Combinatorial analogs of fixed point theorems

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papers

O. R. Musin, Borsuk–Ulam type theorems for manifolds, *Proc. Amer. Math. Soc.* **140** (2012)

O. R. Musin, Extensions of Sperner and Tucker's lemma for manifolds, *J. of Combinatorial Theory Series A*, **132** (2015)

O. R. Musin, Sperner type lemma for quadrangulations, *Mosc. J. of Combinatorics and Number Theory*, **5** (2015).

O. R. Musin and A. Yu. Volovikov, Borsuk–Ulam type spaces, *Mosc. Math. J.*, **15:4** (2015)

O. R. Musin, Homotopy invariants of covers and KKM type lemmas, *Algebraic & Geometric Topology*, **16** (2016)

papers

O. R. Musin, Generalizations of Tucker–Fan–Shashkin lemmas, *Arnold Math J.*, **2:3** 2016

O. R. Musin, KKM type theorems with boundary conditions, *J. Fixed Point Theory Appl.*, **19** (2017)

O. R. Musin and Jie Wu, Cobordism classes of maps and covers for spheres, *Topology Appl.*, **237** (2018)

O. R. Musin and A. Yu. Volovikov, Tucker type lemmas for G-spaces, preprint, arXiv:1612.07314

A. V. Malyutin and O. R. Musin, Neighboring mapping points theorem, preprint, December 2018

Topological Combinatorics

J. Matoušek, *Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry*, Springer, Berlin, 2003

M. de Longueville, *A Course in Topological Combinatorics*, Springer, 2012

Brouwer fixed point theorem (1909)

Brouwer fixed point theorem.
Any continuous map $f : B^d \to B^d$ *must have a fixed point.*

Here B^d denote a d–dimensional disc.

Sperner lemma (1928)

Theorem

(Sperner lemma) *Every Sperner labelling of a triangulation of a d-dimensional simplex contains a cell labelled with a complete set of labels:* $\{1, 2, \ldots, d + 1\}$.

Sperner lemma

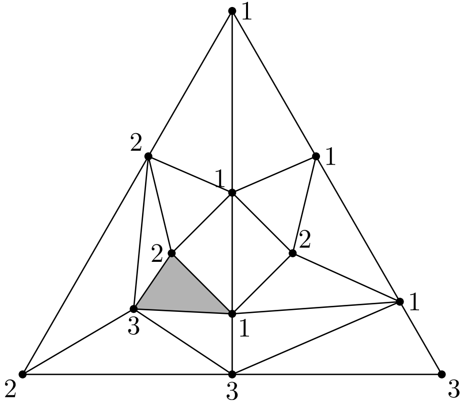


Figure: A 2-dimensional illustration of Sperner's lemma

Möbius band

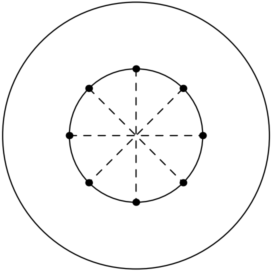
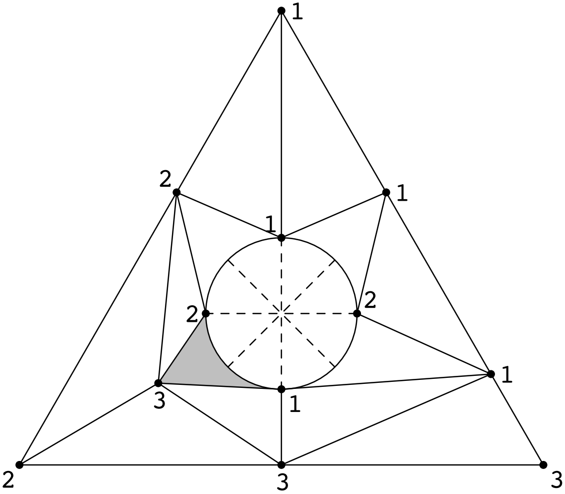


Figure: Möbius band. Diametrically opposite points of the inner boundary circle are to be identified. The outer circle is the boundary of the Möbius band.

Sperner's lemma for the Möbius band



KKM theorem

Theorem (Knaster–Kuratowski–Mazurkiewicz (1929))

If a simplex Δ^m is covered by the closed sets C_i for $i \in I_m := \{0, \dots, m\}$ and that for all $J \subset I_m$ the face of Δ^m that is spanned by vertices v_i for $i \in J$ is covered by C_i for $i \in J$ then all the C_i have a common intersection point.

