Weyl invariant E_8 Jacobi forms

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Outline

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Weyl invariant E_8 Jacobi forms

Let $\varphi : \mathbb{H} \times (E_8 \otimes \mathbb{C}) \to \mathbb{C}$ be a holomorphic function and $k \in \mathbb{Z}$, $t \in \mathbb{N}$. If φ satisfies the following properties

$$\text{(1)} \quad \varphi(\tau, \sigma(\mathfrak{z})) = \varphi(\tau, \mathfrak{z}), \quad \sigma \in W(E_8);$$

$$\text{(2)} \quad \varphi(\tau, \mathfrak{z} + x\tau + y) = e^{-t\pi i((x,x)\tau + 2(x,\mathfrak{z}))}\varphi(\tau, \mathfrak{z}), \quad x, y \in E_8;$$

$$\text{(3)} \quad \varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{\mathfrak{z}}{c\tau + d}\right) = (c\tau + d)^k \exp\left(t\pi i \frac{c(\mathfrak{z}, \mathfrak{z})}{c\tau + d}\right)\varphi(\tau, \mathfrak{z});$$

$$\text{(4)} \quad \varphi(\tau, \mathfrak{z}) = \sum_{n=0}^{\infty} \sum_{\ell \in E_8} f(n, \ell) e^{2\pi i(n\tau + (\ell, \mathfrak{z}))}$$

then φ is called a $W(E_8)$-invariant weak Jacobi form of weight k and index t.

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- If φ further satisfies the condition

$$f(n,\ell) \neq 0 \Longrightarrow 2nt - (\ell,\ell) \geq 0$$

then φ is called a $W(E_8)$-invariant holomorphic Jacobi form.

- If φ further satisfies the stronger condition

$$f(n,\ell) \neq 0 \Longrightarrow 2nt - (\ell,\ell) > 0$$

then φ is called a $W(E_8)$-invariant Jacobi cusp form.

- Denote by

$$J_{k,E_8,t}^{w,W(E_8)} \supsetneq J_{k,E_8,t}^{W(E_8)} \supsetneq J_{k,E_8,t}^{cusp,W(E_8)}$$

the vector space of $W(E_8)$-invariant weak Jacobi forms, holomorphic Jacobi forms and Jacobi cusp forms of weight k and index t.

The structure of $W(E_8)$-invariant Jacobi forms

Theorem (H.Wang, 2018)

The space $J^{w,W(E_8)}_{*,E_8,t}$ is a free module of rank $r(t)$ over the ring M_* of $\mathrm{SL}(2,\mathbb{Z})$ modular forms, where $r(t)$ is given by

$$\frac{1}{(1-x)(1-x^2)^2(1-x^3)^2(1-x^4)^2(1-x^5)(1-x^6)} = \sum_{t\geq 0} r(t)x^t.$$

Equivalently, we have

$$J^{w,W(E_8)}_{*,E_8,*} \subsetneq \mathbb{C}\left(E_4,E_6\right)\left[A_1,A_2,A_3,A_4,A_5,B_2,B_3,B_4,B_6\right],$$

and A_1, A_2, A_3, A_4, A_5, B_2, B_3, B_4, B_6 are algebraically independent over M_*. Here $\mathbb{C}\left(E_4,E_6\right)$ is the fractional field of $\mathbb{C}[E_4,E_6]$.

Crucial facts in the proof

t	1	2	3	4	5	6	7	8	9	10	11	12	13
$r(t)$	1	3	5	10	15	27	39	63	90	135	187	270	364

- For any $m \in E_8$, we define the Weyl orbit of m as

$$\mathrm{orb}(m) := \sum_{\sigma \in W(E_8)/W(E_8)_m} e^{2\pi i(\sigma(m),\mathfrak{z})}.$$

Let w_i be the fundamental weights of E_8. The eight fundamental weyl orbits $\mathrm{orb}(w_i)$ are algebraically independent over $\mathbb{C}$.

