О новом методе ординального анализа теории множеств Крипке-Платека.

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Schmerl formula

The arithmetical Π^0_n uniform reflection for theory T

$$\mathsf{RFN}_{\Pi^0_n}(T) : (\forall \varphi \in \Pi^0_n)(\mathsf{Prv}_T(\varphi) \to \mathsf{Tr}_{\Pi^0_n}(\varphi)).$$

For recursive ordinals α we define r.e. theories $\mathsf{R}^{\alpha}_{\Pi^0_n}(T)$:

$$\mathsf{R}^{\alpha}_{\Pi^0_n}(T) = T + \{\mathsf{RFN}_{\Pi^0_n}(\mathsf{R}^{\beta}_{\Pi^0_n}(T)) \mid \beta < \alpha\}.$$

Formally definition is carried out using Fixed Point Lemma.

EA is a weak fragment of PA proving totality of exponentiation.

Schmerl formula:

$$\mathsf{R}^{\alpha}_{\Pi^0_{n+1}}(\mathsf{EA}) \equiv_{\Pi^0_n} \mathsf{R}^{\omega^{\alpha}}_{\Pi^0_n}(\mathsf{EA}), \text{ for } \alpha > 0.$$

Classifying Π^0_2 consequences of PA in terms of iterated reflection

Ordinal $\omega_n = \underbrace{\omega^{\cdot^{\cdot^{\cdot^{\omega}}}}}_{n \text{ times}}$.

$$\mathsf{PA} \equiv \bigcup_{n\in\mathbb{N}} \mathsf{R}_{\Pi^0_n}(\mathsf{EA}).$$

$$\Downarrow$$

$$\mathsf{PA} \equiv_{\Pi^0_2} \bigcup_{n\in\mathbb{N}} \mathsf{R}^{\omega_n}_{\Pi^0_2}(\mathsf{EA}).$$

$$\Downarrow$$

$$\mathsf{PA} \equiv_{\Pi^0_2} \mathsf{R}^{\varepsilon_0}_{\Pi^0_2}(\mathsf{EA}).$$

From reflection to fast-growing functions

$f_\alpha(x)$ is α'th function from fast-growing hierarchy

For any Δ^0_0 formula $\varphi(x, y)$:

$$\mathrm{R}^{\alpha}_{\Pi^0_2}(\mathrm{EA}) \vdash \forall x \exists y\, \varphi(x, y)$$

$$\Downarrow$$

$$\mathrm{R}^{\alpha}_{\Pi^0_2}(\mathrm{EA}) \vdash \forall x (\exists y < f^n_{2+\beta}(x)) \varphi(x, y), \text{ for some } \beta < \alpha \text{ and } n \in \mathbb{N}$$

Hence

$$\mathrm{PA} \vdash \forall x \exists y\, \varphi(x, y)$$

$$\Downarrow$$

$$\mathrm{PA} \vdash \forall x (\exists y < f_\alpha(x)) \varphi(x, y), \text{ for some } \alpha < \varepsilon_0$$

KPω vs PA

Axioms of KP are: Extensionality, Pair, Union, Δ_0-Separation, Δ_0-Collection, and Foundation.
KPω is KP + Infinity.

Transitive models of KP are known as admissible sets.
Analogies between PA and KPω:

PA	KPω
$\mathbb{N}$	admissible sets with ω
r.e. sets	Σ_1 classes
recursive functions	Σ_1 functions
recursive ordinal notations	Δ_0 class well-orderings
ω	On
ε_0	ε_{On+1}
hierarchies of recursive functions	hierarchies of Σ_1 functions $On \to On$
r.e. theories	class-theories with Σ_1 class of axioms

$\mathsf{KP}_0\omega$

Signature: $x \in y$, $\rho(x)$

Abbreavations:

- $x \subseteq y \overset{\text{def}}{\iff} (\forall z \in y)\ z \in x$; $\quad \mathrm{Trans}(x) \overset{\text{def}}{\iff} (\forall y \in x)\ y \subseteq x$;
- $x \in On \overset{\text{def}}{\iff} \mathrm{Trans}(x) \wedge (\forall y \in x)\mathrm{Trans}(y) \wedge (\forall y \subseteq x)(\exists z \in y)(\forall w \in y)\ \neg w \in z$.