KKM theorem

The two–dimensional case may serve as an illustration. In this case the simplex Δ^2 is a triangle, whose vertices we can label 1, 2 and 3. We are given three closed sets C_1, C_2, C_3 which collectively cover the triangle; also we are told that C_1 covers vertex 1, C_2 covers vertex 2, C_3 covers vertex 3, and that the edge 12 (from vertex 1 to vertex 2) is covered by the sets C_1 and C_2, the edge 23 is covered by the sets C_2 and C_3, the edge 31 is covered by the sets C_3 and C_1. The KKM lemma states that the sets C_1, C_2, C_3 have at least one point in common.

The Borsuk-Ulam theorem (1933)

The Borsuk - Ulam theorem (Borsuk, 1933). Four equivalent statements:
(a) For every continuous mapping $f : \mathbb{S}^n \to \mathbb{R}^n$ there exists a point $x \in \mathbb{S}^n$ with $f(x) = f(-x)$.
(b) For every antipodal (i.e. $f(-x) = -f(x)$) continuous mapping $f : \mathbb{S}^n \to \mathbb{R}^n$ there exists a point $x \in \mathbb{S}^n$ with $f(x) = 0$.
(c) There is no antipodal continuous mapping $f : \mathbb{S}^n \to \mathbb{S}^{n-1}$.
(d) There is no continuous mapping $f : \mathbb{B}^n \to \mathbb{S}^{n-1}$ that is antipodal on the boundary.

Tucker's lemma (1945)

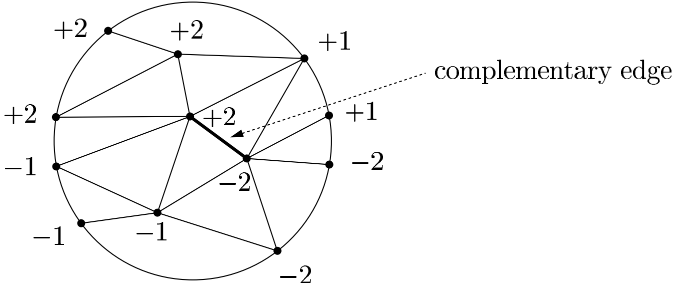
Theorem (Tucker)

Let Λ be a triangulation of the ball $\mathbb{B}^d$ that is antipodally symmetric on the boundary. Let

$$L : V(\Lambda) \to \{+1, -1, +2, -2, \dots, +d, -d\}$$

be a labelling of the vertices of Λ that satisfies $L(-v) = -L(v)$ for every vertex v on the boundary $\mathbb{B}^d$. Then there exists an edge in Λ that is "complementary": i.e., its two vertices are labelled by opposite numbers.

Tucker lemma



Tucker lemma for spheres

Theorem

Let Λ *be an antipodal triangulation of* $\mathbb{S}^d$. *Let*

$$L : V(\Lambda) \to \{+1, -1, +2, -2, \ldots, +d, -d\}$$

be an antipodal labelling of the vertices of Λ *that satisfies* $L(-v) = -L(v)$ *for all vertices. Then* Λ *contains a complimentary edge.*

Borsuk-Ulam theorem for the double torus

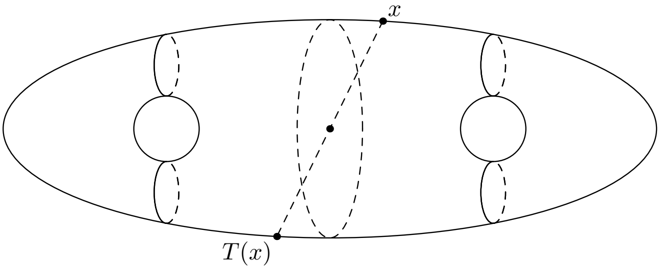


Figure: The double torus that is centrally symmetric embedded to $\mathbb{R}^3$.

Borsuk-Ulam theorem for the double torus

Theorem

Let M_2^2 denote the double torus that is centrally symmetric embedded to $\mathbb{R}^3$. Let $T(x) := -x,\ x \in M_2^2$.

(a) For every continuous mapping $f : M_2^2 \to \mathbb{R}^2$ there exists a point $x \in M_2^2$ with $f(x) = f(T(x))$.

(b) For every antipodal (i.e. $g(T(x)) = -g(x)$) continuous mapping $g : M_2^2 \to \mathbb{R}^2$ there exists a point $x \in M_2^2$ with $g(x) = 0$.

Borsuk-Ulam theorem for manifolds

Our analysis of Bárány's proof shows that it can be extended for a wide class of manifolds. For instance, consider two-dimensional orientable manifolds $X = M_g^2$ of even genus g and non-orientable manifolds $X = N_m^2$ with even m. We can assume that X is "centrally symmetric" embedded to $\mathbb{R}^k$, where $k = 3$ for $X = M_g^2$ and $k = 4$ for $X = N_m^2$.

