Divisibility of quantum channels and entanglement

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Motivations

- Construction of a tool to test if a given channel is a member of a one-parameter semigroup. In other words, if a channel has the form e^L with L Lindblad [1]: A characterization problem.
- Study and classify the convex set of qubit channels from the point of view of the divisibility types: A classification problem.
  - Pauli
  - Unital
  - Study possible relation with *entanglement-breaking*: How are the found divisibility structures related with entanglement breaking channels?

[1] Wolf et al., "Assessing non-Markovian quantum dynamics."; Evans et al., "Dilations of irreversible evolutions in algebraic quantum theory"

Outline

Quantum channel

Formally, Let $\mathcal{B}(\mathcal{H})$ the set of bounded operators acting on the Hilbert space $\mathcal{H}$, *complete positivity and trace preserving operations (quantum channel for short)* are defined as:

Quantum channel

Let $\sigma \in \mathcal{B}(\mathcal{H})$ and $\tilde{\sigma} \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ with $\dim \mathcal{H} = n$, a linear operation $\mathcal{E}$ is CPTP if and only if:

- $\text{tr}\mathcal{E}[\sigma] = \text{tr}\sigma$,
- $\mathcal{E}[\sigma] \geq 0 \ \forall \sigma \geq 0$,
- $\text{id}_n \otimes \mathcal{E}[\tilde{\sigma}] \geq 0 \ \forall \tilde{\sigma}$ (CP).

Properties of quantum channels

The following sentences are equivalent:

- $\mathcal{E} \in$ CPTP.
- $\mathcal{E}[\rho] = \mathrm{tr}_\Sigma \left[U(\rho \otimes \sigma) U^\dagger \right]$ (Stiniespring).
- $\mathcal{E}[\rho] = \sum_i^M K_i \rho K_i^\dagger$, con $\sum_i^M K_i^\dagger K_i = \mathbb{1}$ (Kraus).

Illustrative examples

Unital channels

$\mathcal{E}[\mathbb{1}] = \mathbb{1}$.

- Unitary channels: $\mathcal{E}[\rho] = U\rho U^\dagger$.
- Convex combinations of unitary channels: $\mathcal{E}[\rho] = \sum_i p_i U_i \rho U_i^\dagger$ (every unital qubit channels can be written in such way).

Entanglement-breaking channels

A channel $\mathcal{E}$ is EB if and only if $\rho_2 = (\mathrm{id}_k \otimes \mathcal{E})[\rho_1]$ is a separable state $\forall\ \rho_1 \in \mathcal{T}(\mathcal{H}), k \in \mathbb{Z}^+$.

Dynamical quantum process

Dynamical quantum process

It is defined as a curve inside the convex space of CPTP operations whose one extreme is the identity channel, *i.e.* :

$$\mathcal{E}_t : [0, a] \rightarrow \text{CPTP}$$

with $\mathcal{E}_0 = \text{id}$.

Instantaneous channels

Note chat if such curve is continuous in some interval $\mathcal{I}$, one can write down a master equation for such interval: $\dot{\mathcal{E}}_t = A_t \mathcal{E}_t$.

Examples of dynamical processes

Example 1: Markovian systems

$$\dot{\rho} = i[\rho, H(t)] + \sum_{\alpha,\beta} G_{\alpha\beta}(t) \left(F_\alpha(t)\rho F_\beta^\dagger(t) - \frac{1}{2}\{F_\beta^\dagger(t)F_\alpha(t), \rho\} \right)$$
$$= L_t(\rho),$$

with $G \geq 0$. The differential equation for the dynamical process is $\dot{\mathcal{E}}_t = L_t \mathcal{E}_t$, its formal solution is: $\mathcal{E}_t = \mathcal{T} \exp\left(\int_0^t L_s ds\right)$.

Important remarks:

- It's CP-divisible: $\mathcal{E}_{(t,0)} = \mathcal{E}_{(t,s)}\mathcal{E}_{(s,0)} \forall t \geq s \geq 0$ $[\mathcal{E}_{(t,0)} := \mathcal{E}_t]$.
  - It defines a family of *infinitesimal divisible* channels
- If $L_t = L$, the process is Lindlad type and the formal solution is simply $\mathcal{E}_t = e^{tL} : \left(e^L\right)^t$.

