Vladimir Protasov

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Linear switching systems and several problems of the classical approximation theory

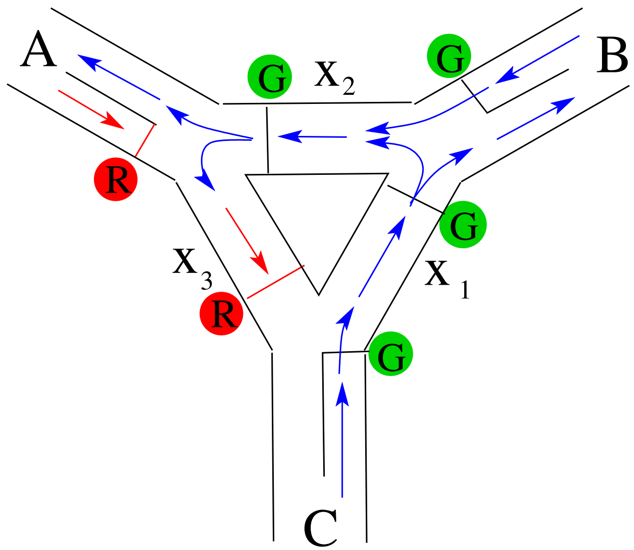
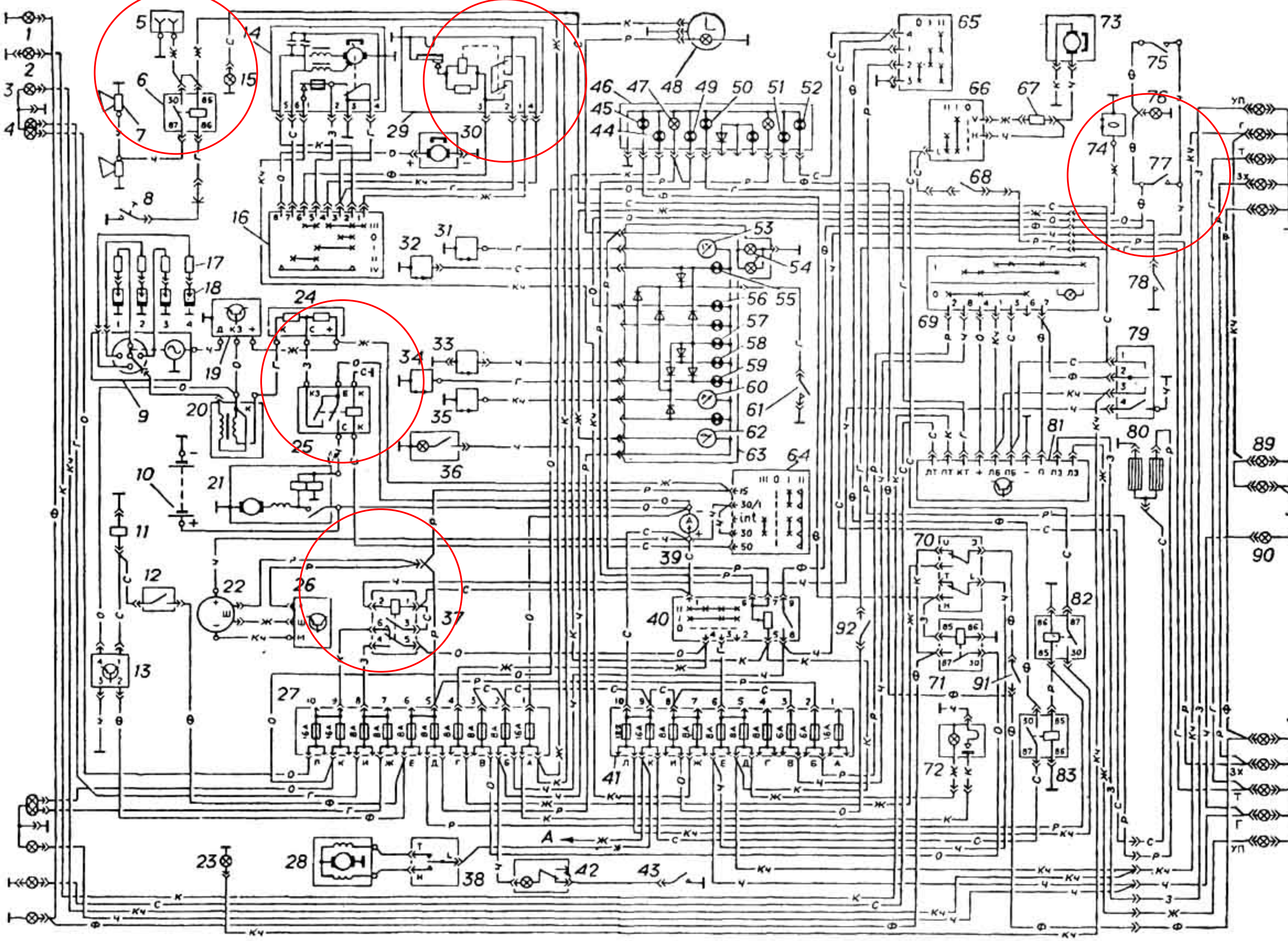


Figure 3: The traffic congestion control problem

Three main roads, 6 traffic lights (red/green), three buffer variables x_i, three symmetric configurations $\sigma \in \{1, 2, 3\}$.

$$\dot{x}(t) = A_{\sigma(t)}x(t) + b$$

For instance: given x_0 find $\sigma(t)$ so as to stabilize.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Thermal system

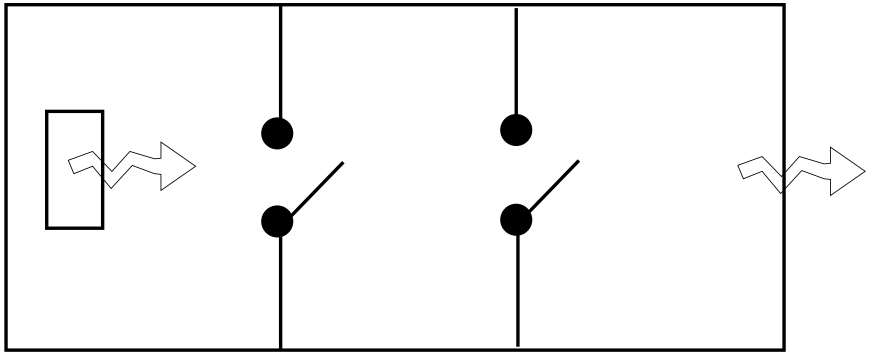


Figure 2: Switching thermal system

x_i, $i = 1,2,3$ temperatures in the three rooms. Two doors (open/closed) $\to \sigma \in \{1,2,3,4\}$.