That means $A(X) = X$, where $A(x) = -x$ for $x \in \mathbb{R}^k$. Then $T := A|_X : X \to X$ is a free involution. It can be shown that there is a projection of $X \subset \mathbb{R}^k$ into a 2-plane R passing through the origin 0 with $|Z_{f_0}| = 2$.

$\mathbb{Z}_2$-maps

Let us consider a closed smooth manifold M with a free smooth involution $T : M \to M$, i.e. $T^2(x) = x$ and $T(x) \neq x$ for all $x \in M$. For any $\mathbb{Z}_2$-manifold (M, T) we say that a map $f : M^m \to \mathbb{R}^n$ is *antipodal* (or equivariant) if $f(T(x)) = -f(x)$.

We say that a closed $\mathbb{Z}_2$-manifold (M, T) is a *BUT (Borsuk-Ulam Type) manifold* if for any continuous map $F : M^n \to \mathbb{R}^n$ there is a point $x \in M$ such that

$$F(T(x)) = F(x).$$

In other words, if a continuous map $f : M^n \to \mathbb{R}^n$ is antipodal, then the set $Z_f := f^{-1}(0)$ is not empty.

BUT manifolds

Theorem

Let M^n be a closed connected manifold with a free involution T. Then the following statements are equivalent:

(a) For any antipodal continuous map $f : M^n \to \mathbb{R}^n$ the set Z_f is not empty.

(b) M admits an antipodal continuous transversal map $h : M^n \to \mathbb{R}^n$ with $|Z_h| = 4k + 2, \ k \in \mathbb{Z}$.

(c) For any equivariant triangulation Λ of M and for any Tucker's labeling of $V(\Lambda)$ there is a complementary edge.

(d) $[M^n, T] = [\mathbb{S}^n, A] + [V^1][\mathbb{S}^{n-1}, A] + \ldots + [V^n][\mathbb{S}^0, A]$ in $\mathfrak{N}_n(\mathbb{Z}_2)$.

BUT manifolds

(e) M is a Lyusternik-Shnirelman type manifold, i.e. for any cover $F_1, \ldots, F_{n+1}$ of M^n by $n+1$ closed (respectively, by $n+1$ open) sets, there is at least one set containing a pair $(x, T(x))$.

(f) M is a Tucker type manifold, i.e. for any equivariant labelling $L : V(\Lambda) \to \{+1, -1, +2, -2, \ldots, +n, -n\}$ of any equivariant triangulation Λ of M there exists a complementary edge.

(g) M is a Ky Fan type manifold, i.e. for any equivariant labelling $L : V(\Lambda) \to \{+1, -1, +2, -2, \ldots, +m, -m\}$ there is a complementary edge or an odd number of n-simplices whose labels are of the form $\{k_0, -k_1, k_2, \ldots, (-1)^d k_n\}$, where $1 \leq k_0 < k_1 < \ldots < k_n \leq m$.

BUT manifolds

(h) **(KKM type)** For any covering of M by a family of $2n$ closed sets $\{C_1, C_{-1}, \ldots, C_n, C_{-n}\}$, where C_i and C_{-i} are antipodal, i. e. $C_{-i} = T(C_i)$, for all $i = 1, \ldots, d$, then there is k such that C_k and C_{-k} have a common intersection point.

(i) **(Kakutani type)** Let $F : M \to 2^{\mathbb{R}^n}$ be a set-valued function on M with a closed graph and the property that for all $x \in M$ the set $F(x)$ is non-empty and convex in $\mathbb{R}^n$ and there is $y \in F(x)$ such that $(-y) \in F(T(x))$. Then there is $x_0 \in M$ such that $F(x_0)$ covers the origin $0 \in \mathbb{R}^n$.

Gale's colored KKM lemma

David Gale in 1984 proved an existence theorem for an exchange equilibrium in an economy with indivisible goods and only one perfectly divisible good, which can be thought of as money. He proved the following lemma:

For $i, j = 1, 2, \ldots, n$ *let* S_j^i *be closed sets such that for each* i, $S_1^i, \ldots, S_n^i$ *is a KKM covering of* Δ^{n-1}. *Then there exists a permutation* π *of* $1, 2, \ldots, n$ *such that*

$$\bigcap_{i=1}^{n} S_{\pi(i)}^i \neq \emptyset.$$

Gale's colored KKM lemma

Gale wrote about his lemma:

"A colloquial statement of this result is the *red, white and blue lemma* which asserts that *if each of three people paint a triangle red, white and blue according to the KKM rules, then there will be a point which is in the red set of one person, the white set of another, the blue of the third.*"

Gale's colored KKM lemma

While there are many real–life examples that can fit into this framework, we will use for concreteness the terminology of room–assignment and rent–division: Several rooms with different characteristics and given capacities are available in a house, and the total rent for the house needs to be divided between the rooms. In this context, envy–freeness boils down to a market clearing condition: A price is assigned to each room such that when each agent chooses his/her favorite room (given the prices) supply exactly equals demand and the market clears. Following Su (1999), we call such a situation *rental harmony*.