Definition of divisibility

Divisible channel

A channel $\mathcal{E}$ is **indivisible** if and only if it cannot be written as a concatenation of two non-unitary channels. In other words, for every decomposition

$$\mathcal{E} = \mathcal{E}_1 \mathcal{E}_2,$$

$\mathcal{E}_1$ or $\mathcal{E}_2$ is unitary. A channel that is not indivisible is **divisible**.

- Full Kraus rank channels are divisible.
- Channels with minimum determinant are indivisible [2].

[2] Wolf et al., "Dividing Quantum Channels"

Infinitely divisible channels

Definition

$\mathcal{E}$ is infinitely divisible if and only if $\forall n \in \mathbb{Z}^+ \; \exists \mathcal{E}_n$ such that $\mathcal{E} = (\mathcal{E}_n)^n$.

- $\mathcal{E}$ can be written as $\mathcal{E}_0 e^L$ with $\mathcal{E}_0 L = \mathcal{E}_0 L \mathcal{E}_0$ [3].
  - $\mathcal{E}_t = \mathcal{E}_0 e^{Lt} \Rightarrow \mathcal{E} = \mathcal{E}_1$.
  - Given that $\mathcal{E}_n = \mathcal{E}_0 e^{L/n}$, $\mathcal{E}$ is infinitesimal divisible.
- In the particular case of $\mathcal{E}_0 = \mathrm{id}$, one recovers Lindblad divisibility $\mathcal{E} = e^L$.
  - Thus $\mathcal{E}$ belongs to a one-parameter semigroup : $\mathcal{E}_t = \mathcal{E}^t = e^{Lt}$ [4].

$$\mathsf{C}^{\mathsf{L}} \subset \mathsf{C}^{\infty}$$

[3] Denisov, "Infinitely Divisible Markov Mappings in Quantum Probability Theory"

[4] Breuer et al., *The Theory of Open Quantum Systems*

Infinitesimal divisible channels

Infinitesimal divisibility in CPTP

Let $\mathcal{L}$ the set of CPTP channels with the property that for every $\epsilon > 0$, there exist a finite number of channels $\mathcal{E}_i \in$CPTP such that $|\mathcal{E}_i - \mathit{id}| < \epsilon$ and $\mathcal{E} = \prod_i \mathcal{E}_i$. It is said that a channel is infinitesimal divisible if it belongs to the closure of $\mathcal{L}$.

$$\mathsf{C}^{\mathsf{Inf}} \equiv \{\mathcal{T} e^{\int_0^t L_\tau d\tau}\} \equiv \{\mathcal{E} = \prod_i e^{L_i}\}\ ^{5} \equiv \mathsf{C}^{\mathsf{CP}}$$

$$\mathsf{C}^{\mathsf{L}} \subset \mathsf{C}^{\infty} \subset \mathsf{C}^{\mathsf{Inf}}$$

[5] Wolf et al., "Dividing Quantum Channels"

Other types of divisibility

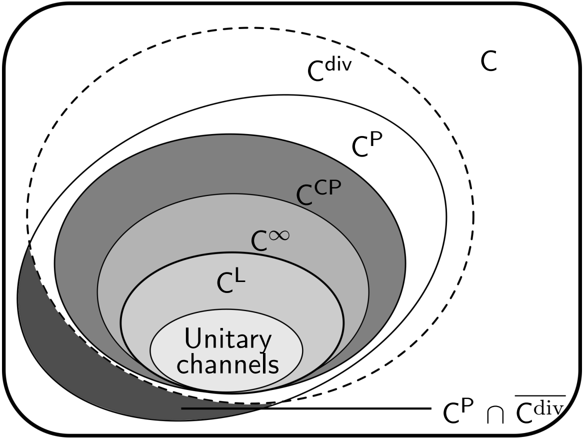
- Infinitesimal divisibility in PTP (positive but not complete positive operations) [6].
  - This type of operations can rise in the case of a system initially correlated with its surroundings, or if the operation is correlated with the initial state [7].
  - They will be called C^{P} .
- Trivially we have the following:

$$C^{CP} \subset C^{P}$$

[6] Wolf et al., "Assessing non-Markovian quantum dynamics."