Crucial facts in the proof

- Every $W(E_8)$-orbit in E_8 meets the set Λ_+ in exactly one point

$$\Lambda_+ = \{m \in E_8 : (\alpha_i, m) \geq 0, 1 \leq i \leq 8\} = \bigoplus_{i=1}^{8} \mathbb{N} w_i.$$

- Let ϕ_t be a $W(E_8)$-invariant weak Jacobi form of index t. Then

$$\sum_{\ell \in E_8} f(0,\ell) e^{2\pi i (\ell, \mathfrak{z})} = \sum_{\substack{m \in \Lambda_+ \\ T(m) \leq t}} c(m) \operatorname{orb}(m),$$

where $m = \sum_{i=1}^{8} x_i w_i$, $c(m)$ are constants and

$$T(m) := 2x_1 + 3x_2 + 4x_3 + 6x_4 + 5x_5 + 4x_6 + 3x_7 + 2x_8.$$

the structure of weak Jacobi forms

- It is well known that $J^{w,W(E_8)}_{*,E_8,1}$ is generated by

$$\vartheta_{E_8}(\tau,\mathfrak{z}) := \sum_{\ell\in E_8} \exp\left(\pi i(\ell,\ell)\tau + 2\pi i(\ell,\mathfrak{z})\right)$$

the theta function of the root lattice E_8 over M_*.

Theorem (H.Wang, 2018)

$$J^{w,W(E_8)}_{*,E_8,2} = M_*\langle \varphi_{-4,2}, \varphi_{-2,2}, \varphi_{0,2}\rangle,$$
$$J^{w,W(E_8)}_{*,E_8,3} = M_*\langle \varphi_{-8,3}, \varphi_{-6,3}, \varphi_{-4,3}, \varphi_{-2,3}, \varphi_{0,3}\rangle,$$
$$J^{w,W(E_8)}_{*,E_8,4} = M_*\langle \varphi_{-2k,4},\ 0\le k\le 8;\ \psi_{-8,4}\rangle,$$

where $\varphi_{k,t}$ is a $W(E_8)$-invariant weak Jacobi form of weight k and index t.

the structure of holomorphic Jacobi forms

Theorem (H.Wang, 2018)

$$J^{W(E_8)}_{*,E_8,2} = M_*\langle A_2, B_2, A_1^2\rangle,$$
$$J^{W(E_8)}_{*,E_8,3} = M_*\langle A_3, B_3, A_1A_2, A_1B_2, A_1^3\rangle,$$

and $J^{W(E_8)}_{*,E_8,4}$ *is generated by two Jacobi forms of weight 4, two Jacobi forms of weight 6, three Jacobi forms of weight 8, two Jacobi forms of weight 10 and one Jacobi form of weight 12 over* M_*.

the structure of Jacobi cusp forms

Theorem (H.Wang, 2018)

Let $t = 2, 3$ or 4. The numbers of generators of indicated weght of $J^{cusp,W(E_8)}_{*,E_8,t}$ are shown in the following table.

weight	8	10	12	14	16
$t=2$	0	0	1	1	1
$t=3$	0	1	2	1	1
$t=4$	1	2	3	2	2

Representative system of Fourier coefficients

Lemma

Let $\varphi \in J^{w,W(E_8)}_{k,E_8,t}$. *Then the coefficients* $f(n,\ell)$ *depend only on the class of* ℓ *in* E_8/tE_8 *and on the value of* $2nt-(\ell,\ell)$. *Besides,*

$$f(n,\ell) \neq 0 \Longrightarrow 2nt-(\ell,\ell) \geq -\min\{(v,v) : v \in \ell + tE_8\}.$$

$$\begin{aligned}
\max\{\min\{(v,v) : v \in l + 2E_8\} : l \in E_8\} &= 4,\\
\max\{\min\{(v,v) : v \in l + 3E_8\} : l \in E_8\} &= 8,\\
\max\{\min\{(v,v) : v \in l + 4E_8\} : l \in E_8\} &= 16,\\
\max\{\min\{(v,v) : v \in l + 5E_8\} : l \in E_8\} &= 22,\\
\max\{\min\{(v,v) : v \in l + 6E_8\} : l \in E_8\} &= 36.
\end{aligned}$$