New York Times' article: April 28, 2014

The New York Times

Science

To Divide the Rent, Start With a Triangle

By ALBERT SUN

APRIL 28, 2014

Last year, two friends and I moved into a small three-bedroom apartment in Manhattan. We chose it for its relatively reasonable price — around $3,000 a month — and its convenient location. Just finding it was a challenge, but then we faced another one: deciding who would get each bedroom.
The bedrooms were different sizes, ranging from small to very small. Two faced north toward the street and had light; the third and smallest faced an alley. The largest had two windows; the midsize room opened onto the fire escape.
Every month, unrelated people move into apartments together to save on rent. Many decide to simply divide the rent evenly, or to base it on bedrooms' square footage or perhaps even on each resident's income.
But as it turns out, a field of academics is dedicated to studying the subject of fair division, or how to divide good and bad things fairly among groups of people. To the researchers, none of the typical methods are satisfactory. They have better ways.

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After his paper on the method was published, Dr. Su worked with one of his students, Elisha Peterson, to create a calculator to promote fair division.

Need to divide your rent? Try our updated version of the rent division calculator implementing Dr. Su's method.

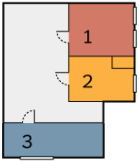
New York Times' article: April 28, 2014

Sperner's Lemma and Rental Harmony

A mathematical theorem called Sperner's Lemma can be used to divide unequal assets fairly.

The Problem

Three friends **Ashwin**, **Bret** and **Chad** want to share an apartment.
The total rent is $3,000 but the rooms are different sizes. How can they choose rooms and divide the rent fairly?



The Solution

This triangle represents every possible combination of prices.
At each **corner**, one room costs $3,000 and the others are free. Not a good solution.
In the **center** the rent is split evenly.
At each point, either **A**shwin, **B**ret or **C**had is asked to choose which room he prefers at the given price.

Sperner's Lemma

Sperner's lemma guarantees that there is a small triangle where every roommate has picked a different room. The "fair" price lies somewhere between the prices at those three corners.

Shapley's KKMS theorem

In 1967 Scarf proved that any non-transferable utility game whose characteristic function is balanced, has a non–empty core. His proof is based on an algorithm which approximates fixed points. Lloyd Shapley (1973) replaced the Scarf algorithm by a covering theorem (the KKMS theorem) being a generalization of the KKM theorem. Now Shapley's KKMS theorem is an important tool in the general equilibrium theory of economic analysis.

Shapley's KKMS theorem

Theorem

Let $\mathcal{K}$ be the collection of all non-empty subsets of $I_{k+1} := \{1, \ldots, k+1\}$. Consider a simplex S in $\mathbb{R}^k$ with vertices $x_1, \ldots, x_{k+1}$. Let $V := \{v_\sigma,\ \sigma \in \mathcal{K}\} \subset \mathbb{R}^k$, where v_σ denotes the center of mass of $S_\sigma := \{x_i,\ i \in \sigma\}$.
Let $\mathcal{C} := \{C_\sigma,\ \sigma \in \mathcal{K}\}$ be a cover of $|\Delta^k|$ such that for every $J \subset I_{k+1}$ the simplex Δ_J that is spanned by vertices from J is covered by $\{C_\sigma,\ \sigma \in J\}$. Then there exists a balanced collection $\mathcal{B}$ in $\mathcal{K}$ with respect to V such that

$$\bigcap_{\sigma \in \mathcal{B}} C_\sigma \neq \emptyset.$$

Shapley's KKMS theorem

L. S. Shapley On balanced games without side payments, in *Mathematical Programming,* Hu, T.C. and S.M. Robinson (eds), Academic Press, New York, 261–290, 1973.

L. S. Shapley and R. Vohra, On Kakutani's fixed point theorem, the KKMS theorem and the core of a balanced game, *Economic Theory*, **1** (1991), 108–116.

H. Komiya, A simple proof of the K–K–M–S theorem, *Economic Theory*, **4** (1994), 463–466.

P. J. J. Herings, An extremely simple proof of the K–K–M–S theorem, *Economic Theory*, **10** (1997), 361–367.

Degree and Sperner's lemma

Theorem (M., 2015)

Let T be a triangulation of a PL orientable d-dimensional manifold M with boundary. Let $L : V(T) \to \{1, 2, \ldots, d+1\}$ be any labelling. Then T contains at least $|\deg(f_L, \partial T)|$ fully labelled d-simplices, where $f_L : T \to \Delta^d$ and Δ^d is a d-dimensional simplex with vertices $1, 2, \ldots, d+1$.