[7] Carteret et al., "Dynamics beyond completely positive maps: Some properties and applications"

Summary of inclusions



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[8] Wolf et al., "Assessing non-Markovian quantum dynamics."; Evans et al., "Dilations of irreversible evolutions in algebraic quantum theory"

Channels belonging to one-parameter semigroups

Definition of the set

Let the set defined by CPTP operations for which there exist at least one natural logarithm, denoted by $L = \log \mathcal{E}$ and with properties:

- Preserves hermiticity: $L[X]^\dagger = L[X^\dagger] \Rightarrow \tau_L$ is hermitian.
- $\omega_\perp \tau_L \omega_\perp \geq 0$ [9] $\iff G \geq 0$ [10].

The set of L-divisible channels (denoted by C^{L}) is given by the closure of the mentioned set.

- Necessity of closure: $L(\rho) = i[\rho, H] + \gamma[H, [H, \varrho]]$.

[9] $\omega_\perp = \mathbb{1} - |\mathsf{Bell}\rangle\langle\mathsf{Bell}|$ and $\tau_L = (\mathsf{id} \otimes L)\,[|\mathsf{Bell}\rangle\langle\mathsf{Bell}|]$

[10] Evans et al., "Dilations of irreversible evolutions in algebraic quantum theory"

Characterization of channels belonging to one-parameter semigroups

The problem is if a HP generator exist and how to calculate it.

Equivalence with another problem

Solve the equation $C = \exp(X)$ with the restriction that X has real entries, given that C has real entries. This problem was solved by Culver [11].

[11] Culver, "On the Existence and Uniqueness of the Real Logarithm of a Matrix"

Hermiticity preserving

Theorem (**Existence of hermiticity preserving generator**)

A non-singular matrix with real entries $\hat{\mathcal{E}}$ has a real generator (i.e. a $\log \hat{\mathcal{E}}$ has real entries) iff the spectrum fulfills the following:

1. *If it contains negative eigenvalues, they have even-fold degeneration and*
2. *if it contains complex eigenvalues, they come in complex conjugate pairs.*

Parametrization problem of L

Freedom in Jordan decomposition

Let $\hat{\mathcal{E}}$ a matrix with real entries whose Jordan normal form is J, such that:

$$\hat{\mathcal{E}} = wJw^{-1} = \tilde{w}J\tilde{w}^{-1},$$

where w and $\tilde{w}$ differ by a matrix factor K that belongs to a continuum of matrices that commute with J, *i.e.* $w = K\tilde{w}$ [12].

- K not necessarily commutes with $\log \hat{\mathcal{E}}$. This leads to a continuous parametrization of the real logarithms of $\hat{\mathcal{E}}$. (in addition to the real branches).

[12] Culver, "On the Existence and Uniqueness of the Real Logarithm of a Matrix"

Characterization of Pauli channels

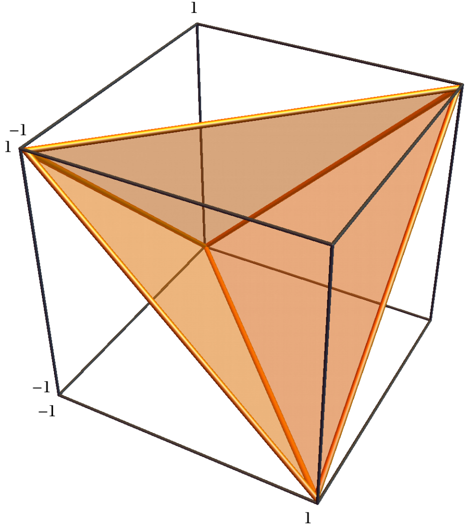
Qubit channels in the Pauli basis are written as:

$$\hat{\mathcal{E}} = \begin{pmatrix} 1 & \vec{0}^T \\ \vec{r} & \Delta \end{pmatrix}.$$