Orbits of vectors under Weyl group

Lemma

Let R_{2n} denote the set of all vectors $\ell \in E_8$ with $(\ell,\ell) = 2n$. The orbits of R_{2n} under the action of $W(E_8)$ are given by

$$
\begin{aligned}
&W(E_8)\backslash R_2 = \{w_8\} && W(E_8)\backslash R_4 = \{w_1\}\\
&W(E_8)\backslash R_6 = \{w_7\} && W(E_8)\backslash R_8 = \{2w_8, w_2\}\\
&W(E_8)\backslash R_{10} = \{w_1 + w_8\} && W(E_8)\backslash R_{12} = \{w_6\}\\
&W(E_8)\backslash R_{14} = \{w_3, w_7 + w_8\} && W(E_8)\backslash R_{16} = \{2w_1, w_2 + w_8\}\\
&W(E_8)\backslash R_{18} = \{w_1 + w_7, 3w_8\} && W(E_8)\backslash R_{20} = \{w_5, w_1 + 2w_8\}\\
&W(E_8)\backslash R_{22} = \{w_6 + w_8, w_1 + w_2\} && W(E_8)\backslash R_{24} = \{2w_7, w_3 + w_8\}.
\end{aligned}
$$

$$\sum_{2} = \mathrm{orb}(w_8) \qquad \sum_{4} = *\,\mathrm{orb}(w_1) \qquad \sum_{6} = *\,\mathrm{orb}(w_7)$$

$$\sum_{8'} = *\,\mathrm{orb}(w_2) \qquad \sum_{8''} = *\,\mathrm{orb}(2w_8) \qquad \sum_{10} = *\,\mathrm{orb}(w_1 + w_8)$$

$$\sum_{12} = *\,\mathrm{orb}(w_6) \qquad \sum_{14'} = *\,\mathrm{orb}(w_3) \qquad \sum_{14''} = *\,\mathrm{orb}(w_7 + w_8)$$

$$\sum_{16'} = *\,\mathrm{orb}(2w_1) \qquad \sum_{16''} = *\,\mathrm{orb}(w_2 + w_8) \qquad \sum_{18'} = *\,\mathrm{orb}(w_1 + w_7)$$

$$\sum_{18''} = *\,\mathrm{orb}(3w_8) \qquad \sum_{20'} = *\,\mathrm{orb}(w_5) \qquad \sum_{20''} = *\,\mathrm{orb}(w_1 + 2w_8)$$

$$\sum_{22'} = *\,\mathrm{orb}(w_1 + w_2) \qquad \sum_{22''} = *\,\mathrm{orb}(w_6 + w_8) \qquad \sum_{24'} = *\,\mathrm{orb}(2w_7)$$

The normalizations of these Weyl orbits are choosen so that they reduce to 240 if one takes $\mathfrak{z} = 0$.

Let φ_t be a $W(E_8)$-invariant weak Jacobi form of index t. Then its q^0-term can be written as

$$[\varphi_2]_{q^0} = 240c_0 + c_1\sum_{2} + c_2\sum_{4},$$

$$[\varphi_3]_{q^0} = 240c_0 + c_1\sum_{2} + c_2\sum_{4} + c_3\sum_{6} + c_4\sum_{8'},$$

$$[\varphi_4]_{q^0} = 240c_0 + c_1\sum_{2} + c_2\sum_{4} + c_3\sum_{6} + c_4'\sum_{8'} + c_4''\sum_{8''} + c_5\sum_{10} + c_6\sum_{12} + c_7\sum_{14'} + c_8\sum_{16'},$$

$$[\varphi_5]_{q^0} = 240c_0 + c_1\sum_{2} + c_2\sum_{4} + c_3\sum_{6} + c_4'\sum_{8'} + c_4''\sum_{8''} + c_5\sum_{10} + c_6\sum_{12} + c_7'\sum_{14'} + c_7''\sum_{14''} + c_8'\sum_{16'} + c_8''\sum_{16''} + c_9\sum_{18'} + c_{10}\sum_{20'} + c_{11}\sum_{22'}.$$

Let $\varphi \in J^{w,W(E_8)}_{2k,E_8,t}$.

