

Symplectic nilmanifolds and applications.

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Abstract.

The talk is devoted to the remarkable towers of bundles

$$M^n \rightarrow M^{n-1} \rightarrow \dots \rightarrow S^1, \quad n \geq 2,$$

with fiber the circle S^1 .

These towers are defined by the nilpotent groups of the polynomial transformations of the real line.

Each M^n , $n \geq 2$, is a smooth nilmanifold with a 2-form which gives a symplectic structure on any M^{2k} .

Such manifolds play an important role in different areas of mathematics. We will discuss the differential-geometric and algebro-topologic results and unsolved problems, concerning these manifolds.

Groups of polynomial transformations.

Put $L^n = \{p_x(t) = t + \sum_{k=1}^n x_k t^{k+1}, x_k \in \mathbb{R}\}$.

We have $L^n \cong \mathbb{R}^n : p_x(t) \Rightarrow x = (x_1, \dots, x_n)$.

We will consider L^n as the n -dim group of polynomial transformations of the real line

$$\mathbb{R} \rightarrow \mathbb{R} : t \mapsto p_x(t),$$

with the multiplication: $x * y = z$, where

$$(p_x * p_y)(t) = p_z(t) = p_y(p_x(t)) \mod t^{n+2}.$$

Example.

For $n = 4$:

$$p_z(t) = (p_x * p_y)(t) = p_x(t) + \sum_{k=1}^4 y_k p_x(t)^{k+1} \mod t^6 :$$

$$z_1 = x_1 + y_1,$$

$$z_2 = x_2 + 2x_1y_1 + y_2,$$

$$z_3 = x_3 + (2x_2 + x_1^2)y_1 + 3x_1y_2 + y_3$$

$$z_4 = x_4 + 2(x_3 + x_1x_2)y_1 + 3(x_2 + x_1^2)y_2 + 4x_1y_3 + y_4.$$

Nilpotent group structure on $\mathbb{R}^n$.

The group $L^n \cong \mathbb{R}^n$ has the structure of *nilpotent* group with the upper central series

$$L_n^n \subset \cdots \subset L_q^n \subset \cdots \subset L_0^n = L^n,$$

where $L_n^n = \{0 \in \mathbb{R}\}$,

$$\mathbb{R}^{n-q} \cong L_q^n = \{p_x(t) = t + \sum_{k=q+1}^n x_k t^{k+1}\}.$$

We have

$$L_q^n = \{x \in L^n \mid \forall y \in L^n : [x, y] \in L_{q+1}^n\}$$

and $L_q^n / L_{q-1}^n \cong \mathbb{R}$ is the center of L^n / L_q^n , $q = 0, \dots, n-1$.

The canonical matrix representation.

The left multiplication $*$ gives the canonical matrix representation

$$(x : v \rightarrow x * v) : \rho : L^n \rightarrow GT(n+1) : \rho(p_x(t)) \begin{pmatrix} 1 \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ x * v \end{pmatrix}$$

into the group of lower triangular $(n+1) \times (n+1)$ -matrices with ones on the diagonal:

$$\rho(p_x(t)) = X = (x_{ik}), \quad i, k = 0, \dots, n,$$

where $x_{i,k} = [p_x(t)^{k+1}]_{i+1}$ is the coefficient of t^{i+1} in $p_x(t)^k$.

Example.

For $n = 4$:

$$\rho(p_x(t)) \begin{pmatrix} 1 \\ v \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ x_1 & 1 & & & \\ x_2 & 2x_1 & 1 & & \\ x_3 & 2x_2 + x_1^2 & 3x_1 & 1 & \\ x_4 & 2(x_3 + x_1x_2) & 3(x_2 + x_1^2) & 4x_1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}.$$

Deformation to the standard group structure.