For Sperner's labelling $\deg(L, \partial T) := \deg(f_L, \partial T) = \pm 1$.

Degree and Sperner's lemma: example

Let $L : K^0 \to \{1,2,3\}$ be a labelling of a planar polygon $K := p_1 p_2 \dots p_k$. Let $V = \{v_1, v_2, v_3\}$ be vertices of a triangle Δ.

$$f_L(p_i) := v_j, \; where \; j = L(p_i)$$

Then f_L is a map from $K = \mathbb{S}^1$ to $\Delta = \mathbb{S}^1$.

$$\deg(f_L) := p_* - n_*,$$

where p_* (respectively, n_*) is the number of (ordering) pairs (p_i, p_{i+1}) such that $L(p_i) = 1$ and $L(p_{i+1}) = 2$ (respectively, $L(p_i) = 2$ and $L(p_{i+1}) = 1$).

For instance, let $L = (122123123211223123 1)$. Then $p_* = 5$ and $n_* = 2$. Thus,

$$\deg f_L = 5 - 2 = 3.$$

Degree and Sperner's lemma

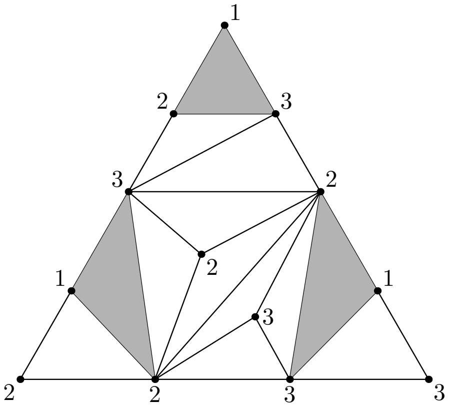


Figure: $\deg(L, \partial T) = 3$. There are three fully labelled triangles.

De Loera - Petersen - Su theorem (2002)

Theorem (M., 2015)

Let P be a convex polytope in $\mathbb{R}^d$ with n vertices. Let T be a triangulation of a compact oriented PL–manifold M of dimension d with boundary. Let $L : V(T) \to \{1, 2, \ldots, n\}$ be a labelling such that $f_{L,P}(\partial M) \subseteq \partial P$. Then T contains at least $(n-d)|\deg(L, \partial T)|$ fully labelled d-simplices.

A generalization of the De Loera - Petersen - Su theorem

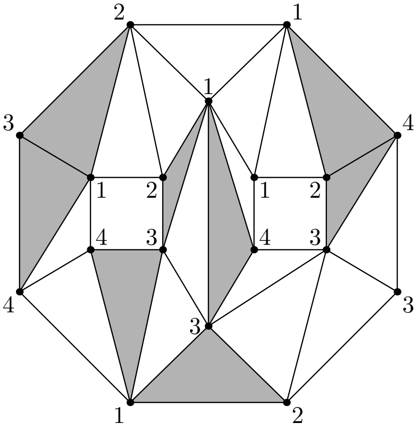


Figure: Octagon with two square holes. Here $n = 4$, $\deg(L, \partial T)| = 4$ and there are eight fully labelled triangles

Generalization of Tucker's lemma

Corollary

Let $V := \{\pm e_1, \ldots, \pm e_n\}$*, where* $e_1, \ldots, e_n$ *is al basis in* $\mathbb{R}^n$*. Let* $\mathcal{F} = \{F_1, F_{-1} \ldots, F_n, F_{-n}\}$ *be a cover of* $\mathbb{B}^k$ *that is not null–homotopic on the boundary. Then there is* i *such that the intersection of* F_i *and* F_{-i} *is not empty.*
In particular, if $\mathcal{F}$ *is antipodally symmetric on the boundary of* $\mathbb{B}^k$*, i. e. for all* i *we have* $S_{-i} = -S_i$*, where* $S_i := F|_{\mathbb{S}^{k-1}}$*, then there is* i *such that* $F_i \cap F_{-i} \neq \emptyset$*.*

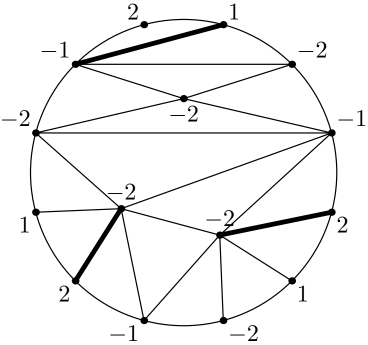


Figure: Since $\deg(L, \partial T) = 3$, there are three complementary edges.