1. Let $t=2$. Then φ is a holomorphic Jacobi form if and only if its q^0-term is a constant. Moreover, φ is a Jacobi cusp form if and only if its q^0-term is 0 and its q^1-term is of the form $c_0 + c_1\sum_2$.
2. Let $t=3$. Then φ is a holomorphic Jacobi form if and only if its q^0-term is a constant and its q^1-term is of the form
$$240c_0 + c_1\sum_2 + c_2\sum_4 + c_3\sum_6.$$
Moreover, a holomorphic Jacobi form φ is a Jacobi cusp form if and only if $c_3=0$ and its q^0-term is 0.
3. Let $t=4$. Then φ is a holomorphic Jacobi form if and only if its q^0-term is a constant and its q^1-term is of the form
$$240c_0 + c_1\sum_2 + c_2\sum_4 + c_3\sum_6 + c_4'\sum_{8'} + c_4''\sum_{8''}.$$
Moreover, a holomorphic Jacobi form φ is a Jacobi cusp form if and only if $c_4' = c_4'' = 0$ and its q^0-term is 0 and its q^2-term does not contain the term $\sum_{16'}$.

Let $\varphi_t \in J^{W(E_8)}_{4,E_8,t}$. In view of the singular weight, we can write

$$\phi_t(\tau,\mathfrak{z}) = q^n \sum_{\substack{\ell\in E_8\\(\ell,\ell)=2nt}} f(n,\ell)e^{2\pi i(\ell,\mathfrak{z})} + O(q^{n+1}) \in J^{W(E_8)}_{4,E_8,t}.$$

Since $\phi_t(\tau,0)=0$, if there exists $\ell\in E_8$ such that $f(n,\ell)\neq 0$, then there exist $\ell_1,\ell_2\in E_8$ satisfying $(\ell_1,\ell_1)=(\ell_2,\ell_2)=2nt$, $\mathrm{orb}(\ell_1)\neq\mathrm{orb}(\ell_2)$ and $(\ell_i,\ell_i)=\min\{(v,v): v\in \ell_i+tE_8\}$, for $i=1,2$. From this, we deduce

$$\dim J^{W(E_8)}_{4,E_8,t}=1,\, t=1,2,3,5, \quad \dim J^{W(E_8)}_{4,E_8,4}=2, \quad 1\le \dim J^{W(E_8)}_{4,E_8,6}\le 2.$$

If $\dim J^{W(E_8)}_{4,E_8,6}=2$, then this Jacobi form has the Fourier expansion

$$F_{4,6}(\tau,\mathfrak{z}) = q^2\Big(\sum_{24'} - \sum_{24''}\Big) + O(q^3).$$

Let $v_2\in E_8$ of norm 2. Then

$$F_{4,6}(\tau,zv_2)/\Delta^2(\tau) = \zeta^{\pm 8}+\ldots+O(q) \in J^{w}_{-20,6}, \quad \textit{contradiction}!$$

Differential operators

Lemma

Let $\varphi(\tau,\mathfrak{z}) = \sum f(n,\ell)e^{2\pi i(n\tau+(\ell,\mathfrak{z}))} \in J^{w,W(E_8)}_{k,E_8,t}$. *Then we have*

$$H_k(\varphi)(\tau,\mathfrak{z}) = H(\varphi) + \frac{4-k}{12}E_2(\tau)\varphi(\tau,\mathfrak{z}) \in J^{w,W(E_8)}_{k+2,E_8,t}$$

$$H(\varphi)(\tau,\mathfrak{z}) = \sum_{n\in\mathbb{N}}\sum_{\ell\in E_8}\left[n - \frac{1}{2t}(\ell,\ell)\right] f(n,\ell)e^{2\pi i(n\tau+(\ell,\mathfrak{z}))},$$

where $E_2(\tau) = 1 - 24\sum_{n\geq 1}\sigma(n)q^n$ *is the Eisenstein series of weight* 2.