The multiplication $*$ on $\mathbb{R}^n$ can be written down as

$$x * y = x + y + A(x)y,$$

where $A(x) = (a_{ik}(x))$ is the lower triangular $(n \times n)$ -matrix with zeros on the diagonal and

$$a_{ik}(x) = x_{i,k} = [p_x(t)^{k+1}]_{i+1}, \quad i \neq k.$$

Any linear transformation $B : \mathbb{R}^n \rightarrow \mathbb{R}^n$ of coordinates in $\mathbb{R}^n$ by $B \in GL(n, \mathbb{R})$ gives a *transformed* multiplication on $\mathbb{R}^n$:

$$\begin{aligned} x *_B y &\stackrel{def}{=} B^{-1}((Bx) * (By)) = \\ &= B^{-1}(Bx + By + A(Bx)By) = \\ &= x + y + (B^{-1}A(Bx)B)y. \end{aligned}$$

In the case of a scalar matrix τE , we obtain

$$x *_\tau y = x + y + A(\tau x)y.$$

This gives a *deformation of multiplication* $*$ ($\tau = 1$) to the standard addition ($\tau = 0$) on $\mathbb{R}^n$.

Example. For $n = 4$:

$$x * y = x + y + \tau A_1(x)y + \tau^2 A_2(x)y,$$

where

$$A_1(x) = \begin{pmatrix} 0 & & & \\ 2x_1 & 0 & & \\ 2x_2 & 3x_1 & 0 & \\ 2x_3 & 3x_2 & 4x_1 & 0 \end{pmatrix}, \quad A_2(x) = \begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ x_1^2 & 0 & 0 & \\ 2x_1x_2 & 3x_1^2 & 0 & 0 \end{pmatrix}.$$

Cocompact lattices.

The multiplication $*$ gives the free actions of L^n on $\mathbb{R}^n$:

The left shift $v \rightarrow x * v$ gives a linear action ρ ,

The right shift $v \rightarrow v * x$ gives a non-linear action.

Let us consider the canonical lattice:

$$\Gamma^n = \{p_x(t) \in L^n : x_i \in \mathbb{Z}\}$$

with the upper central series:

$$\Gamma_n^n \subset \cdots \subset \Gamma_q^n \subset \cdots \subset \Gamma_0^n = \Gamma^n.$$

This lattice $\Gamma^n \cong \mathbb{Z}^n$ is cocompact (uniform).

Nilmanifolds.

With respect to the right shifts we obtain a smooth *closed* and *compact nilmanifold*

$$M^n = \mathbb{R}^n / \Gamma^n.$$

The tangent bundle of M^n is

$$T(M^n) = \mathbb{R}^n \times_{\Gamma^n} \mathbb{R}^n \rightarrow M^n = \mathbb{R}^n / \Gamma^n$$

with respect to the linear action ρ (left shift) on a fiber $\mathbb{R}^n$.

We have the towers of groups

$$L^n \rightarrow L^{n-1} \rightarrow \dots \rightarrow L^1,$$

$$\Gamma^n \rightarrow \Gamma^{n-1} \rightarrow \dots \rightarrow \Gamma^1$$

and the induced tower

$$M^n \rightarrow M^{n-1} \rightarrow \dots \rightarrow M^1 = S^1$$

of bundles $M^n \rightarrow M^{n-1}$ with the fiber S^1 .

For each n the monomorphism holds

$$i_n : L^1 \rightarrow L^n : i_n(x_1) = (x_1, \dots, x_1^k, \dots, x_1^n).$$

Its composition with the projection $L^n \rightarrow L^1$ is the identity map.

Thus for each n the bundle

$$M^1 \rightarrow S^1 \quad \text{with the fiber } L_1^n / \Gamma_1^n$$

has a section.

Left invariant differential operators.

Let us fix the polynomial ring $R[x_1, \dots, x_n]$ as the ring of functions on $L^n \cong \mathbb{R}^n$.

Put for $f(x) \in R[x_1, \dots, x_n]$

$$R_x^y f(x) \stackrel{\text{def}}{=} f(x * y) = \sum_{|I| \geq 0} \mathcal{D}_I(f(x)) y^I$$

where R_x^y is the *right shift* operator,

$I = (i_1, \dots, i_n)$ and $y^I = y_1^{i_1} \dots y_n^{i_n}$.