Axioms:

1. $\forall z\ (z \in x \leftrightarrow z \in y) \to x = y$ (Extensionality);
2. $\exists z\ (x \in z \wedge y \in z)$ (Pair);
3. $\exists y (\forall z \in x)\ z \subseteq y$ (Union);
4. $\exists y \forall z (z \in y \leftrightarrow z \in x \wedge \varphi(z))$, where $\varphi(z)$ is a Δ_0-forumula (Δ_0-Sep);
5. $(\forall x \in x_0) \exists y\ \varphi(x, y) \to \exists y_0 (\forall x \in x_0)(\exists y \in y_0)\ \varphi(x, y)$, where $\varphi(x, y)$ is a Δ_0-formula (Δ_0-Coll);
6. $\alpha \in \rho(x) \leftrightarrow (\exists y \in x)(\alpha \in \rho(y) \vee \alpha = \rho(y))$;
7. $\rho(x) \in On$;
8. $\exists M\ (\mathrm{Trans}(M) \wedge (\forall y \in M)\ \rho(y) \in M \wedge U^{(M)})$, where U is a finite fragment of 1.–7 (Infinity).

Arithmetics in $\mathrm{KP}_0\omega$

Since we have rank function and Δ_0-Sep, any transitive model M satisfies full scheme of Foundation:

$$\forall x((\forall y \in x)\varphi(y) \to \varphi(x)) \to \forall x\, \varphi(x).$$

In $\mathrm{KP} = \mathrm{KP}_0 +$ Foundation we could define natural numbers with addition and multiplication.

Thus we have Δ_1 definition of addition and multiplication on naturals in $\mathrm{KP}_0\omega$. Moreover, since $\mathrm{KP}_0\omega$ proves full scheme of Foundation on transitive models, it proves full scheme of induction on naturals. Since we have a transitive model above naturals, interpretation of any first-order arithmetical formula is Δ_1.

Class-size language in $\mathrm{KP}_0\omega$

We work in $\mathrm{KP}_0\omega$.

We consider language $\mathbf{L}$ that is extension of the language of $\mathrm{KP}_0\omega$ by constants $\hat{x}$, for all sets x.
We denote by $\mathbf{\Pi}_n$, $\mathbf{\Sigma}_n$ the extension of classes Π_n, Σ_n by all the constants $\hat{x}$.

Formally an L formula is a pair $\langle \varphi, ev \rangle$, where ev is a function with $\mathrm{dom}(ev) = n = \{0, \ldots, n-1\}$ and φ is a formula of the language of $\mathrm{KP}_0\omega$ with constants $c_0, \ldots, c_{n-1}$.

$$\boldsymbol{\varphi} = \varphi[\hat{x}_0, \ldots, \hat{x}_{n-1}/c_0, \ldots, c_{n-1}] \rightsquigarrow \langle \varphi, ev \rangle,$$

where $ev(c_i) = x_i$.

Reflection principles for $\mathsf{KP}_0\omega$

There are partial truth definitions that satisfy biconditionals:

$$\mathsf{KP}_0\omega \vdash \forall \vec{x}(\varphi(\vec{x}) \leftrightarrow \mathsf{Tr}_{\mathbf{\Pi}_n}(\varphi[\dot{\vec{x}}/\vec{x}])),$$

where $\varphi(\vec{x})$ is Π_n.
Moreover for $n \geq 1$ the formula $\mathsf{Tr}_{\mathbf{\Pi}_n}$ is from the class Π_n.
The idea is to define $\mathsf{Tr}_{\mathbf{\Delta}_0}(\varphi)$ as

"there exists a set-size partial truth definition A that satisfies compositional axioms for Δ_0 formulas, covers φ, and assign positive truth value to it."

And then use it to define truth definitons for other classes.

