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The Local Theory for Delone and t-bonded Sets

Mikhail Bouniaev (USA, Rio Grande Valley)
University of Texas Rio Grande Valley
e-mail: mikhail.bouniaev@utrgv.edu

Introduction. The overarching goal of this paper is to review main results of the local theory and discuss a potential extension of local theory, that focuses mostly on regular Delone sets and tiling to t-bonded sets, graphs, orthogonal nets, etc., and determine an agenda for research in this area. The main goal of the local theory for crystals is to find the correct statements rigorously explaining why and how the crystalline structure follows from the pair-wise identity of local arrangements around each atom. Before the 70s, there were no rigorously proved mathematical statements until B. Delone, R.Galiulin, and Delone's students N. Dolbilin and M. Stogrin developed a mathematically sound local theory of crystals (see for instance,[1]).

Definition 1. *(Delone Set) Let $\mathbb{R}^d$ be Euclidean space, r and R some positive numbers. A set $X \subset \mathbb{R}^d$, is called a (r, R)* Delone set *if: (i) any open ball of radius r has at most one point from X (r-condition, uniformly discrete set);(ii) any closed ball of radius R has at least one point from X (R-condition).*

Some theorems from the classical local theory for Delone sets have been generalized ([2], [3],[4])for a wider class of sets that we call t-bonded sets.

Definition 2. *(t-bonded Set) We call a uniformly discrete set $X \subset \mathbb{R}^d$ a t-*bonded set, *where t is a positive number, if for any two points x and $x' \in X$ there is a finite sequence $x = x_0, x_1, \dots, x_m = x'$ of points from X such that $|x_{i-1}x_i| \leq t$, $i \in [1, m]$. The sequence $x = x_0, x_1, \dots, x_m = x'$, for which $|x_{i-1}x_i| \leq t$, is called a t-*chain *and denoted by $[x, x']$.Each pair of points x, x' from X with $|xx'| \leq t$ will be called a t-*bond.

Statement 1. Any (r, R) Delone set $X \subset \mathbb{R}^d$ is $2R$-bonded set.

Review of a large variety of proven theorems/facts related to Delone sets, prompted us to extend the concept of the local theory by including statements where a “local” premise allows to make a “global” conclusion.

Local Theorems for Delone Sets; t-bonded Sets. The statements of the local theory require some definitions, we will provide here only the basic ones, all definitions related to local theory may be found in [5],[6].

Definition 3. *(Multi-regular m-Systems) A t-bonded set $X \subset \mathbb{R}^d$ (Delone Set) is a* multi-regular *t-bonded system (multi-regular system or crystal) if there is a finite set $X_0 = \{x_1, ..., x_m\}$ such that $X = \bigcup_{i=1}^{m} \mathrm{Sym(X)} \cdot x_i$. If $m = 1$ a system is called a regular system.*

Definition 4. *(Cluster Equivalence) Given a set X in $\mathbb{R}^d$, $\rho > 0$, and two points $x \in X$ and $x' \in X$, we say that the ρ-cluster $C_x(\rho)$ is* equivalent *to the ρ-cluster $C_{x'}(\rho)$, if there is a space isometry g of $\mathbb{R}^d$, such that $g(x) = x'$ and $g(C_x(\rho)) = C_{x'}(\rho)$.*

Definition 5. *(Cluster Counting Function) For a t-bonded (Delone) set X of finite type the number of equivalence classes of ρ-clusters in X is a function of ρ which is called the* cluster counting function *and denoted by $N(\rho)$.*

Statement 2. *(see, e.g., [6]) If X is a Delone set with $N(2R) < \infty$, then for all $\rho > 0$ the cluster counting function $N(\rho) < \infty$.*

Below are the criteria for multi-regular t-bonded systems and crystals.

Theorem 1. *(Local Criterion for multi-regular t-bonded and Delone Systems) A t-bonded (Delone) set $X \subset \mathbb{R}^d$ is an multi-regular t-bonded system (Crystal) if and only if there is some $\rho_0 > 0$ such that two conditions hold: 1) $N(\rho_0) = N(\rho_0 + t) = m$; ($N(\rho_0) = N(\rho_0 + 2R) = m$); 2) $S_x(\rho_0) = S_x(\rho_0 + t), \forall x \in X$ ($S_x(\rho_0) = S_x(\rho_0 + 2R), \forall x \in X$.)*

A major problem in the local theory is to find the radius of regularity for Delone sets as a function of d for any $\mathbb{R}^d$.

The next recently proved theorem [7] establishes a lower bound for the radius of regularity.

Theorem 2. *(Lower Bound for Radius of Regularity) Suppose R is a fixed positive number. For any $\varepsilon > 0$, there exists a non-regular d-dimensional $(r; R)$-system such that $N(2dR - \varepsilon) = 1$ and X is not a regular set.*

Atomic structures of many crystals are centrally symmetric. New important results related to this case of centrally symmetric structures have been recently proven by N. Dolbilin and A. Magazinov [8].

As far as centrally symmetric sets are concerned, one of the challenges is to prove for t-bonded sets similar theorems that have been proved for Delone sets.

$10R$ and $6t$ are the best known upper bounds for Delone sets and t-bonded sets correspondingly for the radius that guarantee global regularity from regularity of clusters of radius $10R$ and $6t$ correspondingly.

The challenging tasks in the the local theory are to find the radii of regularity for Delone sets and t-bonded sets in $\mathbb{R}^3$.

Local Theorems for Tiling. One of the models of matter's structure is a tiling of 3-space.