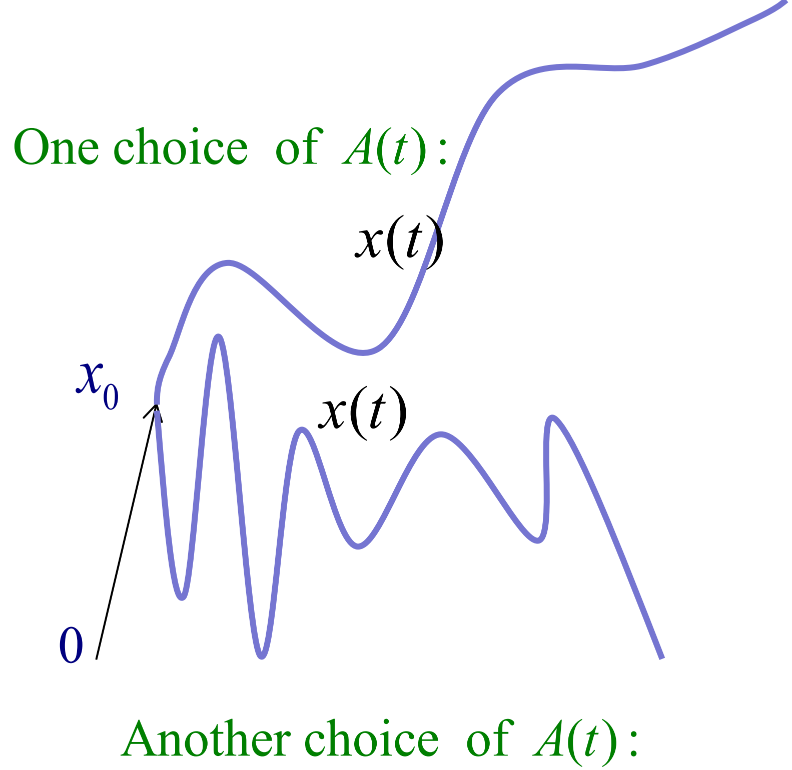
$$\dot{x}(t) = A_{\sigma(t)}x(t) + Bu(t)$$

For instance: "worst" control problem $\max_\sigma x_3(T)$

Linear switching systems

E.Pyatnitsky, V.Opoytsev, A.Molchanov (1980),
N.Barabanov, V.Kozyakin (1988), L.Gurvits (1996)

P.Mason, M.Sigalotti, M.Margaliot, F.Blanchini, S.Miani,
U.Boskian, D.Liberzon, and many others



Consider a system of linear ODE

$$\dot{x}(t) = A(t)x(t), \quad t \in [0,+\infty)$$

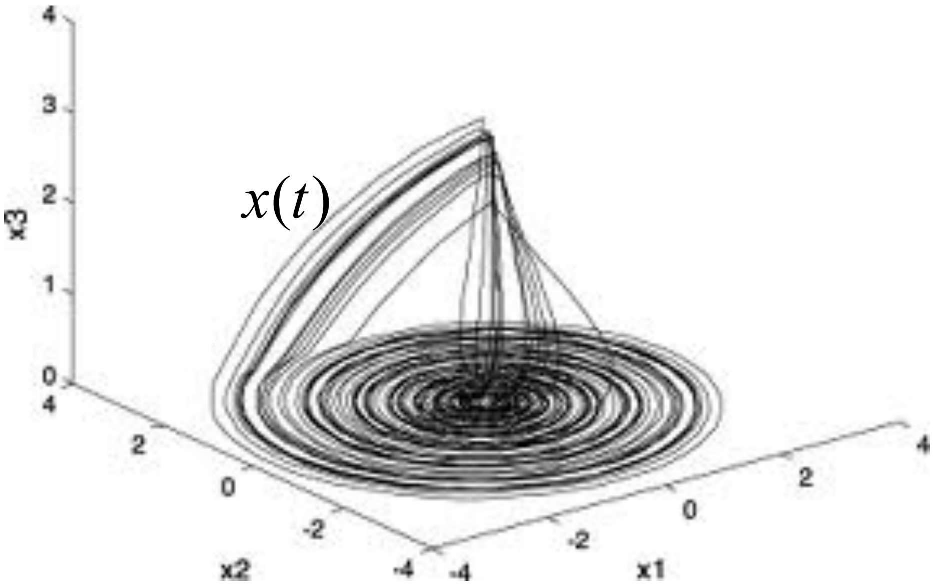
$$x(0) = x_0$$

$$x = \big(x_1(t), \dots, x_d(t)\big), \quad A(t) \text{ is a } d\times d \text{ - matrix,}$$

$$\forall\, t \in [0,+\infty) \quad A(t) \in U$$

U is a compact set of matrices

Example. $U = \{A_1, A_2\}$



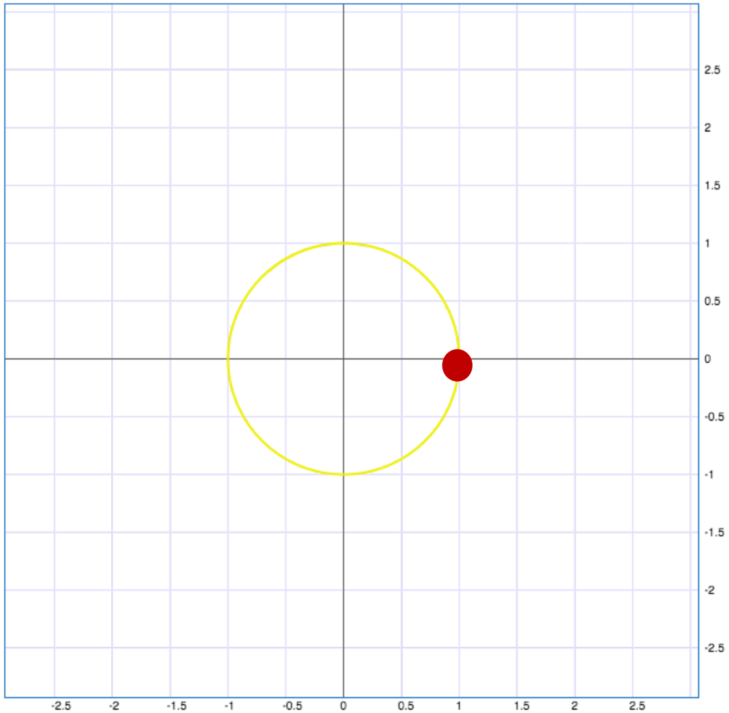
x(t)
x3
4
3
2
1
0
4
2
0
-2
-4
x2
-4
-2
0
2
4
x1

