On phase-lock areas in a model of Josephson effect and double confluent Heun equations

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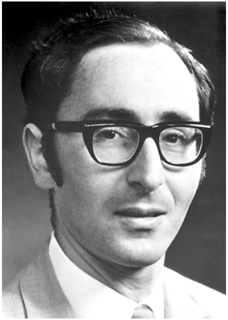
Superconductivity

Occurs in some metals at temperature $\mathbb{T} < \mathbb{T}_{crit}$.
The critical temperature $\mathbb{T}_{crit}$ depends on the metal.
Carried by coherent **Cooper pairs** of electrons.

Brian Josephson. Nobel Prize in physics (1973)
"for theoretic prediction of the Josephson effect"



Born in UK in 1940.

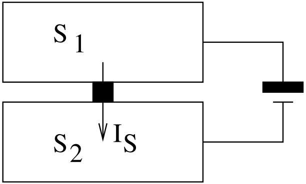
We here present an approach to the calculation of tunneling currents between two metals that is sufficiently general to deal with the case, when both metals are superconducting. In that case new effects are predicted, due to the possibility that electron pairs may tunnel through the barrier leaving the quasi-particle distribution unchanged.

B.Josephson. 1. *"Possible new effects in superconductive tunneling".* Phys. Let., **1 (1962)**, 251–253.
2. *"The discovery of tunnelling supercurrents".* Nobel Lecture, December 12, 1973.

Occurs in some metals at temperature $\mathbb{T} < \mathbb{T}_{crit}$.

The Josephson effect

Let two superconductors S_1, S_2 be separated by a very narrow dielectric, thickness $\leq 10^{-5}$cm ($<<$ distance in Cooper pair).
There exists a **supercurrent I_S through the dielectric.**



Quantum mechanics. State of S_j: wave function $\Psi_j = |\Psi_j| e^{i\chi_j}$;

$$\chi_j \text{ is the } \textit{phase}, \quad \phi := \chi_1 - \chi_2.$$

The first Josephson relation: $I_S = I_c \sin\phi$, $I_c \equiv const$.

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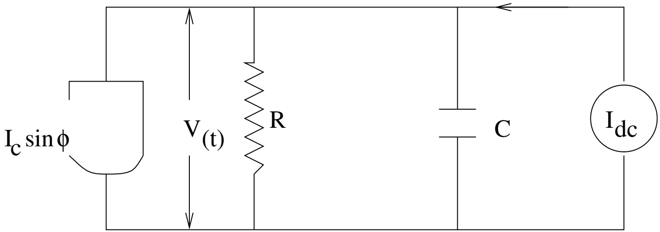
$I_S = I_c \sin\phi$, $I_c \equiv const$.

The supercurrent density through a Josephson junction varies sinusoidally with the phase difference $\phi = \chi_2 - \chi_1$ across the junction in the absence of any scalar and vector potentials.

Equivalent electrical circuit of Josephson junction



See Barone, A. Paterno G. Physics and applications of the Josephson effect 1982, Figure 6.2.

This model electrical scheme is described by the equation

$$\frac{\hbar}{2e} C \frac{d^2\varphi}{dt^2} + \frac{\hbar}{2e} \frac{1}{R} \frac{d\varphi}{dt} + I_c \sin\varphi = I_{dc}$$

Overdamped RSJ case

This scheme is described by the equation

$$\frac{\hbar}{2e}\, C\, \frac{d^2\varphi}{dt^2} + \frac{\hbar}{2e}\, \frac{1}{R}\, \frac{d\varphi}{dt} + I_c \sin\varphi = I_{dc}$$

After appropriate rescaling of the time, we obtain

$$\epsilon\, \frac{d^2\varphi}{d\tau_1^2} + \frac{d\varphi}{d\tau_1} + \sin\varphi = I_c^{-1} I_{dc},$$

$$\epsilon = \frac{\hbar}{2e}\frac{C}{I_c}\left(\frac{2e}{\hbar}\, R\, I_c\right)^2 = \frac{2e}{\hbar}(C\,R)(R\,I_c).$$

Overdamped case: $|\varepsilon| << 1$.

Denote $I_c^{-1} I_{dc} = f(t)$. We obtain

$$\frac{d\phi}{dt} = -\sin\phi + f(t)$$

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We suppose that

- the function $f(t)$ is **periodic with period T;**
- it depends on parameters, one of them is the period T: $f(t) = f(t; u, T)$, $u \in U$.