From the associativity equation $R_x^y R_x^z = R_y^z R_x^y$ we have

$$\sum_{|I| \geq 0} \sum_{|J| \geq 0} \mathcal{D}_I \mathcal{D}_J f(x) y^J z^I = \sum_{|K| \geq 0} \mathcal{D}_K f(x) (y * z)^K.$$

Example $n = 3$. We have $\mathcal{D}_0 f(x) = f(x)$,

$$\mathcal{D}_{(1,0,0)} = \frac{\partial}{\partial x_1} + 2x_1 \frac{\partial}{\partial x_2} + (2x_2 + x_1^2) \frac{\partial}{\partial x_3},$$

$$\mathcal{D}_{(0,1,0)} = \frac{\partial}{\partial x_2} + 3x_1 \frac{\partial}{\partial x_3},$$

$$\mathcal{D}_{(0,0,1)} = \frac{\partial}{\partial x_3}.$$

$$\mathcal{D}_{(1,0,0)} \mathcal{D}_{(0,1,0)} = \mathcal{D}_{(1,1,0)} + 2\mathcal{D}_{(0,0,1)},$$

$$\mathcal{D}_{(0,1,0)} \mathcal{D}_{(1,0,0)} = \mathcal{D}_{(1,1,0)} + 3\mathcal{D}_{(0,0,1)}.$$

The algebra $\mathcal{A}^n$ generated by the operators $\mathcal{D}_I$ is the algebra of all left invariant differential operators on $\mathbb{R}[x_1, \dots, x_n]$ for the left shift L_x^z :

$$L_x^z f(x) = f(z * x),$$

that is

$$L_x^z \mathcal{D}_I f(x) = \mathcal{D}_I L_x^z f(x)$$

for z as parameter.

Algebra of the left invariant operators.

The algebra $\mathcal{A}^n$ is multiplicatively generated by the operators

$$\xi_i = \partial_i + \sum x_{i,q} \partial_q, \quad i = 1, \dots, n,$$

where $\partial_i = \frac{\partial}{\partial x_i}$, and x_{iq} is the coefficient of t^{q+1} in the polynomial $p_x(t)^{i+1}$, as before.

The commutators of this operators are

$$[\xi_i, \xi_j] = (j - i) \xi_{i+j}$$

with $\xi_q = 0$ if $q > n$.

Example. For $n = 3$

$$\mathcal{A}^3 = \mathbb{R}[\xi_1, \xi_2, \xi_3] / ([\xi_1, \xi_2] = \xi_3, [\xi_1, \xi_3] = [\xi_2, \xi_3] = 0).$$

The operators $\{\xi_i\}$ constitute a basis in the Lie algebra $\mathcal{L}^n$ of the left invariant vector fields on the group L^n , and the operator ξ_m corresponds to the one-parameter subgroup $\phi_m(s)$ of polynomials

$$\left\{ \varphi_m(t; s) = t(1 - mst^m)^{-\frac{1}{m}} \mod t^{n+2} \right\}, \quad m = 1, 2, \dots, n.$$

We have

$$\varphi_m(t; s) = t + st^{m+1} + \sum_{k \geq 2} (1+m)(1+2m) \dots (1+(k-1)m) \frac{s^k}{k!} t^{km+1}.$$

Note $\phi_m(t, 1) \notin \Gamma^n$ for $m > 1$, but $\varphi_m(t; m) = \varphi_m(t; 1)^m \in \Gamma^n$.

Example. For $n = 4$

$$\varphi_1(t; s) = t + st^2 + s^2t^3 + s^3t^4 + s_4t^5,$$

$$\varphi_2(t; s) = t + st^3 + \frac{3}{2}s^2t^5,$$

$$\varphi_3(t; s) = t + st^4,$$

$$\varphi_4(t; s) = t + st^5.$$

$$\varphi_1(t; 1) = e_1 * e_2 * e_3^{-2} * e_4^6,$$

where $e_3^{-1}(e_3(t)) = t$.

Cohomology ring of a differential graded algebra.

A differential graded algebra (d. g. a.) (C, d) is a graded algebra

$$C = \sum_{p \geq 0} C^p$$

with a differential $d : C \rightarrow C$ of degree 1, i. e. $d(C^p) \subset C^{p+1}$ and $d^2 = 0$, such that $a \cdot b = (-1)^{pq}ba$ for $a \in C^p$, $b \in C^q$, $dab = (da)b + (-1)^p a(db)$ for $a \in C^p$.