Stability

Consider a linear system of equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ \cos t \end{pmatrix} \qquad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

$$x(0) = 1, \qquad y(0) = 0$$

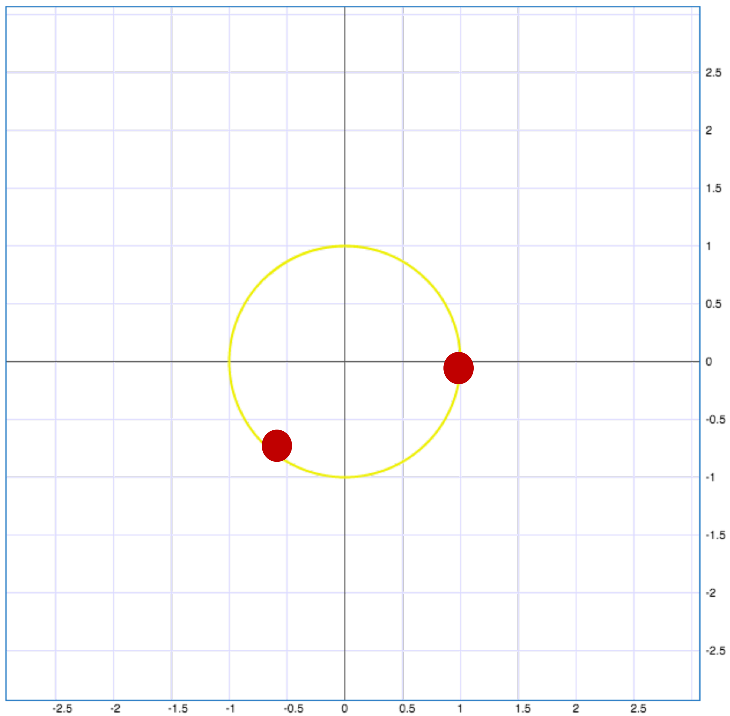


Consider a linear system of equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ \cos t \end{pmatrix}$$

$x(0) = 1 + \varepsilon, \quad y(0) = -\varepsilon \qquad \varepsilon = 10^{-20}$

$t = 10$ sec.

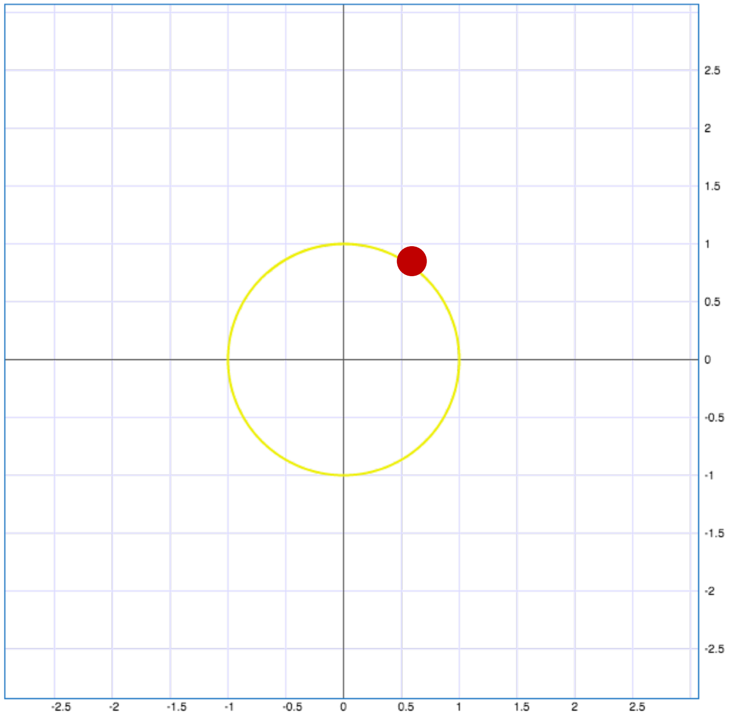


Consider a linear system of equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ \cos t \end{pmatrix}$$

$x(0) = 1 + \varepsilon, \quad y(0) = -\varepsilon \qquad \varepsilon = 10^{-20}$

$t = 20$ sec.

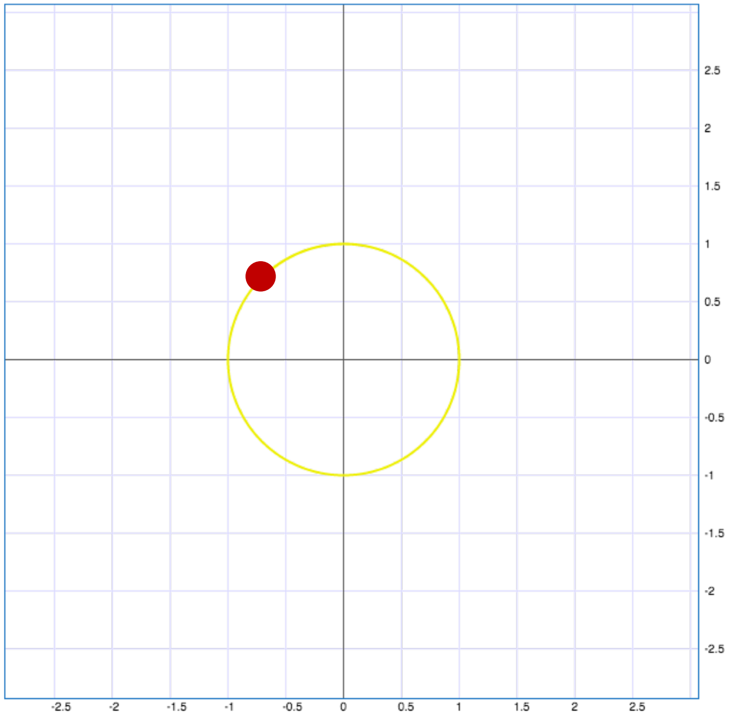


Consider a linear system of equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ \cos t \end{pmatrix}$$

$x(0) = 1 + \varepsilon, \qquad y(0) = -\varepsilon \qquad\qquad \varepsilon = 10^{-20}$

$t = 40$ sec.

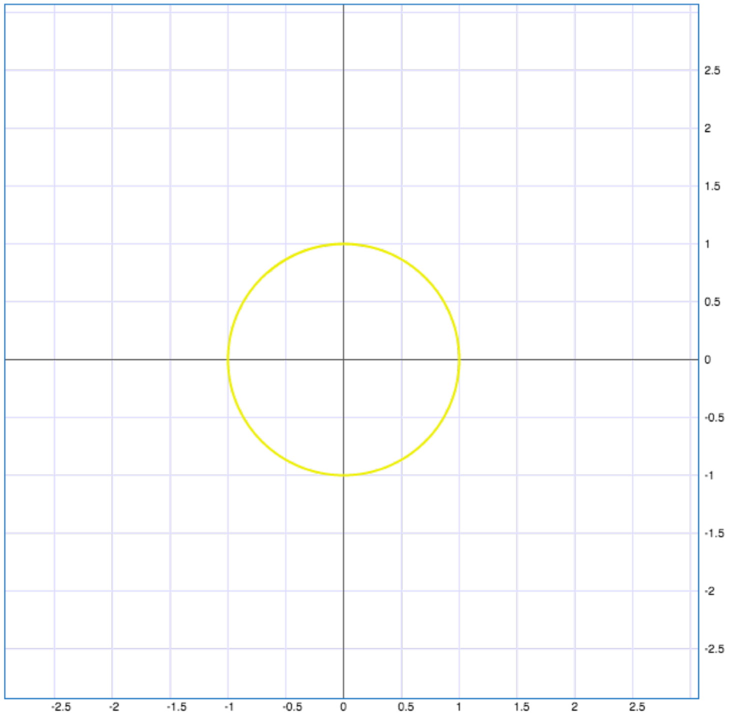


Consider a linear system of equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ \cos t \end{pmatrix}$$

$x(0) = 1 + \varepsilon, \qquad y(0) = -\varepsilon \qquad\qquad \varepsilon = 10^{-20}$

$t = 50$ sec.