We introduce a sequence of functions

$$q_n(\theta) = q_n(\theta; u, T) := f(\theta + (n+1)T) - f(\theta + nT), \ \theta \in [0, 1).$$

Problem. For fixed T, find **domains** of those $u \in U$, for which the sequence $q_n(\theta)$ converges pointwise, as $n \to \infty$.

V.M.Buchstaber, O.V.Karpov, S.I.Tertychnyi (2004): translation of the model with periodic f as a family of dynamical systems on two-torus

$$\mathbb{T}^2_{(\phi,\tau)} = \mathbb{R}^2 / 2\pi\mathbb{Z}^2, \ \tau := \omega t, \ \omega := \frac{2\pi}{T}, \ g(\tau) := f(\omega^{-1}\tau):$$

$$\begin{cases} \dot{\phi} = -\sin\phi + g(\tau) \\ \dot{\tau} = \omega \end{cases}. \tag{1}$$

Buchstaber, Karpov and Tertychnyi (2010) have shown that
if the sequence $q_n(\theta)$ **converges pointwise,** then
its **limit** equals 2π times an **integer** constant $\rho =$ the **rotation number.**

$$\mathbb{T}^2_{(\phi,\tau)} = \mathbb{R}^2/2\pi\mathbb{Z}^2, \ \tau := \omega t, \ \omega := \frac{2\pi}{T}, \ g(\tau; u, \omega) := f(\omega^{-1}\tau; u, T):$$

$$\begin{cases} \dot{\phi} = -\sin\phi + g(\tau) \\ \dot{\tau} = \omega \end{cases} . \quad (1)$$

Consider $\phi = \phi(\tau)$.

The rotation number of flow:

$$\rho(u; \omega) := \lim_{n\to+\infty} \frac{\phi(2\pi n)}{n}.$$

H.Poincaré. *Sur les courbes définies par les équations différentialles.* J. Math. Pures App. **I 167** (1885).

Henri Poincaré (1854–1912)

Our main problem: Consider the space U of parameters
of the T-periodic functions $f(t) = f(t; u, T)$.
Describe the **rotation number** ρ of the flow (1) **with fixed** $\omega = \frac{2\pi}{T}$
as a **function on the space** U**.**

Our main problem: Consider the space U of parameters of the T-periodic functions $f(t) = f(t; u, T)$.
Describe the **rotation number** $\rho = \rho(u) = \rho(u; \omega)$ of the flow (1) **with fixed** $\omega = \frac{2\pi}{T}$ as a **function on the space** U**.**

Rotation number $\simeq$ **average voltage** over a long time interval;
up to a known constant factor.

Phase-lock areas: level sets $\{\rho = r\} \subset U$ with non-empty interiors.

Buchstaber, Karpov, Tertychnyi: Given a connected domain $V \subset U$.
The sequence $q_n(\theta; u)$ **converges** pointwise for every $u \in V$ $<=>$
V lies in a **phase-lock area.**

Quantization effect (Buchstaber, Karpov, Tertychnyi, 2010):
Phase-lock area exist only for **integer rotation values**

Many works concern systems (1) with

$$f(t) = f(t; (B, A); \omega) = B + A\cos(\omega t); \;\; u = (B, A) \in \mathbb{R}^2.$$

Quantization effect (Buchstaber, Karpov, Tertychnyi, 2010):
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Many works concern systems (1) with

$$f(t) = f(t; (B, A); \omega) = B + A\cos(\omega t); \;\; u = (B, A) \in \mathbb{R}^2.$$

$$\frac{d\phi}{dt} = -\sin\phi + B + A\cos\omega t. \qquad (2)$$

Equation (2) occurs in other domains of mathematics.
It occurs, e.g.,

in the investigation of some systems with non-holonomic connections by geometric methods.

It describes a model of the so-called Prytz planimeter.

Analogous equation describes the observed direction to a given point at infinity while moving along a geodesic in the hyperbolic plane.

$$\frac{d\phi}{dt} = -\sin\phi + B + A\cos\omega t; \quad \omega \text{ is fixed.}$$

$$\begin{cases} \dot{\phi} = -\sin\phi + B + A\cos(\tau) \\ \dot{\tau} = \omega \end{cases}, \quad (\phi, \tau) \in \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2; \ \tau := \omega t. \qquad (3)$$

A subfamily of (3) occurred in the work by Yu.S.Ilyashenko and J.Guckenheimer "The duck and the devil: canards on the staircase", MMJ 2001, from the slow-fast system point of view. They have obtained results on its limit cycles, as $\omega \to 0$.