Put $Z^p C = \ker(d : C^p \rightarrow C^{p+1})$ – cocycles group,

$B^p C = \operatorname{Im}(d : C^{p-1} \rightarrow C^p)$ – coboundaries group,

and $H^p C = Z^p C / B^p C$ – cohomology group.

$$\text{Then } H^* C = \sum_{p \geq 0} H^p C$$

is a d.g.a. (with $d = 0$) – *cohomology ring* of C .

Example.

Let X be a smooth n -dimensional compact manifold.
Then we have a d.g.a. of smooth real differential forms

$$C(X) = \sum_{p \geq 0} C^p(X).$$

In a coordinate neighbourhood $U \subset X$ we have for $\omega \in C^p(X)$

$$\omega = \sum u_{i_1 \dots i_p}(x) dx^{i_1} \wedge \dots \wedge dx^{i_p}, \quad x = (x^{i_1}, \dots, x^{i_1}) \in U,$$

$$\begin{aligned} d\omega &= \sum_{i_1 < i_2 < \dots < i_p} du_{i_1 \dots i_p} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} = \\ &= \sum_{i_1 < i_2 < \dots < i_p} \frac{\partial}{\partial x^{i_0}} u_{i_1 \dots i_p} dx^{i_0} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \end{aligned}$$

and $H^*C(X) = H^*(X; \mathbb{R})$.

**Differential graded algebra
of the left invariant differential forms on the nilmanifold.**

Let $\omega_1, \dots, \omega_n$ be the basis of the left invariant differential 1-forms on L^n dual to the basis $\{\xi_i\}$.

Let

$$\omega = \sum_{i=1}^n \omega_i \xi_i$$

be the Maurer-Cartan form taking values in the Lie algebra $\mathcal{L}^n$ of vector fields ξ_i , $i = 1, \dots, n$.

The Maurer–Cartan equation

$$d\omega = -\frac{1}{2}[\omega, \omega]$$

in our case takes the form

$$d\omega_q = \sum_{\{(k,l): k>l>0, k+l=q\}} (k-l) \omega_k \wedge \omega_l. \quad (1)$$

Here $[\omega, \omega](\zeta_1, \zeta_2) = [\omega(\zeta_1), \omega(\zeta_2)]$.

Note that $d\omega_1 = d\omega_2 = 0$ and (1) is independent of n .

Examples: $d\omega_3 = \omega_2 \wedge \omega_1$, $d\omega_4 = 2\omega_3 \wedge \omega_1$, $d\omega_5 = 3\omega_4 \wedge \omega_1 + \omega_3 \wedge \omega_2$.

Bigraded cohomology ring.

We have $H^*(M^n; \mathbb{R}) = H(\Lambda(\omega_1, \dots, \omega_n), d)$
where $\Lambda()$ is the exterior algebra, and d has the form (1).

Set $\text{bideg } \omega_q = (1, -2q)$. It follows from (1) that
the differential complex $(\Lambda(\omega_1, \dots, \omega_n), d)$ can be decomposed as
a sum of differential subcomplexes

$$\Lambda^0 + \sum_{q=1}^n (\Lambda^{-2q}, d),$$

where $\Lambda^0 = \mathbb{R}$ and (Λ^{-2q}, d) is generated by the forms
 $\omega_{i_1} \wedge \dots \wedge \omega_{i_s}$, $s = 1, \dots, n$, $i_1 > i_2 > \dots > i_s > 0$, $i_1 + \dots + i_s = q$.

For any $n \geq 2$ we have:

$$H^1(M^n; \mathbb{R}) = H^{1,-2}(M^n; \mathbb{R}) + H^{1,-4}(M^n; \mathbb{R}) = \mathbb{R} \oplus \mathbb{R}$$

with the generators $[\omega_1]$ and $[\omega_2]$ correspondingly.

Thus $H^1(M^n; \mathbb{Z}) = \mathbb{Z} + \mathbb{Z}$, $n \geq 2$.