Let T be $\mathbf{L}$ theory given by $\mathbf{\Sigma}_1$ formula defining its class of axioms[1].

$$\mathsf{RFN}_{\mathbf{\Pi}_n}(T) : (\forall \varphi \in \mathbf{\Pi}_n)(\mathsf{Prv}_T(\varphi) \to \mathsf{Tr}_{\mathbf{\Pi}_n}(\varphi)).$$

For $n \geq 1$ the principle $\mathsf{RFN}_{\mathbf{\Pi}_n}(T)$ is equivalent to $\mathbf{\Pi}_n$ formula.

[1] $\mathbf{\Sigma}_1$ is complexity of defining formula of the class, axioms could be of arbitrary complexity

Iterated Reflection

Let $\Lambda = (D_\Lambda, \prec_\lambda)$ be Δ_0 class-size linear order

$$\mathsf{KP}_0\omega \vdash \mathsf{LO}(\Lambda).$$

For $\mathbf{L}$ theories T given by $\mathbf{\Sigma}_1$-classes of axioms[2] and $\alpha \in \Lambda$:

$$\mathsf{R}^{\alpha}_{\mathbf{\Pi}_n}(T) \equiv T + \bigcup_{\beta \prec_\Lambda \alpha} \mathsf{RFN}_{\mathbf{\Pi}_n}(\mathsf{R}^{\beta}_{\mathbf{\Pi}_n}(T)).$$

Formally, we define $\mathbf{\Sigma}_1$-classes of axioms for theories $\mathsf{R}^{a}_{\mathbf{\Pi}_n}(T)$ by a joint arithmetical fixed point.

[2] again, $\mathbf{\Sigma}_1$ is complexity of defining formula of the class, axioms could be of arbitrary complexity

Reflexive Induction

Provability predicate $\mathsf{Prv}_{T+\mathsf{Tr}_{\mathbf{\Sigma}_n}}(\varphi)$:

$$(\exists \psi \in \mathbf{\Sigma}_n)(\mathsf{Tr}_{\mathbf{\Sigma}_n}(\psi) \land \mathsf{Prv}_T(\psi \to \varphi)).$$

Note that

$$\mathsf{KP}_0\omega \vdash \neg\mathsf{Prv}_{T+\mathsf{Tr}_{\mathbf{\Sigma}_n}}(\neg\varphi) \leftrightarrow \mathsf{RFN}_{\mathbf{\Pi}_n}(T+\varphi).$$

Theory $\mathsf{KP}_0\omega$ is closed under Reflexive Induction Rule

$$\frac{(\forall \alpha \in \Lambda)((\forall \beta \prec_\Lambda \alpha)\mathsf{Prv}_{\mathsf{KP}_0\omega+\mathsf{Tr}_{\mathbf{\Sigma}_1}}(\varphi(\hat{\beta})) \to \varphi(\alpha))}{(\forall \alpha \in \Lambda)\varphi(\alpha)}.$$

Using reflexive induction it is easy to show that $\mathsf{KP}_0\omega$ proves usual properties of sequences $\mathsf{R}^{\alpha}_{\mathbf{\Pi}_n}(T)$:

- the class of theorems of $\mathsf{R}^{\alpha}_{\mathbf{\Pi}_n}(T)$ is independent of the choice of the solution of the fixed point-equation on its axiomatization;
- $\mathsf{R}^{\alpha}_{\mathbf{\Pi}_n}(T) \sqsupseteq \mathsf{R}^{\beta}_{\mathbf{\Pi}_n}(T)$, for $\alpha \prec_\Lambda \beta$.