Their methods were used by V.A.Kleptsyn, O.L.Romaskevich, I.V.Schurov in their paper "Josephson effect and slow-fast systems", Nanostr., Mat. Fiz. Mod. 2013.

$$\frac{d\phi}{dt} = -\sin\phi + B + A\cos\omega t; \quad \omega \text{ is fixed.}$$

$$\begin{cases} \dot{\phi} = -\sin\phi + B + A\cos(\tau) \\ \dot{\tau} = \omega \end{cases}, \quad (\phi,\tau) \in \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2; \ \tau := \omega t. \qquad (4)$$

$L_r := \{(B,A) \in \mathbb{R}^2 \mid \rho(B,A) = r\}$ **is a phase-lock area** $<=> Int(L_r) \neq \emptyset$.

The **Shapiro step:** the *intersection of the phase-lock area L_r with a line $\{A = A_0\}$.*

ρ is the **average voltage** on the Josephson junction over a long time interval.

Averaging the measured voltage along a wider Shapiro step yields better precision.

$$\begin{cases} \dot{\phi} = -\sin\phi + B + A\cos(\tau) \\ \dot{\tau} = \omega \end{cases}, \quad (\phi, \tau) \in \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2; \ \tau := \omega t.$$

$L_r := \{(B, A) \in \mathbb{R}^2 \mid \rho(B, A) = r\}$ **is a phase-lock area** $<=> Int(L_r) \neq \emptyset$.

Results on the geometry of phase-lock areas.

1) **BKT quantization effect:** Phase-lock areas L_r exist only for $r \in \mathbb{Z}$.

2) There exist functions $\psi_{r,\pm}(A)$ **analytic in** $A \in \mathbb{R}$ such that the **boundary** ∂L_r **is the union of their graphs:**

$$\partial L_r = \partial_+ L_r \cup \partial_- L_r, \ \partial_\pm L_r := \{B = \psi_{r,\pm}(A)\}.$$

$\psi_{r,\pm}(A)$ have **Bessel asymptotics,** as $A \to \infty$.

Observed by Shapiro, Janus, Holly. Proved by A.V.Klimenko and O.L.Romaskevich.

3) Each L_r is an infinite chain (garland) of domains going to infinity, separated by points of intersection $\partial_+ L_r \cap \partial_- L_r$.

Observed numerically by Buchstaber, Karpov, Tertychnyi. Follows from results of Klimenko and Romaskevich.

The separation points with $A \neq 0$ are called **constriction points (constrictions).**

Phase-lock areas for $f(t) = B + A\cos(\omega t)$, $\omega = 2$

Phase-lock areas L_r exist only for $r \in \mathbb{Z}$.
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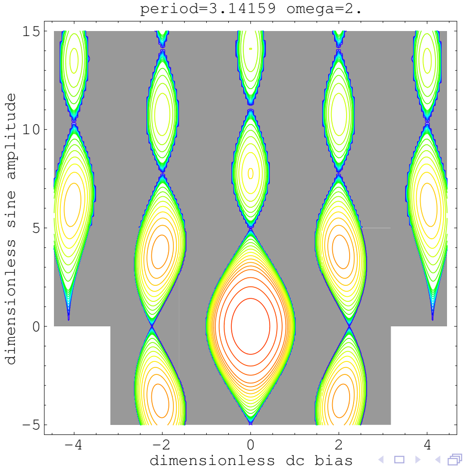
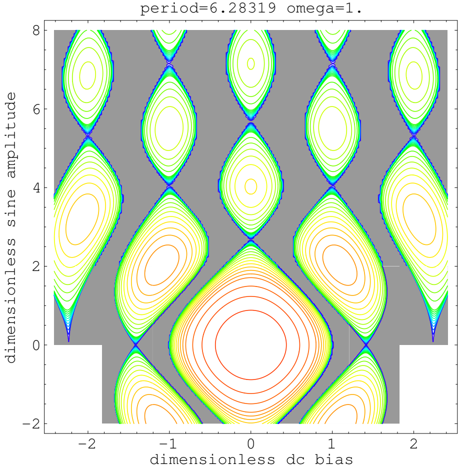


Phase-lock areas for $f(t) = B + A\cos(\omega t)$, $\omega = 1$

Phase-lock areas: level sets $\{\rho(B, A) = r\} \subset \mathbb{R}^2_{B,A}$ with non-empty interiors.