To determine the possible minimum weight (index 3)

Claim: The minimum weight of weak Jacobi forms of index 3 is -8.

If there exists a weak Jacobi form ϕ of weight $2k < -8$ and index 3 whose q^0-term is not zero, then we can construct a weak Jacobi form of weight -10 and index 3 whose q^0-term is not zero. In fact, this function can be constructed as $E_{-10-2k}\phi$ if $2k \leq -14$, or $H_{-12}(\phi)$ if $2k = -12$. We now assume that there exists a weak Jacobi form ϕ of weight -10 and index 3 whose q^0-term is represented as

$$[\phi]_{q^0} = 240c_0 + c_1 \sum_2 + c_2 \sum_4 + c_3 \sum_6 + c_4 \sum_{8'} .$$

By means of the differential operators, we construct $\phi_{-8} = H_{-10}(\phi)$, $\phi_{-6} = H_{-8}(\phi_{-8})$, $\phi_{-4} = H_{-6}(\phi_{-6})$ and $H_{-4}(\phi_{-4})$.

They are respectively weak Jacobi forms of weight -8, -6, -4, -2 with q^0-term of the form (order: $240c_0, c_1\sum_2, c_2\sum_4, c_3\sum_6, c_4\sum_{8'}$)

$$\text{weight} \quad -10: \qquad (a_{1,j})_{j=1}^{9} = (1,1,1,1,1)$$

$$\text{weight} \quad -10+2(i-1): \qquad a_{i,j} = \left(\frac{18-2i}{12} - \frac{j-1}{3}\right) a_{i-1,j}$$

where $2 \le i \le 5$, $1 \le j \le 5$. For these Jacobi forms, if we take $\mathfrak{z} = 0$ then their q^0-terms will be zero. We thus get a system of 5 linear equations with 5 unknowns

$$Ax = 0, \quad A = (a_{i,j})_{5\times 5}, \quad x = (c_0, c_1, c_2, c_3, c_4)^t.$$

By direct calculations, this system has only trivial solution, which contradicts our assumption. Hence the possible minimum weight is -8.

Uniqueness of weak Jacobi form of weight −8 and index 3

Suppose that ϕ is a weak Jacobi form of weight -8 with q^0-term

$$240c_0 + c_1 \sum_2 + c_2 \sum_4 + c_3 \sum_6 + c_4 \sum_{8'} .$$

We can construct weak Jacobi forms of weight -6, -4, -2 with q^0-term

$$\text{weight} \quad -8: \qquad (b_{1,j})_{j=1}^{9} = (1,1,1,1,1)$$

$$\text{weight} \quad -8+2(i-1): \qquad b_{i,j} = \left(\frac{16-2i}{12} - \frac{j-1}{3} \right) b_{i-1,j}$$

where $2 \leq i \leq 4$, $1 \leq j \leq 5$. Then we can build a system of 4 linear equations with 5 unknowns

$$Bx = 0, \quad B = (b_{i,j})_{4\times 5}, \quad x = (c_0, c_1, c_2, c_3, c_4)^t. \tag{1}$$

We found that $(c_0, c_1, c_2, c_3, c_4) = (1, -4, 6, -4, 1)$ is the unique nontrivial solution. Therefore, the weak Jacobi form of weight -8 and index 3 is unique if it exists.