Unital channels upto rotations

- Leave invariant the full mixture $\mathbb{I}/2$: They do not move the center of Bloch sphere ($\vec{r} = 0$).
- An usefull decomposition can be performed using a modification of SVD: $\Delta = R_1 D R_2$, done $D = \text{diag}\,(1, \lambda_1, \lambda_2, \lambda_3)$. $R_{1,2} \in \text{SO}(3)$.
    - Esta descomposición corresponde a $\mathcal{E} = \mathcal{U}_1 \mathcal{D} \mathcal{U}_2$.
    - $\mathcal{D} \in \text{C}^{\text{Inf}} \Leftrightarrow \mathcal{E} \in \text{C}^{\text{Inf}}$.
    - $\mathcal{D} \in \text{C}^{\text{L}} \Rightarrow \mathcal{E} \in \text{C}^{\text{Inf}}$.

Canales unitales (hasta rotaciones)=Pauli channels



Pauli channels with positive eigenvalues

- For positive eigenvalues one has a trivial logarithm given by $L = \log \mathcal{D} = \mathrm{diag}(0, \log \lambda_1, \log \lambda_2, \log \lambda_3)$ (It is unique in case of no degeneration). It is of Lindblad type if and only if:

$$\log \lambda_j - \log \lambda_k - \log \lambda_l \geq 0, \\ \Longrightarrow \frac{\lambda_i}{\lambda_j \lambda_k} \geq 1 \tag{1}$$

Pauli channels with negative eigenvalues

- The only case that has real generator is $\mathcal{E} = \text{diag}\,(1, -\lambda, -\lambda, \eta)$. But the real generator is always non-diagonal:

$$L = K \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \log(|\lambda|) & (2k+1)\pi & 0 \\ 0 & -(2k+1)\pi & \log(|\lambda|) & 0 \\ 0 & 0 & 0 & \log(|\eta|) \end{pmatrix} K^{-1}. \quad (2)$$

- It is parametrized by K and k (Not unique!). A central result is that it is of Lindblad form if and only if:

$$\eta \geq \lambda^2$$

$\Longrightarrow$.

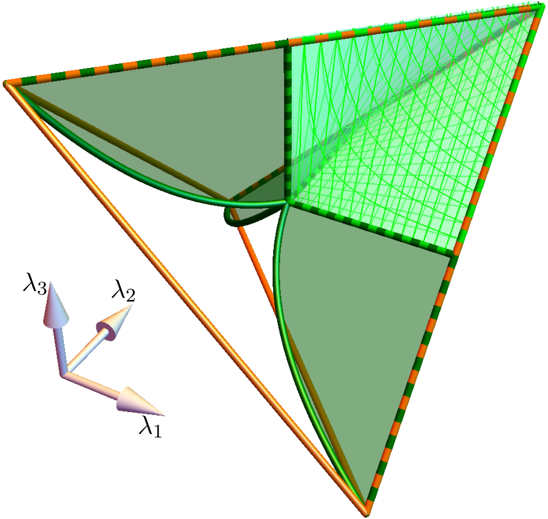
Pauli channels with negative eigenvalues

In summary for Pauli channels with real generators, they are Lindblad if and only if:

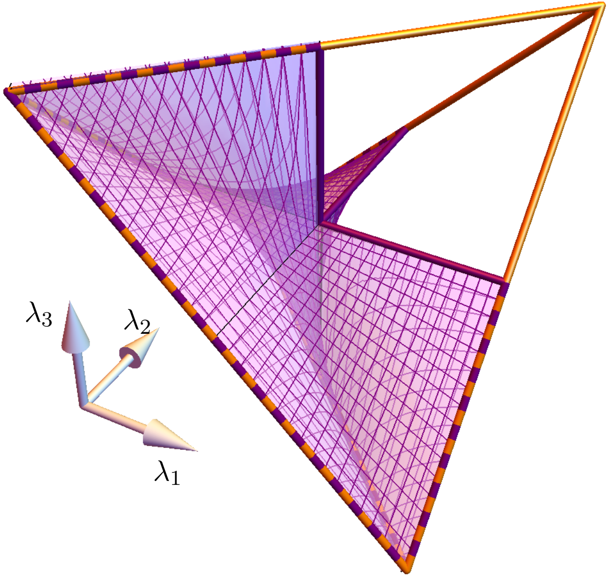
$$\frac{\lambda_i}{\lambda_j \lambda_k} \geq 1,$$

for all combinations of $i \neq j \neq k$.