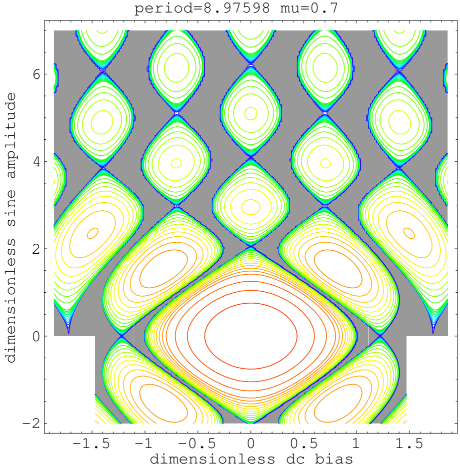
- **quantization:** phase-lock areas exist only for $r \in \mathbb{Z}$.
- infinitely many **constrictions** in every phase-lock area.



Phase-lock areas for $f(t) = B + A\cos(\omega t)$, $\omega = 0.7$

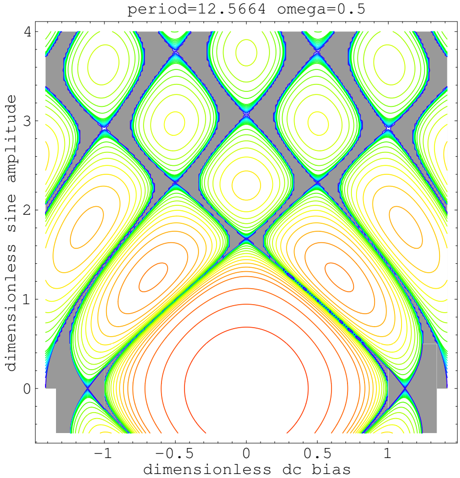
Phase-lock areas: level sets $\{\rho(B, A) = r\} \subset \mathbb{R}^2_{B,A}$ with non-empty interiors.

- **quantization:** phase-lock areas exist only for $r \in \mathbb{Z}$**.**
- infinitely many **constrictions** in every phase-lock areas.



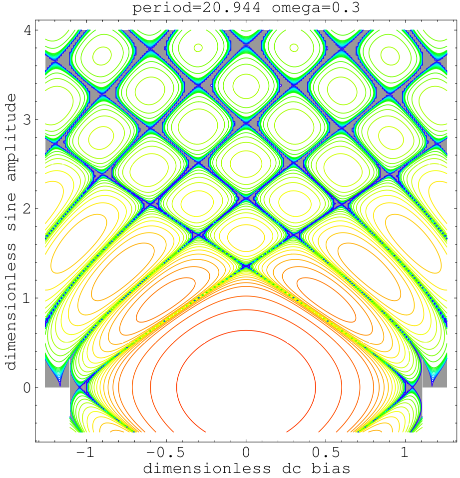
Phase-lock areas for $f(t) = B + A\cos(\omega t), \omega = 0.5$

- **quantization:** phase-lock areas exist only for $r \in \mathbb{Z}$.
- infinitely many **constrictions** in every phase-lock areas.



Phase-lock areas for $f(t) = B + A\cos(\omega t)$, $\omega = 0.3$

- **quantization:** phase-lock areas exist only for $r \in \mathbb{Z}$.
- infinitely many **constrictions** in every phase-lock areas.



- the **portrait** of phase-lock areas is **symmetric:** the transformations $(\phi, \tau) \mapsto (\phi, \tau + \pi)$ and $(\phi, \tau) \mapsto (-\phi, \tau + \pi)$ result in differential equation (2) with changed sign at B (respectively, A).

- Each L_r is an infinite chain (garland) of domains going to infinity, separated by points of intersection $\partial_+ L_r \cap \partial_- L_r$.
- The function $\rho : \rho^{-1}(\mathbb{R} \setminus \mathbb{Z}) \to \mathbb{R} \setminus \mathbb{Z}$ is an **analytic submersion** defining an **analytic fibration by curves.**

Constrictions:= the separation points with $A \neq 0$
The constrictions with $A > 0$ are ordered by their A-coordinates: $\mathcal{A}_{r,1}, \mathcal{A}_{r,2}, \mathcal{A}_{r,3}, \dots$

Open questions based on numerical simulations

Conjecture 1 (quantization of constrictions): *All the constrictions $\mathcal{A}_{r,k}$ lie in the line $\Lambda_r := \{B = r\omega\} :=$* **the axis** *of the phase-lock area L_r.*

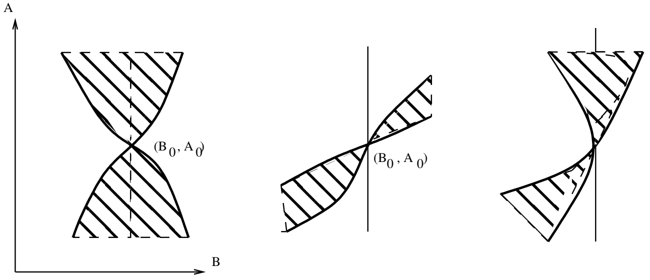
It is based on numerical simulations (Tertychnyi, Filimonov, Kleptsyn, Schurov).

