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On Crystallinity of $2R$-isometric Delone Sets: New Results and Open Problems

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We call a Delone set $X \subset \mathbf{R}^d$ a $2R$-*isometric* if for all points x of X their $2R$-clusters $C_x(2R)$ ($C_x(2R) := \{y \in X : |xy| \leq 2R\}$) are pairwise congruent, i.e. for any two points x and $x' \in X$ there is an Euclidean isometry g such that $g(x) = x'$ and $g(C_x(2R)) = C_{x'}(2R)$.

For last few years we made significant progress in studies of the $2R$-isometric sets. It was shown that the character of a $2R$-isometric set X significantly depends on the $2R$-cluster group $S_x(2R)$.

First, it was proved that if in X the cluster group $S_x(2R)$ contains the central symmetry about the central point x then the Delone set is a regular system, i.e. a Delone set $X \subset \mathbf{R}^d$ whose symmetry group acts transitively on points of the set. Emphasize that this theorem holds for any dimension d of space and plays an essential role in improving the upper bound for the regularity radius.

Second, for dimension $d = 3$ (the most important case for applications) we investigated a large list of finite groups (mostly richest finite groups) of Euclidean isometries of $\mathbf{R}^3$ which could be potential cluster groups $S_x(2R)$ and showed that for them there are two options:

1) some groups from the list can be realised as a group $S_x(2R)$ in some Delone set $X \subset \mathbf{R}^3$, and in this case we showed that the Delone set is a regular system; moreover the Delone set is determined uniquely by its $2R$-cluster;

2) for other groups from the list it was shown. that there is no Delone $2R$-isometric sets with such groups.

In the talk we will explain why these results are of special interest and discuss some open problems, in particular:

a) a link between the 'local' results in terms of the 'emptiness' radius R and the local conditions in terms of the radius of the coordination sphere which are used in crystallography;

b) a problem on generalisation of Bravais' theorem on that no lattice in $\mathbf{R}^3$ with the 5-fold symmetry.

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Cones and polytopes of metrics, hypermetrics, quasimetrics and hemimetrics

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I will present the works that I did with Michel on metric cones and related subjects. Given a finite point set $X = \{1, \ldots, n\}$, we can define the cone of metric on this point set X. That is the set of functions $d : X \times X \mapsto \mathbb{R}$ such that

- $d(x, x) = 0$ for all $x \in X$
- $d(x, y) = d(y, x)$ for all $x, y \in X$
- $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

is a polyhedral cone that we call $\mathrm{MET}(K_n)$. A particular very interesting subset of this cone is the cone of L^1 embeddable metrics, that we call $\mathrm{CUT}(K_n)$. The vertices/facets of those cones are known up to $n = 8$.

The definition of the metric and cut cone can be extended to an arbitrary graph G. The triangle inequalities are replaced by cycle inequalities and non-negative inequalities. In that setting we have $\mathrm{CUT}(G) = \mathrm{MET}(G)$ is and only if G does not have a K_5 minor. This allows to compute the facets of many cut polytope and is a remarkable result.

Another generalization that we consider is to hypermetrics. This generalization is relevant to geometry of numbers and Delaunay polytopes and we computed their dual description up to $n = 8$. We also present the construction of hypermetric polytopes.

One natural generalization of metric is to consider the cone of quasimetrics defined as functions $d : X \times X \mapsto \mathbb{R}$ such that

- $d(x, x) = 0$ for all $x \in X$,
- $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

In that setting we define a notion of metric polytope of a graph that we call $\mathrm{QMET}(G)$ and we give an explicit set of inequalities describing it that generalizes the one for $\mathrm{MET}(G)$. We define the notion of oriented metrics that are weightable and an oriented version of the cuts.

Another generalization is to consider the notion of metrics on more than 2 points, i.e. hemimetrics. In that setting the equivalent of the triangle inequality would be the inequality over a simplex. However, it turns out that this definition is not workable since it does not allow to define the hemimetrics on a simplicial complex. We give another set of inequalities that allow a neat generalization to the case of an arbitrary complex.

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Antipodal covers

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Intuitively, a graph is *distance-regular* if a grouping of vertices, corresponding to their distance from a certain vertex, is so nice that it helps us to calculate all the eigenvalues of the graph. Among such graphs we will study those for which 'being at maximum distance or zero' in an equivalence relation. They are called antipodal graphs and they 'cover' smaller distance-regular graphs. For example, all the 1-skeletons of the Platonic solids are distance-regular and antipodal graphs, and in particular the 3-cube covers the tetrahedron. Most finite objects of sufficient regularity are closely related to certain distance-regular graphs, in particular, antipodal distance-regular graphs give rise, to projective planes, Hadamard matrices and other interesting combinatorial objects. Distance-regular graphs serve as an alternative approach to these objects and allow the use of graph eigenvalues, graph representations, association schemes and the theory of orthogonal polynomials.