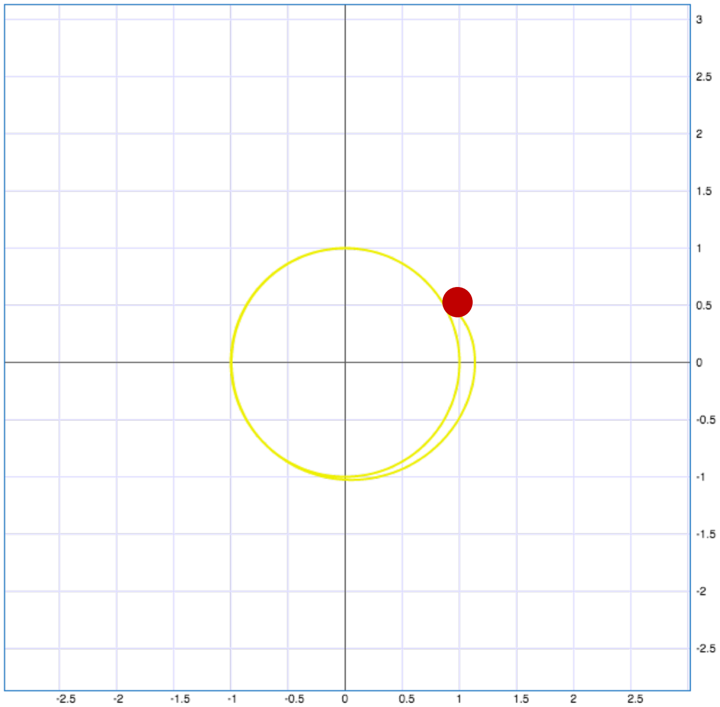


Consider a linear system of equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ \cos t \end{pmatrix}$$

$x(0) = 1 + \varepsilon, \quad y(0) = -\varepsilon \qquad \varepsilon = 10^{-20}$

$t = 45$ sec.

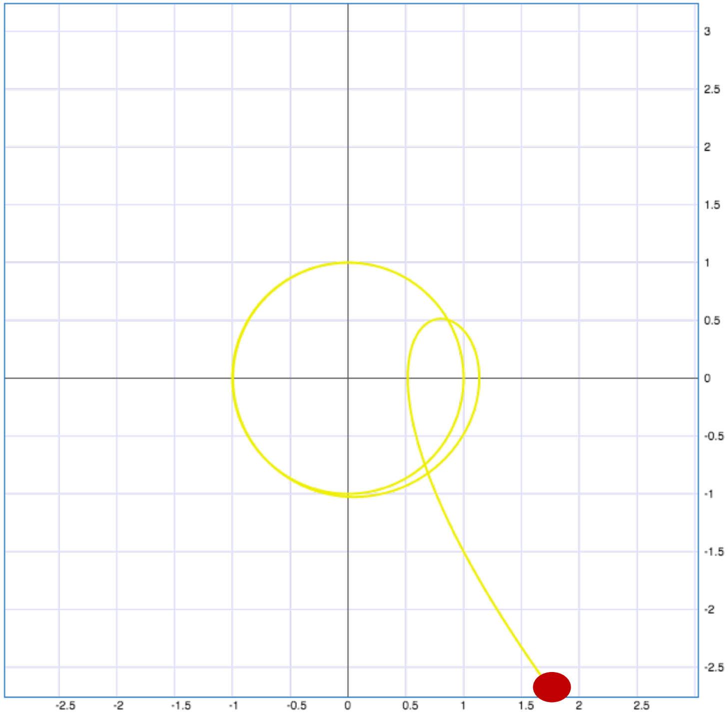


Consider a linear system of equations

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 \\ \cos t \end{pmatrix}$$

$x(0) = 1 + \varepsilon, \qquad y(0) = -\varepsilon \qquad\qquad \varepsilon = 10^{-20}$

$t = 48$ sec.

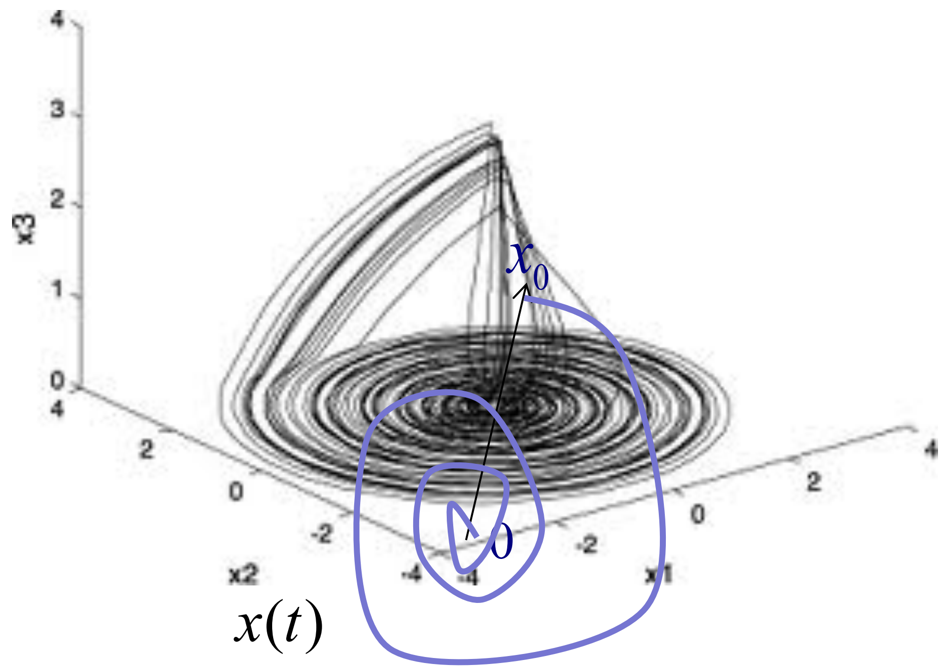


In $t = 60$ sec. the point will be in 14 km. from the center.

$x(t) = \cos t + \varepsilon e^{t}$ $\Rightarrow$ the system is unstable

Def. *The system is asymptotically stable if* $|x(t) - y(t)| \to 0$ *as* $t \to \infty$, *whenever* $y(0)$ *is close to* $x(0)$.