At the moment it is proved that **each constriction $\mathcal{A}_{r,k}$ lies in a line $\{B = \ell\omega\}$, where** $0 \leq \ell \leq r$ **and** $\ell \equiv r(mod 2\mathbb{Z})$ (Filimonov, Glutsyuk, Kleptsyn, Schurov).

Conjecture 1 *All constrictions $\mathcal{A}_{r,k}$ in L_r lie in the* **axis** $\Lambda_r := \{B = r\omega\}$.

Conjecture 2. For $k \geq 2$ the k-th component in L_r contains $(A_{r,k-1}, A_{r,k})$.

Definition. A priori possible types of constrictions:

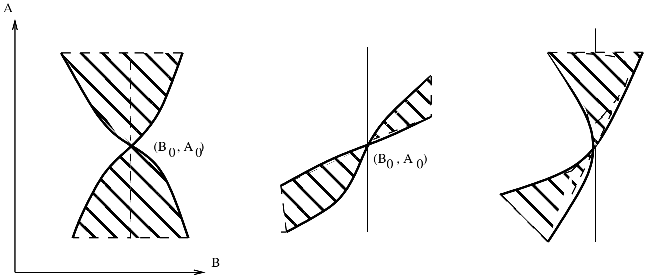


a) Positive b) Negative c) Neutral

Conjecture 3. Each constriction is positive.

Proposition. Conjecture 3 =>Conjecture 1. Conjecture 2 => Conjecture 1.

Definition. A priori possible types of constrictions:



a) Positive b) Negative c) Neutral

Conjecture 3. Each constriction is positive.

Proposition. Conjecture 3 => Conjecture 2 => Conjecture 1.

Theorem 1 (A.Glutsyuk.).
Each constriction is either positive, or negative.

Problem. What happens with the phase-lock area picture, as $\omega \to 0$?

Special double confluent Heun equation

Reduction to special double confluent Heun equation.

$$\frac{d\phi}{d\tau} = \frac{1}{\omega}(-\sin\phi + B + A\cos\tau), \qquad (5)$$

$$z = e^{\tau}, \quad \Phi = e^{i\phi}, \ \ell = \frac{B}{\omega}, \ \mu = \frac{A}{2\omega}, \ \lambda = \frac{1}{4\omega^2} - \mu^2,$$

$$\frac{d\Phi}{dz} = z^{-2}((\ell z + \mu(z^2+1))\Phi - \frac{z}{2i\omega}(\Phi^2 - 1)).$$

This is the projectivization of system of linear equations
in vector function $(u(z), v(z))$ with $\Phi = \frac{v}{u}$:

$$\begin{cases} v' = \frac{1}{2i\omega z} u \\ u' = z^{-2}(-(\ell z + \mu(1+z^2))u + \frac{z}{2i\omega} v) \end{cases} \qquad (6)$$

Reduction of system (5) to a system equivalent to (6) was done in a paper of Buchstaber, Karpov and Tertychnyi in 2010.

Reduction to special double confluent Heun equation

(after papers by Buchstaber and Tertychnyi, 2013-2015). Set

$$E(z) = e^{\mu z} v(z)$$

The system

$$\begin{cases} v' = \frac{1}{2i\omega z} u \\ u' = z^{-2}(-(\ell z + \mu(1 + z^2))u + \frac{z}{2i\omega} v) \end{cases}$$

is equivalent to **special double confluent Heun equation:**

$$z^2 E'' + ((\ell + 1)z + \mu(1 - z^2))E' + (\lambda - \mu(\ell + 1)z)E = 0, \qquad (7)$$

There exist explicit formulas expressing the solution of the non-linear equation

$$\frac{d\phi}{dt} = -\sin\phi + B + A\cos\omega t$$

via solution of equation (7) (Buchstaber - Tertychnyi).

