Explicit counterexamples to Schäffer's conjecture, Fourier coefficients of Blaschke products and Jacobi polynomials with varying parameters

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We discuss the results that we found on our way to a deeper understanding of Schäffer's conjecture about inverse operators. Three topics will be covered:

1. J.J. Schäffer proved that for any induced matrix norm and any $n \times n$ invertible matrix T the inequality

$$|\det T| \|T^{-1}\| \leq \mathcal{S}_n \|T\|^{n-1}$$

holds with $\mathcal{S}_n \leq \sqrt{en}$. He conjectured that the best $\mathcal{S}_n$ was actually bounded. This was rebutted by Gluskin–Meyer–Pajor and by subsequent contributions from J. Bourgain and H. Queffélec that successively improved lower estimates on $\mathcal{S}_n$. Yet the construction of explicit counterexamples to the conjecture remains open since 22 years: The mentioned articles relate Schäffer's conjecture to the theory of power sums of complex numbers. A probabilistic or number theoretic analysis proves the existence of T with growing $\mathcal{S}_n$ but the explicit construction of such T boils down to Turán's power sum problems. We propose a constructive approach to Schäffer's conjecture that is not related to power sum theory. As a consequence we present an explicit sequence of Toeplitz matrices with singleton spectrum $\{\lambda\} \subset \mathbb{D} \setminus \{0\}$ such that $\mathcal{S}_n \geq c(\lambda)\sqrt{n}$. Our framework naturally extends to provide lower estimates on the resolvent $\|(\zeta - T)^{-1}\|$ when $\zeta \neq 0$. We also obtain new upper estimates on the resolvent when the spectrum is given. This yields new upper bounds on $\|T^{-1}\|$ in terms of the eigenvalues of T to slightly refine Schäffer's original estimate.

2. A key ingredient in our approach will be to investigate l_p-norms of Fourier coefficients of powers of a Blaschke factor, which is an interesting and well-studied topic in its own right, initiated by J-P. Kahane in 1956.

3. Finally, on our way, we prove new estimates for the asymptotic behavior of Jacobi polynomials with varying parameters.

The talk is based on a joint work with Oleg Szehr from the University of Vienna.