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On the Hurwitz zeta-function with algebraic irrational parameter

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О дзета-функции Гурвица с алгебраическим иррациональным параметром[1]

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After the Voronin discovery of the universality for the Riemann zeta-function, it turned out that some other zeta- and L-functions are also universal in the Voronin sense. Among them, the Hurwitz zeta-function $\zeta(s,\alpha)$, $s=\sigma+it$, with parameter α, $0<\alpha\leqslant 1$. The function $\zeta(s,\alpha)$ is defined, for $\sigma>1$, by the Dirichlet series

$$\zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s},$$

and has a meromorphic continuation to the whole complex plane. The function $\zeta(s,\alpha)$ depends on the parameter α, and its analytic properties, including the universality, are closely related to arithmetic of α.

Let $D=\{s\in\mathbb{C}:1/2<\sigma<1\}$. Denote by $\mathcal{K}$ the class of compact subsets of the strip D with connected complements, and by $H(K)$ with $K\in\mathcal{K}$ the class of continuous functions on K that are analytic in the interior of K. Then the universality of $\zeta(s,\alpha)$ is contained in the following theorem [4].

Theorem 1. *Suppose that the parameter α is transcendental or rational $\neq 1, 1/2$. Let $K\in\mathcal{K}$ and $f(s)\in H(K)$. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\mathrm{meas}\left\{\tau\in[0,T]:\sup_{s\in K}|\zeta(s+i\tau,\alpha)-f(s)|<\varepsilon\right\}>0.$$

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The universality of $\zeta(s,\alpha)$ in the case of algebraic irrational α is an open problem.

In [2], a certain approximation to the universality of $\zeta(s,\alpha)$ with algebraic irrational α has been obtained. Denote by $H(D)$ the space of analytic functions on D endowed with topology of uniform convergence on compacta.

Theorem 2. *Suppose that the parameter α is algebraic irrational. Then there exists a closed non-empty set $F_\alpha \subset H(D)$ such that if $K \subset D$ is a compact set and $f(s) \in F_\alpha$, then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\right\} > 0.$$

Theorem 2 has the following modiffication [2].

Theorem 3. *Suppose that α is algebraic irrational. Then there exists a closed non-empty set $F_\alpha \subset H(D)$ such that if $K \subset D$ is a compact set and $f(s) \in F_\alpha$, then the limit*

$$\lim_{T\to\infty} \frac{1}{T}\text{meas}\left\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon\right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

The discrete universality of the function $\zeta(s,\alpha)$ is more complicated. The first result in this direction belongs to Bagchi [1].

Theorem 4. *Suppose that α is a rational number, $\alpha \neq 1$, $\alpha \neq 1/2$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$ and $h > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\left\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh,\alpha) - f(s)| < \varepsilon\right\} > 0.$$

The case of transcendental α is more complicated problem, and the known theorems involve some relations between the numbers α and h. For example, the rationality of the number $\exp\{2\pi/h\}$ is required. More general result was considered in [6].

Theorem 5. *Suppose that the set $\{(\log(m+\alpha) : m \in \mathbb{N}_0), 2\pi/h\}$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Let $K \in \mathcal{K}$ and $f(s) \in H(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\left\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh,\alpha) - f(s)| < \varepsilon\right\} > 0.$$

Theorem 1 also is true [5] for α such that the set $L(\alpha) = \{\log(m+\alpha) : m \in \mathbb{N}_0\}$ is linearly independent over $\mathbb{Q}$. The set $L(\alpha)$ is very important in the theory of the function $\zeta(s,\alpha)$. The Cassels theorem asserts that at least 51 percent of elements of the set $L(\alpha)$ with algebraic irrational α are linearly independent over $\mathbb{Q}$ in the sense of density. Thus, the existence of algebraic irrartional α with linearly independent $L(\alpha)$ over $\mathbb{Q}$ does not contradict the Cassels theorem. However, examples of such α are not known.

Now, we state discrete analogues of Theorems 2 and 3 [7].