The ring $H^*(M^n, \mathbb{R})$ has the structure of a bigraded ring

$$\mathbb{R} + \sum_{s=1}^n \sum_{2q=s(s+1)}^{s(2n+1-s)} H^{s,-2q}(M^n; \mathbb{R}).$$

Set $n(s, q) = q - \frac{1}{2}(s-1)(s-2)$. For any $k \geq n(s, q)$ we have

$$H^{s,-2q}(M^k; \mathbb{R}) = H^{s,-2q}(M^{k+1}; \mathbb{R}).$$

Example for $n = 4$.

We have:

$$H^*(M^4; \mathbb{R}) = H^*(\wedge(\omega_1, \omega_2, \omega_3, \omega_4), d),$$

where $d\omega_1 = d\omega_2 = 0$, $d\omega_3 = \omega_2 \wedge \omega_1$, $d\omega_4 = 2\omega_3 \wedge \omega_1$.

$$H^*(M^4; \mathbb{R}) = \mathbb{R} + \sum_{s=1}^4 \sum_{2q=s(s+1)}^{s(9-s)} H^{s, -2q}(M^4; \mathbb{R}).$$

10 differential subcomplexes.

$q \backslash s$	0	1	2	3	4
0	1				
1		ω_1			
2		ω_2			
3		ω_3	$\xrightarrow{1} \omega_2 \wedge \omega_1$		
4		ω_4	$\xrightarrow{2} \omega_3 \wedge \omega_1$		
5			$\omega_4 \wedge \omega_1$		
			$\omega_3 \wedge \omega_2$		
6			$\omega_4 \wedge \omega_2$	$\xrightarrow{-2} \omega_3 \wedge \omega_2 \wedge \omega_1$	
7			$\omega_4 \wedge \omega_3$	$\xrightarrow{-1} \omega_4 \wedge \omega_2 \wedge \omega_1$	
8				$\omega_4 \wedge \omega_3 \wedge \omega_1$	
9				$\omega_4 \wedge \omega_3 \wedge \omega_2$	
10					$\omega_4 \wedge \omega_3 \wedge \omega_2 \wedge \omega_1$

Generators of the Poincare duality.

dim	0	1	2	3	4
		$[\omega_1]$	$\longleftrightarrow$	$[\omega_4 \wedge \omega_3 \wedge \omega_2]$	
		$[\omega_2]$	$\longleftrightarrow$	$[\omega_4 \wedge \omega_3 \wedge \omega_1]$	
			$[\omega_1 \wedge \omega_4]$		
			$\updownarrow$		
			$[\omega_2 \wedge \omega_3]$		
	1		$\longleftrightarrow$		$[\omega_4 \wedge \omega_3 \wedge \omega_2 \wedge \omega_1]$

Toric bundles.

For each n and $q < \left\lfloor \frac{n+1}{2} \right\rfloor$ there are exact sequences

$$0 \rightarrow \mathbb{R}^{q+1} \rightarrow L^{n+1} \rightarrow L^{n-q} \rightarrow 0,$$

$$0 \rightarrow \mathbb{Z}^{q+1} \rightarrow \Gamma^{n+1} \rightarrow \Gamma^{n-q} \rightarrow 0,$$

which give a smooth bundle

$$\pi_n^q : M^{n+1} \rightarrow M^{n-q}$$

with fibre torus $\mathbb{T}^{q+1}$.

Symplectic nilmanifolds M^n .

A smooth manifold M is called *symplectic* if it carries a *nondegenerate closed* 2-form Ω which is called a *symplectic* form.

Consider the smooth bundle with fibre circle S^1

$$\pi_n = \pi_n^0 : M^{n+1} \rightarrow M^n.$$

The left invariant 1-form ω_{n+1} is a *connection* in the bundle π_n . The *curvature* form of this bundle is

$$\Omega_n = \sum_{\{(k,l): k+l=n+1, k>l>0\}} (k-l)\omega_k \wedge \omega_l$$

and we have $\pi^*\Omega_n = d\omega_{n+1}$.

The nilmanifold M^{2n} with the form Ω_{2n} is *symplectic*.

Conjecture. Ω_n is an integer form for any n .