Familes of Heun equations

General 6-parametric family of Heun equations

$$z(z-1)(z-t)E''+$$
$$(c(z-1)(z-t)+dz(z-t)+(a+b+1-c-d)z(z-1))E'+$$
$$+(abz-\nu)E=0. \quad (8)$$

Four Fuchsian singularities: 0, 1, t, ∞.
Parameters: $a, b, c, d; t, \nu$.

Double confluent Heun equation

$$z^2E''+(-z^2+cz+t)E'+(-az+\lambda)E=0$$

is its limit with pairs of confluenting singularities $(0,1)$, (t,∞).

It has only two singularities: 0 and ∞; both are *irregular.*

Two confluent families of Heun equations

Biconfluent Heun equation with two singularities: 0 (regular) and ∞ (irregular).

$$zE'' - (z^2 + tz - c)E' - (az - \lambda)E = 0. \tag{9}$$

It is a limit of confluence of three singular points: t, 1, ∞.
Four-parameter family of equations.

Relation to Stark effect.
In 1919 Stark was awarded the Nobel Prize for Physics for his discovery of the Doppler effect in canal rays and the splitting of spectral lines in electric fields.

The study of **Stark effect for atom of hydrogenium.**
leads to a **Biconfluent Heun equation** (9).

Double confluent Heun equation. Two **irregular** singularities: 0 and ∞.

$$z^2E'' + (-z^2 + cz + t)E' + (-az + \lambda)E = 0 \tag{10}$$

It is the limit of confluence of pairs of singularities $(0, 1)$, (t, ∞).
Four-parameter family of equations.

Problems on special double confluent Heun equations

$$z^2E'' + ((\ell+1)z + \mu(1-z^2))E' + (\lambda - \mu(\ell+1)z)E = 0, \tag{11}$$

This **three-parameter** family of so-called **special double confluent Heun equations** is equivalent to a subfamily of the 4-parameter family of double confluent Heun equations.

It has also two singularities: 0 and ∞;
both are *irregular and non-resonant of Poincaré rank 1.*

Well-known problems on double confluent Heun equations.

Find polynomial solutions.

Find entire solutions.

Geometry of phase-lock areas and special double confluent Heun equations.

Family of **dynamical systems on torus:**

$$\frac{d\phi}{d\tau} = \frac{1}{\omega}(-\sin\phi + B + A\cos\tau)$$

$$z = e^{\tau}, \quad \Phi = e^{i\phi}, \ \ell = \frac{B}{\omega}, \ \mu = \frac{A}{2\omega}, \ \lambda = \frac{1}{4\omega^2} - \mu^2,$$

Family of corresponding **special double confluent Heun equations:**

$$z^2 E'' + ((\ell+1)z + \mu(1-z^2))E' + (\lambda - \mu(\ell+1)z)E = 0. \qquad (12)$$

It is enough to consider the points (B, A) with $\ell = \frac{B}{\omega} \geq 0$ (portrait symmetry).

$(B, A) \in \mathbb{R}^2$ **constriction**, $B > 0 =>$ (12) has **entire solution** (B., Tertychnyi). There is explicit **transcendental** equation $\xi_\ell(\lambda, \mu) = 0$ on the parameters (λ, μ) for which (12) has entire solution (B.-Tertychnyi, B.-Glutsyuk).

The function $\xi_\ell(\lambda, \mu)$ is holomorphic in $(\lambda, \mu) \in \mathbb{C}^2$. Constructed via an infinite product of explicit matrix functions in (λ, μ^2) whose elements are affine functions.

Equation on parameters.

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Polynomial solutions.

Let $\ell \geq 0$. Then

- equation **(12) cannot have polynomial solutions;**

- equation **(12)$_-$** obtained from (12) by $\ell \mapsto -\ell$:

$$z^2 E'' + ((-\ell + 1)z + \mu(1 - z^2))E' + (\lambda + \mu(\ell - 1)z)E = 0 \qquad (12)_-$$

may have polynomial solutions only for $\ell \in \mathbb{Z}_{\geq 0}$.
Existence of polynomial solutions of $(12)_-$ <=> polynomial equation $P(\lambda, \mu^2) = 0$.

$$z^2E'' + ((-\ell+1)z + \mu(1-z^2))E' + (\lambda + \mu(\ell-1)z)E = 0. \qquad (12)_-$$

may have polynomial solutions only for $\ell \in \mathbb{Z}_{\geq 0}$.