For linear systems, the stability is equivalent to the following property:
All trajectories $x(t)$ of the system converge to zero as $t \to \infty$.



x3
x2
x1
x_0
0
$x(t)$

How to decide the stability ?

$$\dot{x}(t) = A(t)x(t), \quad t \in [0,+\infty)$$
$$x(0) = x_0$$

$x = \left(x_1(t),\ldots,x_d(t)\right)$, $A(t)$ is a measurable control function,

$A(t) \in U$ for almost all $t \in [0,+\infty)$

If U consists of one matrix A, then $x(t) = e^{tA}x_0$

the system is stable $\Leftrightarrow$ Re $(\lambda) < 0$, for all eigenvalues λ of A

(i.e., A is a Hurwitz stable matrix)

What to do if Card (U) ≥ 2 ?

Necessary condition:

if the system is stable, then all matrices from co (U) are Hurwitz stable.

Not sufficient already for d =2 (L.Gurvits, 1999)

Conjecture 1 (P.Mason, R.Shorten, 2003) Sufficient for positive systems.

The system is positive if all matrices $A \in U$ are Metzler $\Leftrightarrow$

$A_{ij} \geq 0$ for all i,j $\in \{1,\ldots,d\}$, provided $i \neq j$
(all off-diagonal elements are non-negative) $\Leftrightarrow$

$e^{tA} \geq 0, \quad$ for all $t \in \mathbb{R}$

Example.

$$A = \begin{pmatrix} -100 & 1 & 2 \\ 0 & -15 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

The conjecture is proved for $d = 2$ by P.Mason and R.Shorten (2003)
and disproved for $d \geq 3$ by L.Faishil, M.Margaliot and P.Chigansky (2011)

$$A_0 = \begin{pmatrix} -1 & 0 & 0 \\ 10 & -1 & 0 \\ 0 & 0 & -10 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -10 & 0 & 10 \\ 0 & -10 & 0 \\ 0 & 10 & -1 \end{pmatrix}.$$

Another approach: the Lyapunov function

The Lyapunov function

Definition. A continuous function $f : \mathbb{R}^d \to \mathbb{R}_+$ is called Lyapunov function if

1) $f(x) > 0, \ x \neq 0,$

2) $f(\alpha x) = \alpha f(x) , \quad \alpha \geq 0,$

3) $f(x(t))$ is decreasing in t, for everty trajectory $x(t)$ of the system.

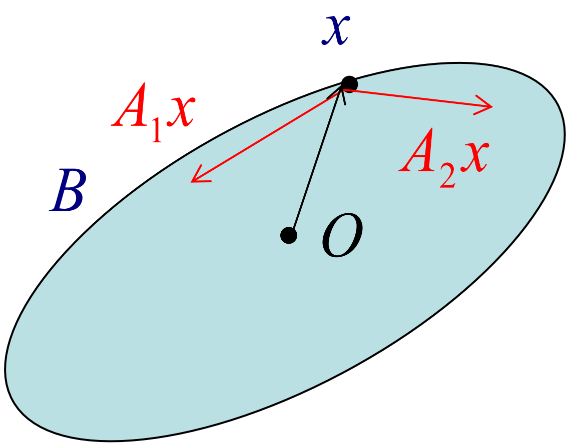
There is a Lyapunov function $\Rightarrow$ the system is stable.

The system is stable $\Rightarrow$ there is a convex symmetric Lyapunov function (norm).

(L.Opoitsev (1977), A.Molchanov, E.Pyatnitsky (1980), N.Barabanov (1989)).

The Lyapunov norm

Take a unit ball of that norm: $B = \{ x \in \mathrm{R}^d \mid f(x) \leq 1 \}$



A norm $f(x)$ is a Lyapunov norm

⇕

for every $x \in \partial B$ and for every $A \in U$

the vector Ax starting at the point x is "directed inside" B.

A quadratic Lyapunov function

Thus, to prove the stability it suffices to present a Lyapunov function $f(x)$.

How to find $f(x)$?

This is equivalent to constucting a convex body B.

The most natural choice is a quadratic function $f(x) = \sqrt{x^T M x}$, where M is p.s.d. matrix.

A matrix $M \succ 0$ defines a Lyapunov function $\Leftrightarrow$ $A^T M + M A \prec 0 \quad \forall A \in U$

This is an s.d.p. problem, it can be efficiently solved.

However, this is just a sufficient condition. In practice, it is far from being necessary.

Very often a quadratic Lyapunov function does not exists, although the system is stable.

There are other types of Lyapunov functions in the literature
(piecewise-quadratic, polyhedral, sum-of-squares, etc.)

Definition. The Lyapunov exponent $\sigma(A)$ is the infimum of numbers α such that
$\|x(t)\| \;\le\; C\, e^{\alpha t}$ for all trajectories $x(t)$.

The system is stable if and only if $\sigma(A) \;<\; 0$.

Theorem (N.Barabanov, 1989). For an arbitrary irreducible system there exists an invariant Lyapunov norm $f(x) = \|x\|$, for which two conditions are satisfied:

1) $\|x(t)\| \;\le\; \|x(0)\|\, e^{\sigma t}$ for all trajectories $x(t)$.

2) There is a trajectory $x(t)$ such that $\|x(t)\| \;=\; \|x(0)\|\, e^{\sigma t}$ for all t.

In case $\sigma = 0$

(The geometric interpretation). There is a symmetric about the origin convex body $G \subset \mathbb{R}^d$ such that all tracterures started in G never leave it, and there is at least one trajectory that entirely lies on the boundary of G.