Homotopy invariants of covers and KKM type lemmas

Homotopy invariants of covers and Sperner type lemmas

With any labelling of a simplicial complex T we associate certain homotopy classes of maps T into spheres. These homotopy invariants can be considered as obstructions for extensions of covers of a subspace A to a space X. We using these obstructions for generalizations of Sperner's lemma. In particular, we show that in the case when A is a k–sphere and X is a $(k+1)$–disk there exist Sperner type lemmas for covers by $n+2$ sets if and only if the homotopy group $\pi_k(\mathbb{S}^n) \neq 0$.

Homotopy invariants of covers and Sperner type lemmas

Example 1: Let T be a triangulation of a tetrahedron S ($S = \mathbb{B}^3$). $L : V(\partial T) \to \{1, 2, 3\}$. If in $\partial T = \mathbb{S}^2$ there are no fully labeled triangles, then $f_L : \mathbb{S}^2 \to \mathbb{S}^1$. So f_L is null–homotopic and f_L can be extended to $f_L : \mathbb{B}^3 \to \mathbb{S}^1$.

Homotopy invariants of covers and Sperner type lemmas

Example 2: Let T be a triangulation of a $(k+1)$–simplex Δ^{k+1}. $L : V(\partial T) \to \{1, 2, 3, 4\}$. If in $\partial T = \mathbb{S}^d$ there are no fully labeled simplices, then we have $f_L : \mathbb{S}^k \to \mathbb{S}^2$. If $k \geq 2$, $\pi_k(\mathbb{S}^2) \neq 0$, then there are non null–homotopic maps (labelings). Thus, we have a Sperner type lemma.

For the case $k = 3$ we can construct a labeling from the Hopf fibration (map) $p : \mathbb{S}^3 \to \mathbb{S}^2$.

Homotopy invariants of covers and Sperner's type lemma

Theorem (M., 2016)

Let T be a triangulation of Δ^{k+1}. Let $L : V(T) \to \{1, 2, \ldots, m+2\}$ be any labeling. Suppose $[f_L] \neq 0$ in $\pi_k(\mathbb{S}^m)$. Then T must contain a fully labeled $(m+1)$-simplex.

Labeling and covering

Let K be a simplicial complex. Denote by $\mathrm{St}(u)$ the open star of a vertex $u \in \mathrm{Vert}(K)$. In other words, $\mathrm{St}(u)$ is $|S| \setminus |B|$, where S is the set of all simplices in K that contain u, and B is the set of all simplices in S that contain no u.

Let

$$L : \mathrm{Vert}(K) \to \{0, 1, \ldots, m\}$$

be a labeling of vertices of K. There is a natural open cover of $|K|$

$$\mathcal{U}_L(K) = \{U_0(K), \ldots, U_m(K)\},$$

where

$$U_\ell(K) := \bigcup_{u \in W_\ell} \mathrm{St}(u), \quad W_\ell := \{u \in \mathrm{Vert}(K) : L(u) = \ell\}.$$

Partition of unity

Let $\mathcal{U} = \{U_0, \ldots, U_m\}$ be a collection of open sets whose union contains a space X. In other words, $\mathcal{U}$ is an *open cover* of X.

A collection of functions $\Phi = \{\varphi_0, \ldots, \varphi_m\}$ is called a *partition of unity subordinate to* $\mathcal{U}$, if

1. $\varphi_i(x) \geq 0$, $i = 0, \ldots, m$, for all $x \in X$.
2. $\text{supp}(\varphi_i) \subset U_i$, $i = 0, \ldots, m$.
3. $\varphi_0(x) + \ldots + \varphi_m(x) = 1$ for all $x \in X$.

Homotopy invariants of covers

Let $\Phi = \{\varphi_0, \ldots, \varphi_m\}$ be a partition of unity subordinate to $\mathcal{U}$. Suppose the intersection of all the U_i is empty. Let

$$\rho_{\mathcal{U},\Phi}(x) := \sum_{i=0}^{m} \varphi_i(x) v_i, \; x \in X$$

where $v_0, \ldots, v_m$ are vertices of an m–dimensional simplex Δ^m that are considered as vectors in $\mathbb{R}^m$. Then $\rho_{\mathcal{U},\Phi}$ is a continuous map from X to $\partial\Delta^m = \mathbb{S}^{m-1}$.

$$[\rho_{\mathcal{U},\Phi}] = [\mathcal{U}] \in [X, \mathbb{S}^{m-1}].$$

If $X = \mathbb{S}^k$, then $[\mathcal{U}] \in \pi_k(\mathbb{S}^{m-1})$.