Definition 6. *A face-to-face tiling T is called isohedral if its symmetry group G operates on tiles from T transitively.*

A theory similar for the local theory for Delone sets has also been developed for tiling. Function $N(k)$ (where k is a natural number), that plays a role similar to that of $N(\rho)$, can be introduced for tiling. A condition similar to $S_x(\rho_0) = S_x(\rho_0 + t)$ can be introduced for tiling in terms of function $M(k)$, which is an order of the group of symmetries of some k-dependant "surrounding" of tile P.

Theorem 3. *(Local Criterion for Isohedral Tiling) The tiling T is isohedral if and only if there exists $k_0 \in \mathbb{N}$ such that the following two conditions hold: (1)$N(k_0+1) = 1$; (2)$M(k_0) = M(k_0+1)$.*

Local Theorems for Graphs. Many chemists and crystallographers promote a combinatorial approach to modeling matter's structure as graphs, where vertices represent atoms/moleculars and edges represent chemical bonds. In this respect we would like to mention work of L. Danzer and N.Dolbilin ([9]), as well as recent research of Dr. I. Baburin from the Technological University of Dresden, who has been working on the following problems.

$1B-\Gamma$: Given a finite graph Δ, under which conditions can it be uniquely extended to a vertex-transitive graph Γ with a polynomial growth such that the "neighbourhoods" of all its vertices are isomorphic to Δ.

$2B-\Gamma$: Given a vertex-transitive graph Γ with polynomial growth, how to find its finite subgraph that "determines" the structure of Γ.

Orthogonal Networks, Strucrural Automata, and Local Problems. Recent research of cristallographers, who use a combinatorial model for crystals are also closely related to local problems or/and the need to prove some local theorems. First of all, we mean research related to modeling crystal growth through cellular automata **CA**, deterministic finite automation **DFA**, and structural automata **SA**. In this discussion we will follow S. Krivovichev's findings ([10]).

For modeling crystal growth and studying complexity of crystals V. Shevchenko, S. Krovovichev, and A. Mackay, [11] suggested using **SA**. In this model all states correspond to vertices in a graph (orthogonal net) with a certain configuration of adjacent edges. Symbols of $\mathbf{v} \in \mathbf{\Sigma}$ (or letters in the language) are vectors (directed edges) of standard orthonormal basis and of their opposite vectors $\underline{\mathbf{v}}$. A transition occurs from state q_i to g_j by moving (adding a vector from Σ) from one vertex to an adjacent vertex. Any state from Q may be initial, and any state from Q is an accepting state. The simplest example of this construction is a primitive cubic net **pcu**.

Pcu can be represented as $\mathbb{Z}^3$, where all nods are connected by unit vectors, i.e., for every nod there are six orthogonal edges connecting this nod with adjacent ones. Combining orthogonal nets (subgraphs of **pcu**) with **SA** is a very productive technique to study graphs of crystals, however, in crystallographic literature it is used mostly to study particular chemical elements.

For instance S. Krivovichev [10] discussed an example of the tetrahedral layer in the structure of RUB-15 and its orthogonal representation. The orthogonal network contains three different vertex configurations, graph of RUB-15 can be embedded into **pcu**, and the image can be generated by structural automata with three states. In this particular example it is sufficient to start with one vertex and applying transition rules generate the entire orthogonal net that represents RUB-15. From this example several local problems arise. (1) How to determine the size of a orthogonal subnetwork and **SA′s** states for a given network that allow to generate the entire network.(2) Not any periodic graph can be embedded to **pcu**. A major problem is to find sufficient conditions for the graph's embedding. From the practical point of view these conditions should be local conditions, and the proof should be constructive.

Local Problems and Fuctionnals on Triagnulations. N. Dolbilin, H. Edelsbrunner, A. Glazyrin, and O. Musin came across local problems in the paper “ Fuctionals on Triangulations of Delaunay Sets” (2014) .

Meyer Sets and Quasicrystals. A Delone set $X \subset \mathbb{R}^n$ is called a Meyer set if $X - X$ is a Delone set. Meyer sets play an important role in the theory of quasicrystals. The challenge is to find the local conditions equivalent to the definition of the Meyer set.

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Базис Шафаревича формальных модулей групп Любина–Тейта и Хонды

С. В. Востоков (Россия, Санкт-Петербург)
Санкт-Петербургский государственный университет
e-mail: sergei.vostokov@gmail.com
Р. П. Востокова (Россия, Санкт-Петербург)
Санкт-Петербургский государственный университет
e-mail: rvostokova@yandex.ru

Shafarevich Basis of the formal modules of the Lubin–Tate and Honda groups

S. V. Vostokov (Russia, Saint-Petersburg)
St. Petersburg state University
e-mail: sergei.vostokov@gmail.com
R. P. Vostokova (Russia, Saint-Petersburg)
St. Petersburg state University
e-mail: rvostokova@yandex.ru

В докладе излагается теория базиса Шафаревича, которая была создана в 1950 году для получения результатов в направлении Явного закона взаимности в локальных полях.

Для нужд арифметической геометрии и изучения как явного спаривания обобщённого символа Гильберта, настала необходимость такого же типа формальных модулях, построенных на максимальных идеалах колец целых локальных полей. Это нужно также для исследования эллиптических кривых.

Самыми важными типами формальных групп являются группы Любина-Тейта и группы Хонды. Именно для такого типа групп мы и получаем аналог базиса Шафаревича на формальных модулях.

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УДК 511.9

$L-$функций, кратные дзета значения, и приложения

Н. М. Глазунов (Украина, г. Киев)
Национальный Авиационный Университет
e-mail: glanm@yahoo.com