Theorem 6. *Suppose that α is an algebraic irrational number, and $h > 0$. Then there exists a non-empty closed set $F_{\alpha,h} \subset H(D)$ such that, for every compact set $K \subset D$, $f(s) \in F_{\alpha,h}$ and $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\left\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh,\alpha) - f(s)| < \varepsilon\right\} > 0.$$

Moreover, for every compact set $K \subset D$ and $f(s) \in F_{\alpha,h}$, the limit

$$\lim_{N\to\infty} \frac{1}{N+1}\#\left\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh,\alpha) - f(s)| < \varepsilon\right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

Theorem 6 can be generalized for some compositions $\Phi(\zeta(s,\alpha))$, where $\Phi : H(D) \to H(D)$ is a certain operator.

Theorem 7. *Suppose that α is an algebraic irrational number, $h > 0$ and $\Phi : H(D) \to H(D)$ is a continuous operator. Then there exists a non-empty closed subset $F_{\alpha,h} \subset H(D)$ such that, for every compact set $K \subset D$, $f(s) \in \Phi(F_{\alpha,h})$ and $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |\Phi(\zeta(s+ikh,\alpha)) - f(s)| < \varepsilon \right\} > 0.$$

Moreover, for every compact set $K \subset D$ and $f(s) \in \Phi(F_{\alpha,h})$, the limit

$$\lim_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant k \leqslant N : \sup_{s\in K} |\Phi(\zeta(s+ikh,\alpha)) - f(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

For Hurwitz zeta-functions, the joint universality also is considered. The most general result was obtained in [5].

Theorem 8. *Suppose that the set $\{\log(m+\alpha_j) : m \in \mathbb{N}_0,\ j = 1,\dots,r\}$ is linearly independent over $\mathbb{Q}$. For $j = 1,\dots,r$, let $K_j \in \mathcal{K}$ and $f_j(s) \in H(K_j)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+i\tau,\alpha_j) - f_j(s)| < \varepsilon \right\} > 0.$$

In [3], the following result without using any independence condition is presented.

Theorem 9. *Suppose that α_j, $0 < \alpha_j < 1$, $\alpha_j \neq 1/2$, $j = 1,\dots,r$, are arbitrary numbers. Then there exists a closed non-empty set $F_{\alpha_1,\dots,\alpha_r} \subset H^r(D)$ such that, for every compact sets $K_1,\dots,K_r \subset D$, $(f_1,\dots,f_r) \in F_{\alpha_1,\dots,\alpha_r}$ and $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+i\tau,\alpha_j) - f_j(s)| < \varepsilon \right\} > 0.$$

Moreover, for every compact sets $K_1,\dots,K_r \subset D$ and $(f_1,\dots,f_r) \in F_{\alpha_1,\dots,\alpha_r}$, the limit

$$\lim_{T\to\infty} \frac{1}{T} \mathrm{meas} \left\{ \tau \in [0,T] : \sup_{1\leqslant j\leqslant r} \sup_{s\in K_j} |\zeta(s+i\tau,\alpha_j) - f_j(s)| < \varepsilon \right\} > 0$$

exists for all but at most countably many $\varepsilon > 0$.

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Weighted universality of the Hurwitz zeta-function

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In mathematics, often weighted statements are discussed. They generalize usual statements when the weight function is identically 1. In probability theory, we have weighted limit theorems, in number theory the weighted sieves are used. Analytic number theory considers weighted universality theorems for zeta- and L-functions.

The first weighted universality theorem was obtained for the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, in [3]. Let $T_0 > 0$ be fixed. Suppose that $w(t)$ is a positive function of bounded variation on $[T_0, \infty)$ such that, for every subinterval $[a, b] \subset [T_0, \infty)$, the variation $V_a^b w$ satisfies the inequality $V_a^b w \leqslant cw(a)$ with a certain constant $c > 0$, and that

$$\lim_{T\to\infty} U(w, T) = \lim_{T\to\infty} \int_{T_0}^{T} w(\tau)\, \mathrm{d}\tau = +\infty.$$

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