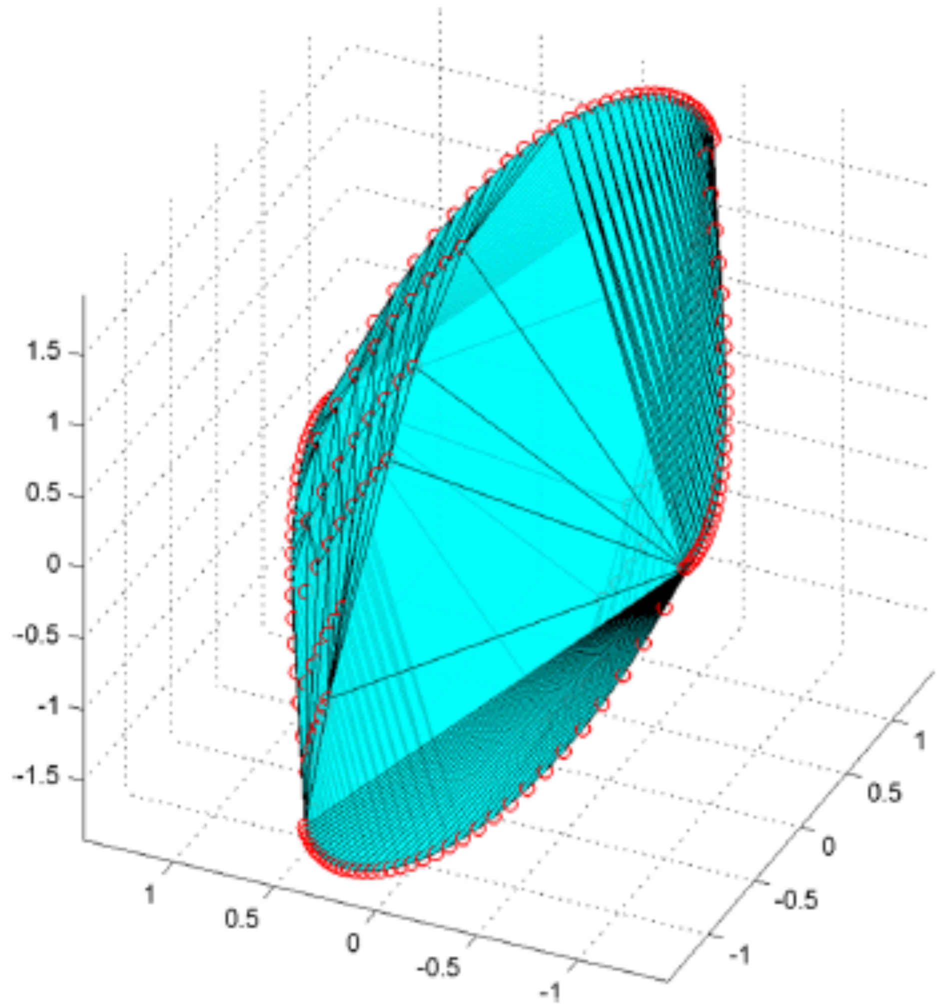
The invariant norm may not be well-approximated by quadratic functions

Polytope (piecewise-linear) Lyapunov function :

$$f(x) = \max_{i=1,\ldots,N} (a_i , x)$$

Theorem (F. Blanchini, S. Miani, 1996) For any stable LSS there exists a polytope Lyapunov norm.

The polytope norm is extremely difficult to compute already in the dimension 3



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0.5
0
-0.5
-1
-1.5
1
0.5
0
-0.5
-1
1
0.5
0
-0.5
-1

Consider a discrete systems

$$x_{k+1} \; = \; A\, x_k\,, \quad k \in \mathbb{Z}_+ \quad x_0 \text{ is given}$$

A system is stable if all trajectories tend to zero
(Schur stability)

Theorem (N.Barabanov, 1988) A discrete system is stable if and only if its joint spectral radius is smaller than one.

The Joint spectral radius (JSR)

$A_1, \cdots A_m$ are linear operators in $\mathbb{R}^d$

$$\hat{\rho}(A_1, \cdots, A_m) = \lim_{k\to\infty} \max_{d_1,\ldots,d_k \in \{1,\ldots,m\}} \left\| A_{d_1} \cdots A_{d_k} \right\|^{1/k}$$

J.C.Rota, G.Strang (1960) -- Normed algebras

N.Barabanov, V.Kozyakin, E.Pyatnitsky, V.Opoytsev, L.Gurvits, …(1988)

Linear switching systems

C.Micchelli, H.Prautzch, W.Dahmen, A.Levin, N.Dyn, P.Oswald,…… (1989)

Subdivision algorithms

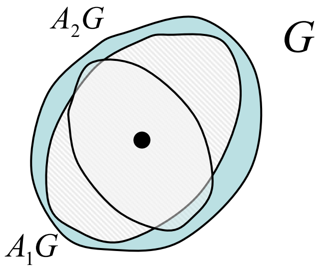
I.Daubechies, J.Lagarias , C.Heil, D.Strang, … (1991)

Wavelets

The Joint spectral radius (JSR)

$A_1, \quad , A_m$ are linear operators in $\mathbb{R}^d$

$$\hat{\rho}(A_1, \quad , A_m) = \lim_{k\to\infty} \max_{d_1,\dots,d_k \in \{1,\dots,m\}} \left\| A_{d_1} \quad A_{d_k} \right\|^{1/k}$$



The geometric sense:

$\hat{\rho} < 1 \iff$ there exists a norm $\|\bullet\|$ in $\mathbb{R}^d$

such that $\|A_i\| < 1$ for all $i = 1, \dots, m$

JSR is the measure of simultaneous contractibility

Taking the unit ball in that norm:

The JSR is smaller than 1 if and only if there is convex body G such that
$A_k G \subset int\, G$, k = 1, .., m

Example. If all the matrices $A_1, \dots, A_m$ are symmetric, then
one can take G a Euclidean ball $\Rightarrow$ $\hat{\rho} = \max\{\rho(A_1), \dots, \rho(A_m)\}$

The Joint spectral radius (JSR)

$A_1, \cdots, A_m$ are linear operators in R^d

$$\hat{\rho}(A_1 \cdots, A_m) = \lim_{k\to\infty} \max_{d_1,\dots,d_k \in \{1,\dots,m\}} \left\| A_{d_1} \cdots A_{d_k} \right\|^{1/k}$$

Example 1. If m = 1, we have a family of one matrix {A};

then $\hat{\rho}(A) = \lim_{k\to\infty} \left\| A^k \right\|^{1/k} = \max_{j=1,\dots,d} |\lambda_j|$

Example 2. If all the matrices $A_1, \dots, A_m$ are orthogonal, then $\left\| A_{d_1} \quad A_{d_k} \right\| = 1$,

hence $\hat{\rho} = 1$

Example 3. If all the matrices $A_1, \dots, A_m$ are diagonal, then

$$\hat{\rho} = \max \{\rho(A_1), \dots, \rho(A_m)\}$$

The same is true if all the matrices

- commute
- are upper (lower) triangular
- are symmetric

In general, however, $\hat{\rho} > \max \{\rho(A_1), \dots, \rho(A_m)\}$

Other applications of the Joint Spectral Radius

- Probability
- Combinatorics
- Number theory
- Mathematical economics
- Discrete math

How to compute or estimate ?

Blondel, Tsitsiklis (1997-2000).

- **The problem of JSR computing for nonnegative rational matrices in NP-hard**

- **The problem, whether JSR is less than 1 (for rational matrices) is algorithmically undecidable whenever d > 46.**

- **There is no polynomial-time algorithm, with respect to both the dimension d and the accuracy**

Sometimes easier to prove more

George Polya «Mathematics and Plausible Reasoning» (1954)

When trying to *prove* something, often a good strategy is to try to *prove more*.