KKM type lemma

Theorem

Let $\mathcal{F} = \{F_0, \ldots, F_m\}$ be an open or closed cover of $\mathbb{B}^{k+1}$ that extends a cover $\mathcal{S} = \{S_0, \ldots, S_m\}$ of $\partial\mathbb{B}^{k+1} = \mathbb{S}^k$. If the intersection of all S_i is empty and $[\mathcal{S}] \neq 0$ in $\pi_k(\mathbb{S}^{m-2})$, then all the F_i have a common intersection point.

Eilenberg – Pontryagin spaces

Definition

We say that a pair of spaces (X, A), where $A \subset X$, belongs to EP_n and write $(X, A) \in \mathrm{EP}_n$ if there is a continuous map $f : A \to \mathbb{S}^n$ with $[f] \neq 0$ in $[A, \mathbb{S}^n]$ that cannot be extended to a continuous map $F : X \to \mathbb{S}^n$ with $F|_A = f$.

We denoted this class of pairs by EP after S. Eilenberg and L. S. Pontryagin who initiated obstruction theory in the late 1930s. Note that
(i) if $\pi_k(\mathbb{S}^n) \neq 0$, then $(\mathbb{B}^{k+1}, \mathbb{S}^k) \in \mathrm{EP}_n$,
(ii) if X is an oriented $(n+1)$–dimensional manifold and $A = \partial X$, then $(X, A) \in \mathrm{EP}_n$.

Homotopy invariants of covers and Gale's lemma

Theorem

Let A be a subspace of a space X. Let $(X, A) \in \mathrm{EP}_{n-2}$. Let $\mathcal{S}^i = \{S_1^i, \ldots, S_n^i\}$, $i = 1, \ldots, n$, be covers of (X, A). Let

$$F^i := \bigcup_{j=1}^{n} S_j^i, \quad \mathcal{F} := \{F^1, \ldots, F^n\}, \quad \mathcal{C} := \mathcal{F}|_A.$$

Suppose $\mathcal{C}$ is not null–homotopic. Then there exists a permutation π of $1, 2, \ldots, n$ such that

$$\bigcap_{i=1}^{n} S_{\pi(i)}^i \neq \emptyset.$$

Rental Harmony Theorem

Denote by S^i_j a set of price vectors p in Δ^{n-1} such that housemate i likes room j at these prices. Consider the following conditions:

(C1) In any partition of the rent, each person finds some room acceptable. In other words, $\mathcal{S}^i = \{S^i_1, \ldots, S^i_n\}$, $i = 1, \ldots, n$, is a (closed or open) cover of Δ^{n-1}.
(C2) Each person always prefers a free room (one that costs no rent) to a non–free room. In other words, for all i and j, S^i_j contains $\Delta^{n-1}_j := \{(x_1, \ldots, x_n) \in \Delta^{n-1} \mid x_j = 0\}$.

Rental Harmony Theorem [Su, 1999] *Suppose n housemates in an n–bedroom house seek to decide who gets which room and for what part of the total rent. Also, suppose that the conditions (C1) and (C2) hold. Then there exists a partition of the rent so that each person prefers a different room.*

Rental Harmony Theorem

Suppose there are some constraints $f_i(p) \leq 0$, $i = 1, \ldots, n$, for price vectors. Let $M := \{p \in \Delta^{n-1} \mid f_1(p) \leq 0, \ldots, f_k(p) \leq 0\}$ be a manifold of dimension $n - 1$. Let S_j^i be sets of price vectors p in M such that housemate i likes room j at these prices. Consider the following conditions:

(A1) $\mathcal{S}^i = \{S_1^i, \ldots, S_n^i\}$, $i = 1, \ldots, n$, is a cover of M.

(A2) $\deg \mathcal{C} \neq 0$, where $\mathcal{C} := \mathcal{F}|_{\partial M}$, $\mathcal{F} := \{F^1, \ldots, F^n\}$, and $F^i := \bigcup_{j=1}^n S_j^i$.

Theorem

Suppose n housemates in an n–bedroom house seek to decide who gets which room and for what part of the total rent. Also, suppose that the conditions (A1) and (A2) hold. Then there exists a partition of the rent so that each person prefers a different room.

$V := \{v_1, \ldots, v_m\}$ in $\mathbb{R}^n$. Denote by c_V the center of mass of V, $c_V := (v_1 + \ldots + v_m)/m$.