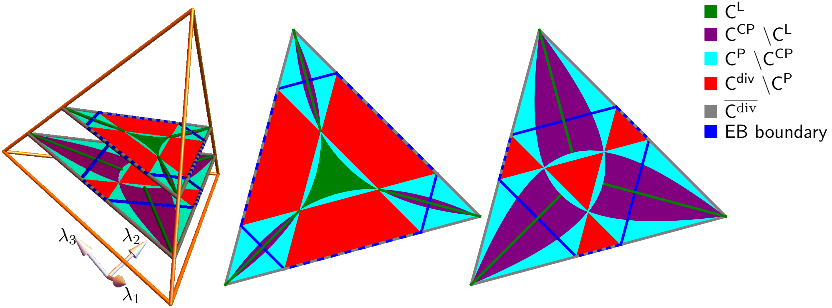
Pauli channels belonging to C^L



Pauli channels belonging to C^{CP}



Transversal slice



Some specific results

- Every unital qubit belonging to C^{Inf} , $\mathcal{E}$, has the form:

$$\mathcal{E} = \mathcal{U}_1 e^L \mathcal{U}_2.$$

- $C^L{}_{Pauli} = C^{\infty}{}_{Pauli}$
  - Easy to prove knowing that the only idempotent Pauli channels different from identity are: $\sigma_{x,y,z}$: $\rho \mapsto \rho_{\text{diag}}$.
  - Those channels are included already in the closure of $C^L{}_{Pauli}$.
- We conjectured and proved that every qubit full Kraus rank channel belonging to $C^{div} \setminus C^{Inf}$, is *entanglement-breaking*.
  - Proof consist on writing the Jamiołkowski-Choi state of the target channel as [13]

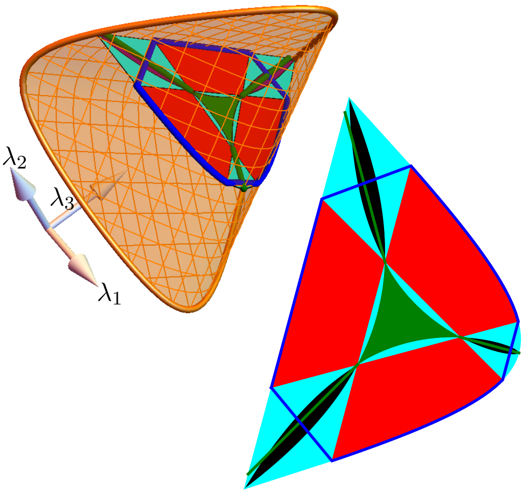
$$\tau_{\mathcal{E}} \propto (X_1 \otimes X_2)\, \tau_{\mathcal{G}}\, (X_1 \otimes X_2)^{\dagger}.$$

  - It means that qubit channels with negative determinant are always entanglement-breaking.

[13] Ziman et al., "Concurrence versus purity: Influence of local channels on Bell states of two qubits"

Slice of non-unital channels

The volume is defined by $\vec{r} = (1/2, 0, 0)$:



Summary

- We introduced a tool to test if a channel belongs to a one-parameter semigroup, including the problematic case negative eigenvalues.
- We characterized Pauli channels and some non-unital channels:
  - We observed the non-convexity of the divisibility sets.
  - Qubit channels in $\mathsf{C}^{\mathsf{Inf}}$ have a relatively simple form: $\mathcal{U}_1 e^L \mathcal{U}_2$.
  - We proved that $\mathsf{C}^{\mathsf{div}} \setminus \mathsf{C}^{\mathsf{Inf}} \subset \mathsf{C}^{\mathsf{EB}}$ for qubit channels.
  - and $\mathsf{C}^{\infty}{}_{\mathsf{Pauli}} = \mathsf{C}^{\mathsf{L}}{}_{\mathsf{Pauli}}$.