Foundation and iterated reflection

$$\mathsf{KP}\omega = \mathsf{KP}_0\omega + \mathsf{Foundation} \equiv \bigcup_{n\in\mathbb{N}} \mathsf{R}^{On+1}_{\mathbf{\Pi}_n}(\mathsf{KP}_0\omega).$$

Foundation $\Rightarrow$ iterated reflection:
In $\mathsf{KP}\omega$ by induction over ordinals α we have

$$\mathsf{RFN}_{\mathbf{\Pi}_n}(\mathsf{KP}_0\omega + \bigcup_{\beta<\alpha} \mathsf{RFN}_{\mathbf{\Pi}_n}(\mathsf{R}^{\beta}_{\mathbf{\Pi}_n}(\mathsf{KP}_0\omega))).$$

To justify the step by induction over naturals we show that true $\mathbf{\Pi}_n$-axiomatized theory could prove only true $\mathbf{\Pi}_n$ conclusions.

Iterated reflection $\Rightarrow$ foundation:
For any Π_n formula $\varphi(x)$ by reflexive induction theory $\mathsf{KP}_0\omega$ proves that for all ordinals α

$$\mathsf{R}^{\alpha}_{\mathbf{\Pi}_{n+3}}(\mathsf{KP}_0\omega) \to (\forall x((\forall y \in x)\varphi(y) \to \varphi(x)) \to \forall x(\rho(x) \leq \hat{\alpha} \to \varphi(x))).$$

Thus

$$\mathsf{R}^{On+1}_{\mathbf{\Pi}_{n+3}}(\mathsf{KP}_0\omega) \vdash \forall x((\forall y \in x)\varphi(y) \to \varphi(x)) \to \forall x\, \varphi(x).$$

Reduction property

For $n \geq 2$ theory $\mathsf{KP}_0\omega$ proves that for $T \sqsupseteq \mathsf{KP}_0\omega$

$$\mathsf{KP}_0\omega + \mathsf{RFN}_{\mathbf{\Pi}_{n+1}}(T) \equiv_{\mathbf{\Pi}_n} \mathsf{R}^{\omega}_{\mathbf{\Pi}_n}(T).$$

$\sqsupseteq$:

For any $\mathbf{\Pi}_n$ sentence φ:

$$\begin{aligned} \mathsf{KP}_0\omega + \mathsf{RFN}_{\mathbf{\Pi}_{n+1}}(T) + \varphi &\vdash \mathsf{RFN}_{\mathbf{\Pi}_{n+1}}(T+\varphi) \\ &\vdash \mathsf{RFN}_{\mathbf{\Pi}_n}(T+\varphi). \end{aligned}$$

Proof of: $\mathsf{KP}_0\omega + \mathsf{RFN}_{\mathbf{\Pi}_{n+1}}(T) \sqsupseteq_{\mathbf{\Pi}_n} \mathsf{R}^{\omega}_{\mathbf{\Pi}_n}(T)$

We work in PA and prove the inclusion theories in any finite sublanguage of **L**.
We define forcing notion $\Vdash$.
Forcing conditions are $\mathbf{\Pi}_{n-1}$ formulas with free variables

$$\varphi \Vdash A \overset{\mathrm{def}}{\iff} \mathsf{R}^{\omega}_{\mathbf{\Pi}_n}(T) + \varphi \vdash A,$$

for atomic **L** formulas A.

$$\varphi \Vdash \bot \overset{\mathrm{def}}{\iff} \mathsf{R}^{\omega}_{\mathbf{\Pi}_n}(T) + \varphi \vdash \bot.$$

$$\varphi \preceq \psi \overset{\mathrm{def}}{\iff} \mathsf{R}^{\omega}_{\mathbf{\Pi}_n}(T) + \psi \vdash \varphi.$$

We expand $\Vdash$ to all formulas in a standard manner:

- $\varphi \Vdash \psi \wedge \chi \overset{\mathrm{def}}{\iff} \varphi \Vdash \psi$ and $\varphi \Vdash \chi$;
- $\varphi \Vdash \psi \vee \chi \overset{\mathrm{def}}{\iff} \varphi \Vdash \psi$ or $\varphi \Vdash \chi$;
- $\varphi \Vdash \psi \to \chi \overset{\mathrm{def}}{\iff}$ for each $\varphi' \succeq \varphi$: $\varphi' \Vdash \psi$ implies $\varphi' \Vdash \chi$;
- $\varphi \Vdash \exists x\, \psi \overset{\mathrm{def}}{\iff}$ for some variable y: $\varphi \Vdash \psi[y/x]$;
- $\varphi \Vdash \forall x\, \psi \overset{\mathrm{def}}{\iff}$ for all variables y: $\varphi \Vdash \psi[y/x]$.