Uniqueness of weak Jacobi form of weight -6 and index 3

If there exists another weak Jacobi form of weight -6, we assume

$$[f]_{q^0} = 240c_0 + c_1 \sum_2 + c_2 \sum_4 + c_3 \sum_6 \neq 0.$$

Once again, we construct weak Jacobi forms of weight -4, -2 and 0 with q^0-terms

$$\text{weight} \quad -6: \qquad (c_{1,j})_{j=1}^4 = (1,1,1,1)$$

$$\text{weight} \quad -6+2(i-1): \qquad c_{i,j} = \frac{9-i-2j}{6} c_{i-1,j}$$

where $2 \leq i \leq 4$, $1 \leq j \leq 4$. For each Jacobi form of negative weight, if we take $\mathfrak{z} = 0$ then its q^0-term will be zero. Hence we have

$$\sum_{j=1}^4 c_{i,j} c_{j-1} = 0, \quad 1 \leq i \leq 3.$$

Lemma

Let φ be a $W(E_8)$-invariant weak Jacobi form of weight 0 and index t. Then

$$2t\sum_{\ell\in E_8} f(0,\ell) = 3\sum_{\ell\in E_8} f(0,\ell)(\ell,\ell).$$

For the Jacobi form of weight 0, by the lemma we have

$$\sum_{j=1}^{4}(12-6j)c_{4,j}c_{j-1} = 0.$$

We thus get a system of linear equations of 4×4. By direct calculations, we obtain $c_j = 0$ for $0 \le j \le 3$, which contradicts our assumption.

The proof of the case of index 3

Our main theorem shows that $J^{w,W(E_8)}_{*,E_8,3}$ is a free M_*-module generated by five weak Jacobi forms. It is obvious that $\varphi_{-8,3}$, $\varphi_{-6,3}$ and $\varphi_{-4,3}$ are generators. Since $\varphi_{-2,3}$ is independent of $E_6\varphi_{-8,3}$ and $E_4\varphi_{-6,3}$, the function $\varphi_{-2,3}$ must be a generator. Moreover, $\varphi_{0,3}$ is also a generator on account of $[\varphi_{0,3}]_{q^0}(\tau,0)\neq 0$. We then conclude the eager result:

$$J^{w,W(E_8)}_{*,E_8,3} = M_*\langle \varphi_{-8,3}, \varphi_{-6,3}, \varphi_{-4,3}, \varphi_{-2,3}, \varphi_{0,3}\rangle.$$

Pull-backs of Jacobi forms

Trouble: When the index is larger than 4, the absolute value of minimal weight will be less than the maximal norm of Weyl orbits appearing in q^0-terms, which causes our previous approach to not work well because there is not enough linear equation in this case.

Let $\phi \in J^{w,W(E_8)}_{k,E_8,t}$ and v_4 be a vector of norm 4 in E_8. Then the function

$$\phi(\tau, zv_4)$$

is a weak Jacobi form of weight k and index $2t$.

Let $\sum_v$ be a Weyl orbit associated to v:

$$\sum_v = \frac{240}{\#W(E_8)} \sum_{\sigma\in E_8} \exp(2\pi i(\sigma(v), \mathfrak{z})).$$

Since the Weyl group $W(E_8)$ acts transitively on the set R_4 of vectors of norm 4 in E_8, we have

$$\begin{aligned}\sum_v(zv_4) =& \frac{240}{\#W(E_8)} \sum_{\sigma\in E_8} \exp(2\pi i(\sigma(v), v_4)z) \\ =& \frac{240}{\#W(E_8)} \sum_{\sigma\in E_8} \exp(2\pi i(v, \sigma(v_4))z) \\ =& \frac{240}{\#R_4} \sum_{l\in R_4} \exp(2\pi i(v, l)z).\end{aligned}$$

