirals (8) turn into a periodic trajectory (cycle) for the case $K = 0$ as well as for $K \neq 0$. Then our main result follows from the invariant manifold theorems for the limit cycle [5].

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Динамическая реконструкция входов диффузионной стохастической системы (Dynamical reconstruction of inputs in a stochastic diffusion system)*

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Задачи реконструкции входов динамических систем на основе неточной и/или неполной информации о фазовом состоянии, возникающие во многих научных и прикладных исследованиях, как правило, являются некорректными и требуют применения регуляризующих процедур. Подход к решению, предложенный в работах А.В. Кряжимского и Ю.С. Осипова [1] изначально для обыкновенных дифференциальных уравнений (ОДУ) и получивший название метода динамического обращения, основан на сочетании принципов теории позиционного управления и идей теории некорректных задач. Задача восстановления сводится к задаче управления по принципу обратной связи вспомогательной динамической системой (моделью), при этом адаптация модельных управлений к результатам текущих наблюдений обеспечивает аппроксимацию неизвестных входных воздействий. Обзор алгоритмов динамического восстановления входов для систем ОДУ приведен в [2].

В докладе с позиций указанного подхода исследуется задача для системы стохастических дифференциальных уравнений (СДУ) с диффузией, зависящей от фазового состояния, в постановке, в которой

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