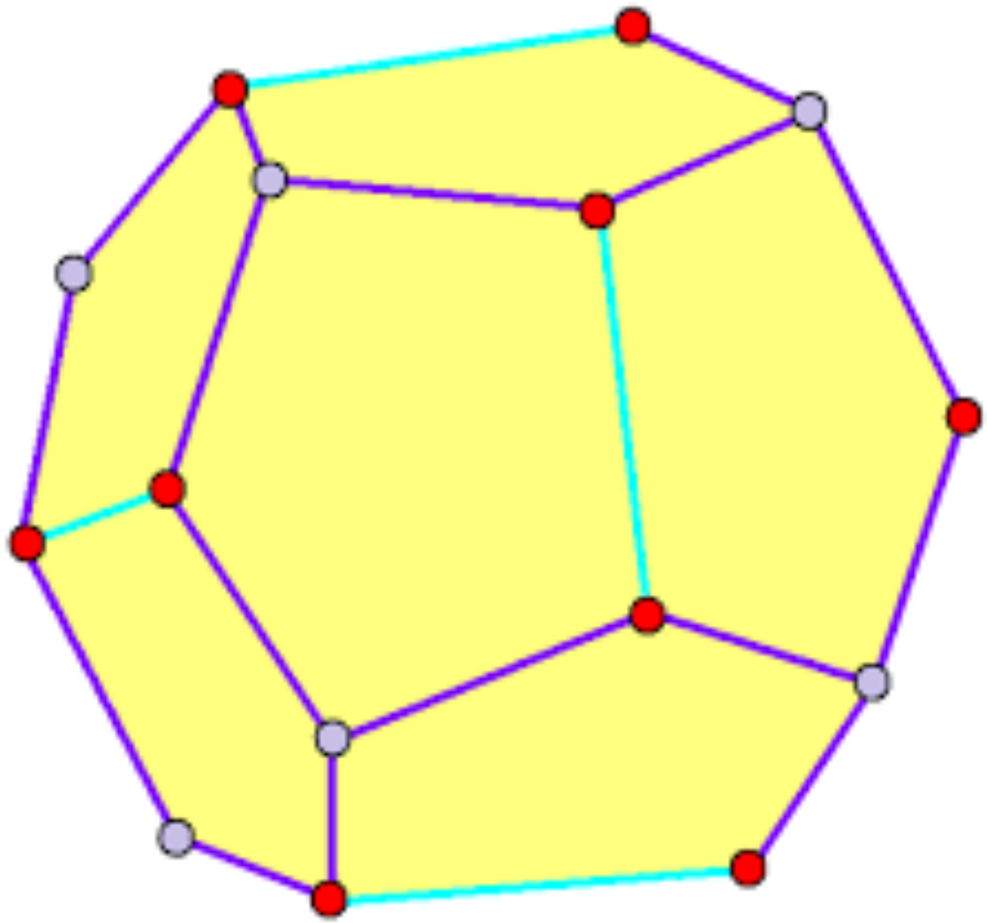
When trying to compute something approximately, often a good strategy is to…

find it precisely.

The invariant polytope algorithm.

Normalize all operators so that $\rho(A_1, \dots, A_m) = 1$

and step-by-step construct a polytope G such that $A_k G \subset G$



Discrete systems
and the Markov-Bernstein inequality for exponents.

Discretization of a linear switching system

We make the discretization wirh the stepsize $\tau > 0$

$$x_k = x(k\tau) \ ; \quad A_k = A(k\tau) \, , \quad k \in \mathbb{N}$$

$$\dot{x}(k\tau) \approx \frac{x(k\tau + \tau) - x(k\tau)}{\tau} = \frac{x_{k+1} - x_k}{\tau}$$

and obtain the discretized system:

$$x_{k+1} = (\mathrm{I} + \tau A_k) x_k \, , \quad k \in N$$

x_0 is given, $A_k \in U$

How to decide the stability of the discretized system ?

$$x_{k+1} = (I + \tau A_k) x_k , \quad k \in N$$
$$x_0 \text{ is given, } A_k \in U$$

Denote $I + \tau A_k = B_k$. Then $x_{k+1} = B_k \cdots B_0 x_0$.

The problem becomes: to determine, whether $\max\limits_{B_i \in I + \tau U} \| B_k \cdots B_0 \| \to 0 \ \ as \ \ k \to \infty$?

Answer: when the joint spectral radius (JSR) of the set $I + \tau U$ is smaller than 1.

Theorem 3 (N.Barabanov, 1988). The discrete system is stable $\Leftrightarrow$ $\hat{\rho}(I + \tau U) < 1$.

How small must be τ ?

It turns out that τ can be found from the following problem:

$$p(t) = \sum_{k=1}^{d} c_k e^{-\alpha_k t}, \qquad \alpha_1, \ldots, \alpha_d > 0$$

Find the minimal constant C such that

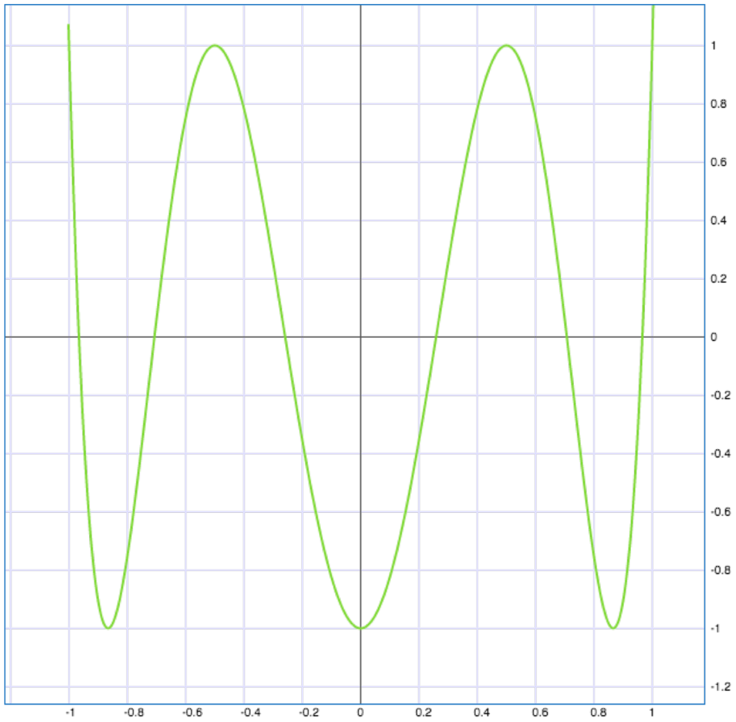
$$||p'|| \leq C\,||p||$$

In the norm $C[0, +\infty]$, for all p.