Example. For $n = 3$, $q = 1$ we have the smooth bundle

$$\pi_3^1 : M^4 \rightarrow M^2 = \mathbb{T}^2$$

with the fibre $\mathbb{T}^2$ and the symplectic form:

$$\Omega_4 = 3\omega_4 \wedge \omega_1 + \omega_3 \wedge \omega_2.$$

The base is the symplectic manifold with the form $\Omega_2 = \omega_2 \wedge \omega_1$ and $(\pi_3^1)^*\Omega_2 = 0$.

The manifold $M^3 \times S^1$ is symplectic with the form

$$2\omega_3 \wedge \omega_1 + \omega_2 \wedge dt.$$

The manifold M^{2n-1} has the form

$$\Omega_{2n-1} = \sum_{\{(k,l): k+l=2n, k>l>0\}} (k-l)\omega_k \wedge \omega_l.$$

For $n > 2$ the form ω_n is not closed, thus the 2-form

$$\Omega = \Omega_{2n-1} + \omega_n \wedge dt$$

is not closed on $M^{2n-1} \times S^1$ but Ω^n is closed and gives the fundamental cocycle on this manifold.

Nonformality of nilmanifolds.

A simplicial complex X is called *formal* if its rational homotopy type is a formal consequence of its cohomology ring.

Theorem. (F. E. A. Johnson, E. G. Rees, 1989)

If G is a nilpotent Lie group and $\Gamma \subset G$ is a discrete cocompact subgroup, then G/Γ is *formal* if and only if G is abelian.

Corollary. The symplectic nilmanifolds M^{2m} are *nonformal*, $m \geq 2$, and $M^2 = T^2$ is *formal*.

Realizing nilmanifolds as symplectic submanifolds of complex projective spaces $\mathbb{C}P^N$, denote by $X_m(N)$ the *symplectic* blow up of $\mathbb{C}P^N$ along M^{2n} .

Theorem. (I.K. Babenko, I.A. Taimanov, 1999)

For $m \geq 2$ and $N \geq 2m + 1$ the symplectic manifolds $X_m(N)$ are *simply connected* and *nonformal*.

The proof of this result makes use of the fact that in the cohomology ring $H^*(M^{2n})$ there are *nontrivial* Massey products.

Universal properties of M^n .

The manifold $M^n = K(\Gamma^n, 1)$ is the *Eilenberg-MacLane space* and thus for any CW -complex X

$$[X, M^n] = H^1(X, \Gamma^n).$$

The manifold M^n is the *classifying space* for the discrete group Γ^n , that is $M^n = B\Gamma^n$ and thus $[X, M^n]$ is the set of isomorphism classes of principal Γ^n -bundles over a CW -complex X ; we have

$$[X, M^n] = \text{Hom}(\pi_1(X), \Gamma^n),$$

$$H_k(M^n; \mathbb{Z}) = H_k(\Gamma^n; \mathbb{Z}), \quad H^k(M^n; \mathbb{Z}) = H^k(\Gamma^n; \mathbb{Z}).$$

Cellular subdivision of M^n .

Consider the cellular subdivision

$$(pt) = M_0^n \subset M_1^n \subset \cdots \subset M_{n-1}^n \subset M_n^n = M^n,$$

where $M_1^n = \vee_{i=1}^n S_i^1$, $M_{k+1}^n/M_k^n = \vee S^{k+1}$, $M_n^n/M_{n-1}^n = \vee S^n$.

For the $\mathbb{Z}$ -homology groups of pair we obtain the exact sequence

$$0 \rightarrow H_2(M^n) \rightarrow H_2(M^n/M_1^n) \rightarrow \bigoplus_{i=1}^n \mathbb{Z} \rightarrow H_1(M^n) \rightarrow 0.$$

Using that $M^n = K(\Gamma^n; 1)$ and $M_1^n = K(\vee_{i=1}^n \mathbb{Z}; 1)$

for the homotopy groups of pair we obtain the exact sequence

$$0 \rightarrow R_n \rightarrow \vee_{i=1}^n \mathbb{Z} \rightarrow \Gamma^n \rightarrow 0.$$

Here $\vee_{i=1}^n \mathbb{Z}$ is the free product of $\mathbb{Z}$

and $R_n = \pi_2(M^n, M_1^n)$ is its subgroup. It is a free group.