Proof of: $\mathsf{KP}_0\omega + \mathsf{RFN}_{\mathbf{\Pi}_{n+1}}(T) \sqsupseteq_{\mathbf{\Pi}_n} \mathsf{R}^{\omega}_{\mathbf{\Pi}_n}(T)$

As usual $\neg\varphi$ is $\varphi \to \bot$.
Negative translation ψ^N is contruted by putting $\neg\neg$ prefix over all atomic formulas, all disjunctions, and all existential quantifiers, e.g. $(\exists x\, \psi')^N$ is $\neg\neg\exists x(\psi')^N$.

For each Π_n formula $\psi(\vec{x})$ in $\mathsf{KP}_0\omega$ we could prove that for all vectors of parameters $\vec{p}$ and forcing conditions φ:

$$\varphi \Vdash \psi(\vec{\hat{p}}) \iff \mathsf{R}^{\omega}_{\mathbf{\Pi}_n}(T) + \varphi \vdash (\psi(\vec{\hat{p}}))^N.$$

Proof of: $KP_0\omega + RFN_{\mathbf{\Pi}_{n+1}}(T) \sqsupseteq_{\mathbf{\Pi}_n} R^{\omega}_{\mathbf{\Pi}_n}(T)$

We reformulate reflection principle $RFN_{\mathbf{\Pi}_{n+1}}(T)$:

$$\forall x(x \in \mathbf{\Pi}_n \wedge Tr_{\mathbf{\Pi}_n}(x) \to Con(T + Tr_{\mathbf{\Pi}_n}(\hat{x}))).$$

We claim $\top \Vdash (RFN_{\mathbf{\Pi}_{n+1}}(T))^N$. Consider arbitrary $\varphi \in \mathbf{\Pi}_{n-1}$:

$$\begin{array}{c} \varphi \Vdash (x \in \mathbf{\Pi}_n \wedge Tr_{\mathbf{\Pi}_n}(x))^N \Rightarrow \varphi \Vdash (Con(T + Tr_{\mathbf{\Pi}_n}(\hat{x})))^N \\ \Updownarrow \\ R^{\omega}_{\mathbf{\Pi}_n}(T)+\varphi \vdash x \in \mathbf{\Pi}_n \wedge Tr_{\mathbf{\Pi}_n}(x) \Rightarrow R^{\omega}_{\mathbf{\Pi}_n}(T)+\varphi \vdash Con(T+Tr_{\mathbf{\Pi}_n}(\hat{x})) \\ \Uparrow \\ R^{n}_{\mathbf{\Pi}_n}(T)+\varphi \vdash x \in \mathbf{\Pi}_n \wedge Tr_{\mathbf{\Pi}_n}(x) \Rightarrow R^{n+1}_{\mathbf{\Pi}_n}(T)+\varphi \vdash Con(T+Tr_{\mathbf{\Pi}_n}(\hat{x})). \end{array}$$

The last implication holds since $R^{n+1}_{\mathbf{\Pi}_n}(T) + \varphi \vdash Con(R^{n}_{\mathbf{\Pi}_n}(T) + \varphi)$.

Now by induction on cut-free proofs we show that for all sequents of $\mathbf{\Pi}_{n+2}$ formulas Γ:

$$\vdash \Gamma \Rightarrow \top \Vdash \bigvee \Gamma.$$

Thus for any $\mathbf{\Pi}_n$ formula φ:

$$KP_0\omega + RFN_{\mathbf{\Pi}_n}(T) \vdash \varphi \Rightarrow \top \Vdash \varphi \Rightarrow R^{\omega}_{\mathbf{\Pi}_n}(T) \vdash \varphi.$$

Schmerl Formula

The order ω^Λ:

- domain consists of terms $\omega^{\alpha_0} + \ldots + \omega^{\alpha_{m-1}}$, where $\alpha_0 \succeq_\Lambda \ldots \succeq_\Lambda \alpha_{m-1}$ and $m \geq 1$;
- $\omega^{\alpha_0} + \ldots + \omega^{\alpha_{m-1}} \prec_{\omega^\Lambda} \omega^{\beta_0} + \ldots + \omega^{\beta_{k-1}}$ iff $\langle \alpha_0, \ldots, \alpha_{m-1} \rangle$ is $\prec_\Lambda$-lexicographically less than $\langle \beta_0, \ldots, \beta_{k-1} \rangle$;

By reflexive induction using reduction property we prove Schmerl formula for $n \geq 2$:

$$\mathsf{R}^{\alpha}_{\mathbf{\Pi}_{n+1}}(\mathsf{KP}_0\omega) \equiv \mathsf{R}^{\omega^\alpha}_{\mathbf{\Pi}_n}(\mathsf{KP}_0\omega).$$

ε_{On+1}

ε_Λ is the naturally defined order on terms

- 0;
- ε_α, for $\alpha \in \Lambda$;
- $\omega^{t_0} + \ldots + \omega^{t_{m-1}}$, where t_i are terms and $t_0 \succeq_{\varepsilon_\Lambda} \ldots \succeq_{\varepsilon_\Lambda} t_{m-1}$.

$\mathsf{KP}_1\omega = \mathsf{KP}_0\omega + \Sigma_1$-Foundation. It proves totality of ε_x function. Hence in $\mathsf{KP}_1\omega$ we could construct a bijection between ε_{On} and On. Thus we could show that any proper initial fragment of ε_{On+1} is embeddable in ω_n^{On+1}, for some n.

Reformulating KPω in terms of iterated reflection

For any $m \geq 2$:

$$\mathsf{KP}\omega \equiv \bigcup_{n\in\mathbb{N}} \mathsf{R}^{On+1}_{\mathbf{\Pi}_n}(\mathsf{KP}_0\omega) \sqsubseteq \bigcup_{n\in\mathbb{N}} \mathsf{R}^{\omega_n^{On+1}}_{\mathbf{\Pi}_m}(\mathsf{KP}_0\omega) \sqsubseteq_{\Pi_m} \mathsf{R}^{\varepsilon_{On+1}}_{\mathbf{\Pi}_m}(\mathsf{KP}_0\omega).$$

$$\begin{aligned}
\mathsf{R}^{\varepsilon_{On+1}}_{\mathbf{\Pi}_m}(\mathsf{KP}_0\omega) \sqsubseteq \mathsf{R}^{\varepsilon_{On+1}}_{\mathbf{\Pi}_m}(\mathsf{KP}_1\omega) &\equiv_{\Pi_m} \bigcup_{n\in\mathbb{N}} \mathsf{R}^{\omega_n^{On+1}}_{\mathbf{\Pi}_m}(\mathsf{KP}_1\omega) \\
&\equiv_{\Pi_m} \bigcup_{n\in\mathbb{N}} \mathsf{R}^{On+1}_{\mathbf{\Pi}_{n+m}}(\mathsf{KP}_1\omega) \\
&\equiv \mathsf{KP}\omega.
\end{aligned}$$

Thus for any $m \geq 2$:

$$\mathsf{KP}\omega \equiv_{\Pi_m} \mathsf{R}^{\varepsilon_{On+1}}(\mathsf{KP}_0\omega).$$

Provably total functions of collection

Theory $\mathsf{BST}_0\omega$ is Π_2 axiomatizable theory which is $\mathsf{KP}_0\omega$ with collection replaced by collection rule

$$\frac{\forall x \exists y\ \varphi(x,y)}{\forall x_0 \exists y_0 (\forall x \in x_0)(\exists y \in y_0)\ \varphi(x,y)}, \text{ where } \varphi(x,y) \text{ is } \Delta_0 \quad (\Delta_0\text{-CollR})$$