In view of this fact, we define

$$\max(\sum_v, v_4) := \max(v, R_4) = \max\{(v, l) : l \in R_4\}.$$

$$\max(\sum_{2}, v_4) = 2 \qquad \max(\sum_{4}, v_4) = 4 \qquad \max(\sum_{6}, v_4) = 4$$

$$\max(\sum_{8'}, v_4) = 5 \qquad \max(\sum_{8''}, v_4) = 4 \qquad \max(\sum_{10}, v_4) = 6$$

$$\max(\sum_{12}, v_4) = 6 \qquad \max(\sum_{14'}, v_4) = 7 \qquad \max(\sum_{14''}, v_4) = 6$$

$$\max(\sum_{16'}, v_4) = 8 \qquad \max(\sum_{16''}, v_4) = 7 \qquad \max(\sum_{18'}, v_4) = 8$$

$$\max(\sum_{18''}, v_4) = 6 \qquad \max(\sum_{20'}, v_4) = 8 \qquad \max(\sum_{20''}, v_4) = 8$$

$$\max(\sum_{22'}, v_4) = 9 \qquad \max(\sum_{22''}, v_4) = 8 \qquad \max(\sum_{24'}, v_4) = 8$$

$$\max(\sum_{24''}, v_4) = 9 \qquad \max(\sum_{26'}, v_4) = 10 \qquad \max(\sum_{26''}, v_4) = 9$$

Possible minimum weight in the case of index 5

(I) Assume that $\phi = \sum_{22'} + \cdots + O(q) \in J^{w,W(E_8)}_{-2k,E_8,5}$ with $k > 0$. Then we have

$$\phi(\tau, zv_4) = \zeta^{\pm 9} + \cdots + O(q) \in J^{w}_{-2k,10}.$$

Since $J^w_{-2k,10} = \phi^k_{-2,1} \cdot J^w_{0,10-k}$, we have $10 - k \geq 0$. But when $k = 9$ or 10, the spaces $J^w_{-20,10}$ and $J^w_{-18,10}$ are all generated by one function with leading Fourier coefficient $\zeta^{\pm 10}$, which contradicts the Fourier expansion of $\phi(\tau, zv_4)$. Therefore, we get $k \leq 8$ i.e. $-2k \geq -16$.

(II) Assume that $\phi \in J^{w,W(E_8)}_{-2k,E_8,5}$ has no Fourier coefficient $\sum_{22'}$ in its q^0-term. The function $\Delta^2\phi \in J^{W(E_8)}_{24-2k,E_8,5}$. Thus we have $24 - 2k \geq 6$ i.e. $-2k \geq -18$.

(III) Assume that $\phi \in J^{w,W(E_8)}_{-2k,E_8,5}$ has no Fourier coefficients $\sum_{22'}$ and $\sum_{20'}$ in its q^0-term. Then the function $\eta^{44}\phi$ is a $W(E_8)$-invariant Jacobi cusp form of weight $22-2k$ and index 5 with a character. From the singular weight, it follows that $22-2k > 4$ i.e. $-2k \geq -16$.

(IV) Assume that $\phi \in J^{w,W(E_8)}_{-2k,E_8,5}$ has no Fourier coefficients $\sum_{22'}$, $\sum_{20'}$ and $\sum_{18'}$ in its q^0-term. Then the function $\eta^{40}\phi$ is a $W(E_8)$-invariant Jacobi cusp form of weight $20-2k$ and index 5 with a character. It follows that $20-2k > 4$ i.e. $-2k \geq -14$.

Theorem

$$\dim J^{w,W(E_8)}_{-2k,E_8,5} = 0, \quad if \quad -2k \le -20,$$
$$\dim J^{w,W(E_8)}_{-18,E_8,5} \le 1,$$
$$\dim J^{w,W(E_8)}_{-16,E_8,5} \le 3.$$

Moreover, if the $W(E_8)$-invariant weak Jacobi form of weight -18 and index 5 exists, then its q^0-term has no Fourier coefficient $\sum_{22'}$ and contains Fourier coefficient $\sum_{20'}$.

Remark: We do not know if the $W(E_8)$-invariant weak Jacobi form of weight -18 and index 5 exists. But the $W(E_8)$-invariant weak Jacobi forms of weight -16 and index 5 do indeed exist.

Outlook

- Determine the generators of the space of $W(E_8)$-invariant weak Jacobi forms of index 5, 6, ...
- Want to know whether the bigraded ring $J^{w,W(E_8)}_{*,E_8,*}$ is finitely generated over M_* or not.

Thanks for your attention!