Theorem (B.-Tertychnyi + B.-Glutsyuk). Let $\ell \in \mathbb{Z}_{\geq 0}$.
$(12)_-$ has a **polynomial solution,** $=>$ **(12) has no entire solution.**

Definition. (B, A) := **simple intersection,** if $\ell = \frac{B}{\omega} \in \mathbb{Z}_{\geq 0}$,
$(B, A) \in \partial L_r$, $r \equiv \ell (\mathrm{mod}\, 2)$**,** and (B, A) is **not a constriction.**

Theorem (B.-Tertychnyi + B.-Glutsyuk). Let $\ell \in \mathbb{Z}_{\geq 0}$.
Equation $(12)_-$ has a **polynomial solution,** $<=>$ (B, A) is a **simple intersection.**

$$\#(\text{simple intersections contained in } \Lambda_\ell = \{B = \ell\omega\}) \leq \ell.$$

$\mathcal{P}_r :=$ the simple intersection in $\Lambda_r = \{B = r\omega\}$ with the biggest ordinate A.

$$Sr := \{r\omega\} \times [A(\mathcal{P}_r), +\infty) \subset \Lambda_r, \quad L_r^+ := L_r \cap \{A \geq 0\}$$

Conjecture. $L_r^+ \cap \Lambda_r = Sr$ for every $r \in \mathbb{Z} \setminus \{0\}$

Theorem (B.-Tertychnyi + B.-Glutsyuk). Let $\ell \in \mathbb{Z}_{\geq 0}$.
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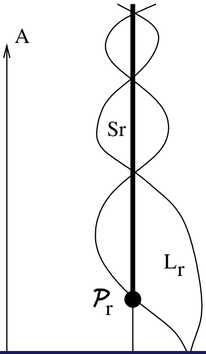
$\mathcal{P}_r :=$ the simple intersection in $\Lambda_r = \{B = r\omega\}$ with the biggest ordinate A.

$$Sr := \{r\omega\} \times [A(\mathcal{P}_r), +\infty) \subset \Lambda_r, \quad L_r^+ := L_r \cap \{A \geq 0\}$$



Conjecture (A.Glutsyuk): $L_r^+ \bigcap \Lambda_r = Sr$ **for nonzero integer r.**

Theorem (A.Glutsyuk) $L_r^+ \bigcap \Lambda_r \supset Sr$

Deduced from Glutsyuk's theorem on absence of neutral constrictions and Bessel asymptotics of boundaries of phase–lock areas (Klimenko – Romaskevich)

Geometry of phase-lock areas and invariants of linear systems

$$\begin{cases} v' = \frac{1}{2i\omega z} u \\ u' = z^{-2}(-(\ell z + \mu(1+z^2))u + \frac{z}{2i\omega} v) \end{cases}. \qquad (6)$$

$M :=$ **monodromy operator around 0 of the equation (6).**

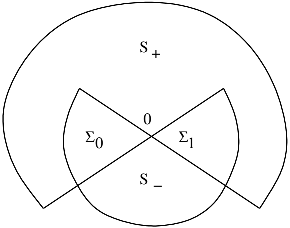
It is a linear operator in the two-dimensional space of germs of solutions at $z_0 \neq 0$:

germ $\mapsto$ its analytic extension along a positive loop.

Fact (Filimonov-Glutsyuk-Kleptsyn-Schurov):

- (B, A) is a **constriction** $<==>$ system (6) has **trivial monodromy:** $M = Id$.

Stokes operators. General theory (Jurkat, Lutz, Peyerimhoff, Sibuya, Balser)



System (6). Canonical fundamental solution matrices $W_\pm(z)$ "at zero" in $S_\pm$:

$$W_\pm(z) = H_\pm(z)F(z), \;\; F(z) = \begin{pmatrix} z^{-\ell}e^{\mu(\frac{1}{z}-z)} & 0 \\ 0 & 1 \end{pmatrix}, H_\pm \in \mathcal{O}(S_\pm) \cap C^\infty(\overline{S}_\pm).$$

$H_\pm(0) = Id$

$$\text{For } \ell \in \mathbb{N}: \quad W_-(z) = W_+(z)C_0 \text{ on } \Sigma_0; \quad W_+(z) = W_-(z)C_1 \text{ on } \Sigma_1,$$

$$C_0 = \begin{pmatrix} 1 & c_0 \\ 0 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix} \quad := \textbf{ Stokes matrices}.$$

Classical theorem: $M = C_1^{-1}C_0^{-1}\,\mathrm{diag}(e^{-2\pi i\ell}, 1)$.

$$C_0 = \begin{pmatrix} 1 & c_0 \\ 0 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix} \quad := \textbf{ Stokes matrices}.$$

Classical theorem: $M = C_1^{-1} C_0^{-1} \operatorname{diag}(e^{-2\pi i \ell}, 1)$.