Theorem (A.Markov, S.Bernstein, 1889) For an algebraic polynomial of degree d, we have

$$\|p'\|_{C[-1,1]} \leq d^2 \|p\|_{C[-1,1]}$$

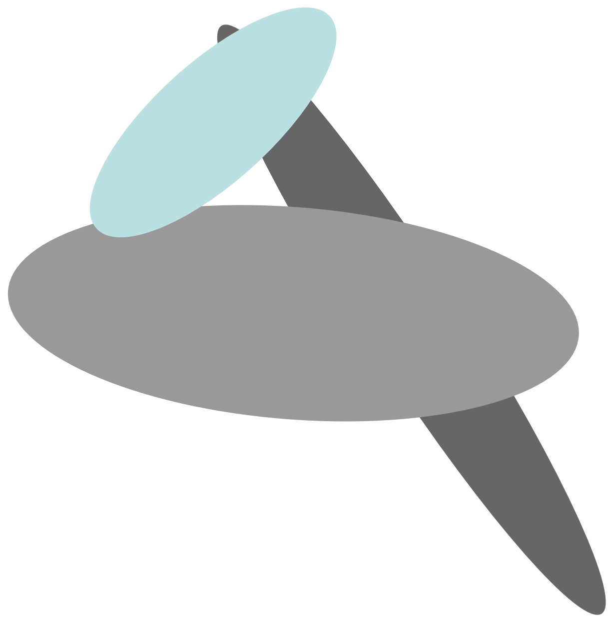
The equality holds only for polynomials proportional to the Chebyshev polynomial T_d



$$\mathbf{T_n (x)} = \cos\,(\boldsymbol{n\ arccos\ x})$$

$$
\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \\
T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1 \\
T_9(x) &= 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x \\
T_{10}(x) &= 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1 \\
T_{11}(x) &= 1024x^{11} - 2816x^9 + 2816x^7 - 1232x^5 + 220x^3 - 11x
\end{aligned}
$$

Theorem of Helly (1914) . A finite family of convex sets is given in the d-dimensional space. Then if every d+1 sets of the family possesses a nonempty intersections, then the whole family does.



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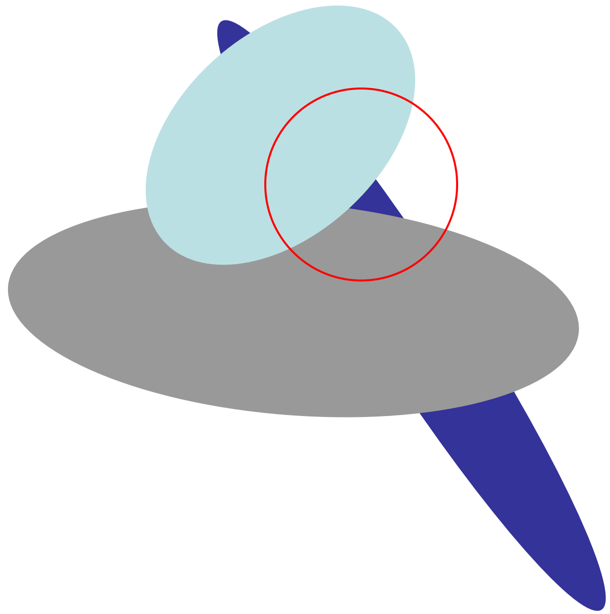


Refinement theorem (теорема об очистке)

(L.Schnirelman, 1938, V.Levin, 1967) .

If a function F(x,t) : L x T → R , where $\dim L = d,$ and T is compact

is convex in x ∈ L and is continuous in t ∈ T, then

one can choose at most n+1 points $t_1,\ldots,t_m$, $m \leq n + 1$ such that

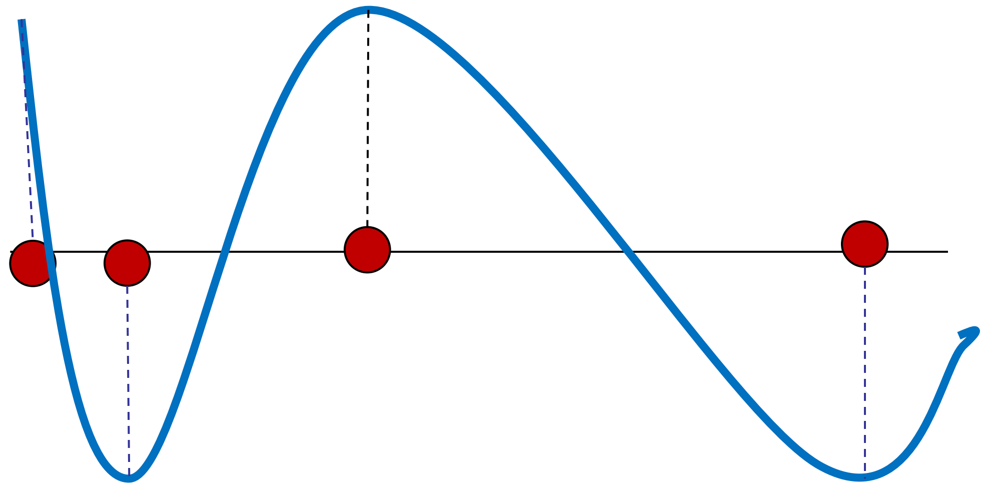
$$\min_{x\in L} \max_{t\in T} F(x,t) = \min_{x\in L} \max_{t\in\{t_1,\ldots,t_m\}} F(x,t)$$

Proof. Clearly, ≥. Let us show that ≤ .
For every t ∈ T, the set $A_t = \{x\in L \mid F(x,t) \leq a\}$ is convex.
If for every $t_1,\ldots,t_m$, the intersection of A_t is nonempty, then
the left hand side ≤ a . Thus, if the right hand side is ≤ a ,
Then so is the left hand side. Hence, ≤ .

Let T = [0,1] , L = { $x(t) = \sum_{k=0}^{n} x_k f_k(t)$ }, F(t,x) = | x(t)|

Thus, L = P_n is the space of polynomials on the system
f_0 , ……., f_n

$$\min_{p\in P_n} \| f(t) - p(t) \| = \min_{p\in P_n} \max_{k=0,..,n} | f(t_k) - p(t_k) |$$



$$\|p'\|_{C[-1,1]} \leq d^2 \|p\|_{C[-1,1]}$$

This inequality can be extended to every Chebyshev system of functions.
In particular, to the sum of real exponents:

$$p(t) = \sum_{k=1}^{d} c_k e^{-\alpha_k t}, \qquad \alpha_1, \ldots, \alpha_d > 0$$

Theorem (P.B. Borwein, T. Erdélyi, 1995) For an exponentioal polynomial of degree d, we have

$$\|p'\|_{C[0,+\infty)} \leq c\,a\,d\,\|p\|_{C[0,+\infty)},$$

where $a = \max\{\alpha_1, \cdots, \alpha_d\}$, $c > 0$ is a constant

The sharp estimates for the constant c have been found by V.Sklyarov (2010).

We apply this inequality to exponential polynomials for the numbers

$$\alpha_k = -\lambda_k \,, \quad k=1,....,d, \quad \text{where } \{\lambda_1,....,\lambda_d\} = \mathrm{sp}(A)$$

However, this method is applicable to matrices with a real spectrum only!

For general complex numbers this does not work because:

Complex exponents do not form a Chebyshev system

How to solve the problem

$$\dot{p}(0) \to \max$$

$$\|p\|_{C[0,+\infty)} \leq 1$$

$$p(t) = \sum_{k=1}^{d} e^{-\alpha_k t}$$

For arbitrary complex numbers $\alpha_1,\cdots,\alpha_d \in \mathbb{C}$?

Neither alternance idea nor Remez type of algorithms work here

Thank you!