Lemma

Suppose φ is Π_2 sentence. Then

$$\mathsf{KP}_0\omega + \varphi \equiv_{\Pi_2} \mathsf{BST}_0\omega + \varphi + \Delta_0\text{-CollR}.$$

Fundamental sequences for ε_{On+1}

For $t \in \varepsilon_{On+1}$ we define fundamental sequnces $t[\xi]$, with $\xi < \tau_t$:

- if t is 0, we put $\tau_t = 0$;
- if t is $\omega^{v_0} + \ldots + \omega^{v_{m-1}} + \omega^0$, we put $\tau_t = 1$ and $t[0] = \omega^{v_0} + \ldots + \omega^{v_{m-1}}$;
- if t is $\omega^{v_0} + \ldots + \omega^{v_{m-1}}$ and $\tau_{v_{m-1}} = 1$, we put $\tau_t = \omega$ and $t[n] = \omega^{v_0} + \ldots + \omega^{v_{m-2}} + \omega^{v_{m-1}[0]}n$;
- if t is $\omega^{v_0} + \ldots + \omega^{v_{m-1}}$ and $\tau_{v_{m-1}} > 1$, we put $\tau_t = \tau_{v_{m-1}}$ and $t[\xi] = \omega^{v_0} + \ldots + \omega^{v_{m-2}} + \omega^{v_{m-1}[\xi]}$;
- if t is ε_α and α is $\beta + 1$, we put $\tau_t = \omega$ and $t[n] = \omega_n^{\varepsilon_\beta}$;
- if t is ε_α and α is limit or $\alpha = On$, we put $\tau_t = \alpha$ and $t[\xi] = \varepsilon_\xi$.

We have

$$\alpha = \sup_{\xi < \tau_\alpha} \alpha[\xi]$$

Hierarchies of ordinal function

We use hierarchy $\mathbf{F}_\alpha$, $\alpha \in \varepsilon_{On+1}$ that is closely connected to fast-growing hierarchy f_α:

$f_\alpha : \mathbb{N} \to \mathbb{N}$	$\mathbf{F}_\alpha : On \to On$
$f_0(n) = n + 1$	$\mathbf{F}_0(x) = x + 1$
$f_{\alpha+1}(n) = f_\alpha^n(n)$	$\mathbf{F}_{\alpha+1}(x) = \sup\limits_{n<\omega} \mathbf{F}_\alpha^n(x)$
	$\mathbf{F}_\alpha(x) = \sup\limits_{\xi<\tau_\alpha} \mathbf{F}_{\alpha[\xi]}(x)$ if $\tau_\alpha < On$
$f_\lambda(n) = f_{\lambda[n]}(n)$	$\mathbf{F}_\alpha(x) = \mathbf{F}_{\alpha[x]}(x)$ if $\tau_\alpha = On$

Ordinal bounds for Π_2 theorems of KPω

Recall that

$$\mathsf{KP}\omega \equiv_{\mathbf{\Pi}_2} \mathsf{R}^{\varepsilon_{\mathcal{O}n+1}}_{\mathbf{\Pi}_2}(\mathsf{KP}_0\omega).$$

By reflexive induction we show

$$\mathsf{RFN}^{\alpha}_{\mathbf{\Pi}_2}(\mathsf{KP}_0\omega) \vdash \text{“}\mathbf{F}_\beta \text{ is total”, for } \beta < 1 + \alpha$$

For any $\mathbf{\Delta}_0$ formula $\varphi(x, y)$

$$\mathsf{R}^{\alpha}_{\mathbf{\Pi}_2}(\mathsf{KP}_0\omega) \vdash \forall x \exists y\ \varphi(x, y)$$
$$\Downarrow$$
$$\mathsf{R}^{\alpha}_{\mathbf{\Pi}_2}(\mathsf{KP}_0\omega) \vdash \forall x \exists y \big(\rho(y) \leq \mathbf{F}^{n}_{\beta}(\rho(x)) \wedge \varphi(x, y)\big)$$
$$\text{for some } \beta < 1 + \alpha \text{ and } n \in \mathbb{N}$$

Спасибо!