Corollary (Filimonov, Glutsyuk, Kleptsyn, Schurov).

$$(B, A) \in \mathbb{R}^2 \text{ is a constriction } <=> \begin{cases} c_0 = c_1 = 0 \\ \ell \in \mathbb{Z} \end{cases}.$$

=> abscissas of constrictions: $B = \ell\omega \in \mathbb{Z}\omega$. **Quantization of abscissas.**

Theorem. (A.Glutsyuk). Each constriction is either positive, or negative.

Proof. Fix a constriction (B, A_0), $B = \ell\omega$, $\ell \in \mathbb{N}$. Set $A = A_0 + s$; $c_j = c_j(s)$.
$\det M = 1$, $\operatorname{tr} M = 2 + c_0(c)c_1(s)$ for every $s \in \mathbb{R}$.

(B, A) in a phase-lock area $<==>$ $\operatorname{tr} M > 2$ $<==>$ $c_0(s)c_1(s) \geq 0$.

Suffices to show: $c_0(s)c_1(s)$ **does not change sign,** as s crosses 0.

Buchstaber's suggestion: To study real family of dynamical system on torus it would be helpful to study **the transition matrix** of the linear system (6):

The transition matrix between the fundamental matrices W_+ at 0 and $\hat{W}_-$ at ∞:

$$W_+(z) = \hat{W}_-(z)Q, \;\; Q = \begin{pmatrix} -a & b \\ -c & a \end{pmatrix}.$$

Main results on the transition matrix Q (A.Glutsyuk):
1) $ac_0c_1 = bc_1 - cc_0$; **Theorem (A.G.):** $c_0(s), c_1(s) \in \mathbb{R}$.

2) Key result: $b, c \neq 0$.

Our goal: $c_0(s)c_1(s)$ does not change sign at 0.

Miracle: 1) + 2) $==>$ $c_0(s) \simeq \nu c_1(s)$**, as** $s \to 0$**;** $\nu = \frac{b}{c} \neq 0$**.**

$==>$ $c_0(s)c_1(s) \simeq ps^{2k}$, $p \neq 0$

$==>$ $c_0(s)c_1(s)$ **does not change sign, since** $2k$ **is even!**

Conjecture (A.Glutsyuk). $\nu = \frac{b}{c} > 0$ for the transition matrix corresponding to every constriction. **It would imply: all constrictions are positive.**

Summary

The methods of **real** dynamical systems on torus
and linear **complex** differential equations
have led to solutions of some problems
on phase-lock areas in a model of Josephson effect
and double confluent Heun equations.

The problems of **phase-lock areas** in a model of Josephson effect
have led to the study of an interesting family of dynamical systems
that is transformed to a family of
classical complex linear equations on the Riemann sphere
well-known as **double confluent Heun equations.**

V.M. Buchstaber, O.V. Karpov, S.I.Tertychnyi. *Electrodynamic properties of a Josephson junction biased with a sequence of δ-function pulses.* J. Exper. Theoret. Phys., **93** (2001), No.6, 1280–1287.

Buchstaber, V.M.; Karpov, O.V.; Tertychnyi, S.I. *The rotation number quantization effect.* Theoret and Math. Phys., **162** (2010), No. 2, 211–221.

A.A. Glutsyuk, V.A. Kleptsyn, D.A. Filimonov, I.V. Schurov. *On the adjacency quantization in an equation modeling the Josephson effect.* Funct. Analysis and its Appl. **45** (2011), No. 3, 192–203.

Buchstaber, V.M.; Tertychnyi, S.I. *Holomorphic solutions of the double confluent Heun equation associated with the RSJ model of the Josephson junction.* Theoret. and Math. Phys., **182:3** (2015), 329–355.

V.M.Buchstaber, A.A.Glutsyuk. *On monodromy eigenfunctions of Heun equations and boundaries of phase-lock areas in a model of overdamped Josephson effect.* Proc. Steklov Inst. of Math., **297** (2017). Volume dedicated to D.V.Anosov.

Glutsyuk, A.A. *On constrictions of phase-lock areas in model of overdamped Josephson effect and transition matrix of the double-donfluent Heun equation.* J. Dyn. Control Syst. Published First Online: https://doi.org/10.1007/s10883-018-9411-1

HAPPY BIRTHDAY,

YULY SERGEEVICH!