Let $\mathcal{U} = \{U_1, \ldots, U_m\}$ be an open cover of a space T and $\Phi = \{\varphi_1, \ldots, \varphi_m\}$ be a partition of unity subordinate to $\mathcal{U}$. Let

$$\rho_{\mathcal{U},\Phi,V}(x) := \sum_{i=1}^{m} \varphi_i(x) v_i.$$

Suppose c_V lies outside of the image $\rho_{\mathcal{U},\Phi,V}(T)$ in $\mathbb{R}^n$. Let for all $x \in T$

$$f_{\mathcal{U},\Phi,V}(x) := \frac{\rho_{\mathcal{U},\Phi,V}(x) - c_V}{||\rho_{\mathcal{U},\Phi,V}(x) - c_V||}.$$

Then $f_{\mathcal{U},\Phi,V}$ is a continuous map from T to $\mathbb{S}^{n-1}$.
Actually, the homotopy class $[f_{\mathcal{U},\Phi,V}] \in [T, \mathbb{S}^{n-1}]$ does not depend on Φ and then the homotopy class $[f_{\mathcal{U},V}]$ in $[T, \mathbb{S}^{n-1}]$ is well define.

KKMS theorem

Definition

Let I be a set of labels of cardinality m. Let $V := \{v_i,\ i \in I\}$, be a set of points in $\mathbb{R}^n$. Then a nonempty subset $\mathcal{B} \subset I$ is said to be balanced with respect to V if for all $i \in \mathcal{B}$ there exist non-negative λ_i such that

$$\sum_{i \in \mathcal{B}} \lambda_i v_i = c_V, \quad \text{where} \quad \sum_{i \in \mathcal{B}} \lambda_i = 1 \quad \text{and} \quad c_V := \frac{1}{m} \sum_{i \in I} v_i.$$

In other words, $c_V \in \mathrm{conv}\{v_i,\ i \in \mathcal{B}\}$, where $\mathrm{conv}(Y)$ denote the convex hull of Y in $\mathbb{R}^n$.

KKMS theorem

Theorem

Let $V := \{v_1, \ldots, v_m\}$ *be a set of points in* $\mathbb{R}^n$. *Let* A *be a subspace of a space* X. *Let* $(X, A) \in \mathrm{EP}_{n-1}$. *Let* $\mathcal{F} = \{F_1, \ldots, F_m\}$ *be a cover of* (X, A). *Suppose* $\mathcal{S} := \mathcal{F}|_A$ *is not null–homotopic. Then there is a balanced subset* $\mathcal{B}$ *in* I_m *with respect to* V *such that*

$$\bigcap_{i \in \mathcal{B}} F_i \neq \emptyset.$$

KKMS theorem

Corollary

Let $V := \{v_1, \dots, v_m\}$ be a set of points in $\mathbb{R}^n$. Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a cover of $\mathbb{B}^k$ that is not null–homotopic on the boundary. Then there is a balanced with respect to V subset $\mathcal{B}$ in $I_m := \{1, \dots, m\}$ such that the intersection of all F_i, $i \in \mathcal{B}$, is not empty.

If $V = \mathrm{Vert}(\Delta^n)$, then this corollary implies the KKM theorem.

Let $\mathcal{K}$ be the collection of all non-empty subsets of I_{k+1}. If $V := \{v_\sigma,\ \sigma \in \mathcal{K}\} \subset \mathbb{R}^k$, where v_σ denotes the center of mass of $S_\sigma := \{x_i,\ i \in \sigma\}$, then the corollary implies the KKMS theorem.

Neighboring mapping points theorem

Theorem [Malytuin & M, Dec 2018]. *Let M be a contractible metric space and let $f : S^n \to M$ be a continuous map. Then there are two points p and q in S^n and a sphere S_R of radius $R \geq 0$ in M such that the Euclidean distance $\|p - q\|$ between p and q is at least $\sqrt{\frac{n+2}{n}}$, both $f(p)$ and $f(q)$ lie on S_R, and there are no points of $f(S^n)$ inside of S_R (in the case $R = 0$, this means that $f(p) = f(q)$).*

Neighboring mapping points theorem

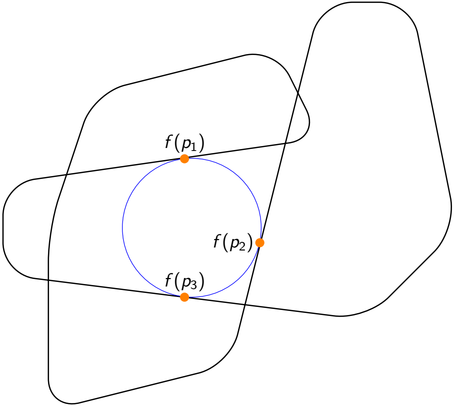


Figure: f–neighbors

Neighboring mapping points theorem

Theorem

Let X be a normal topological space and M be a contractible metric space. Let $C := \{C_1, \ldots, C_m\}$ be a non null–homotopic closed cover of X. Then for every continuous map $f : X \to M$ there exist (not necessarily distinct) points $p_1, \ldots, p_m$ with $p_i \in C_i$ for all $i = 1, \ldots, m$ such that they are f–neighbors.

Thank you