The multiplicative generators of the group $\Gamma^n \subset L^n$ are $e_k(t) = t + t^{k+1}$, $k = 1, \dots, n$. Put $e_0(t) = t$. Note $\varphi_q(t; 1) = e_q(t)$ for $q > [\frac{n}{2}]$.

It is clear that if $e^I = e_1^{i_1} * \dots * e_n^{i_n} = e_0$ where $I = (i_1, \dots, i_n) \in \mathbb{Z}^n$, then $I = 0$.

We have

$$[e_k, e_{k+2}] = e_{2k+2}^2 * e_i * \dots, \quad i > 2k + 2, \quad k \geq 1,$$

$$[e_k, e_{k+1}] = e_{2k+1} * e_j * \dots, \quad j > 2k + 1, \quad k \geq 1.$$

Thus the group $H_1(M^n; \mathbb{Z}) = \Gamma_n / [\Gamma_n, \Gamma_n]$ has only 2-torsion.

Hopf's integral homology formula.

Let $G = F/R$ and F is a free group. Then

$$H_2(G, \mathbb{Z}) \cong (R \cap [F, F])/[F, R].$$

Thus

$$H_2(M^n, \mathbb{Z}) \cong (R_n \cap [F^n, F^n])/[F^n, R_n],$$

where $F^n = \bigvee_{i=1}^n \mathbb{Z}$ and

$$0 \rightarrow R_n \rightarrow \bigvee_{i=1}^n \mathbb{Z} \rightarrow \Gamma^n \rightarrow 0,$$

and therefore to each element $a \in H_2(M^n, \mathbb{Z})$ corresponds an element

$$g = [a_1, b_1] \cdot \dots \cdot [a_g, b_g] \in (R_n \cap [F^n, F^n]).$$

Example $n = 3$.

Γ^3 has the generators e_1, e_2, e_3 and the relations

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_0, \quad [e_2, e_3] = e_0.$$

Thus $H_1(M^3, \mathbb{Z}) = \Gamma^3 / [\Gamma^3, \Gamma^3] = \mathbb{Z} \oplus \mathbb{Z}$.

In this case F^3 has the generators c_1, c_2, c_3 ,

R_3 has the generators r_1, r_2, r_3 and

$$R_3 \rightarrow F^3 : \quad r_1 \mapsto [c_1, c_3], \quad r_2 \mapsto [c_2, c_3], \quad r_3 \mapsto [c_1, c_2]c_3^{-1}.$$

We have $r_3 \notin [F^3, F^3]$.

The generators of

$$H_2(M^3; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$$

correspond to r_1 and r_2 .

Example $n = 4$.

Γ^4 has the generators e_1, e_2, e_3, e_4 and the relations

$$[e_1, e_2] = e_3 * e_4, \quad [e_1, e_3] = e_4^2, \quad [e_1, e_4] = e_0,$$

$$[e_i, e_j] = e_0, \quad i, j = 2, 3, 4.$$

Thus $H_1(M^4, \mathbb{Z}) = \Gamma^4 / [\Gamma^4, \Gamma^4] = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$.

Consider an oriented 2-dimensional surface S_g^2 of genus g .

We have $S_g^2 = K(G_g, 1)$,
where $G_g = \pi_1(S_g^2)$ is the group with the generators $a_1, b_1, \dots, a_g, b_g$
and a single relation $[a_1, b_1] \cdot \dots \cdot [a_g, b_g] = 1$,
that is

$$0 \rightarrow \mathbb{Z} \rightarrow \bigvee_{i=1}^{2g} \mathbb{Z} \rightarrow G_g \rightarrow 0.$$

We have

$$[S_g^2, M^n] = \text{Hom}[G_g, \Gamma^n].$$

Corollary.

Each element $a \in H_2(M^n, \mathbb{Z})$, $n \geq 2$, is realised by a smooth mapping

$$f_a : S_g^2 \rightarrow M^n, \quad (f_a)_*([S_g^2]) = a$$

for some g .

The form Ω_n is integer if and only if

$$\langle f_a^* \Omega_n, [S_g^2] \rangle \in \mathbb{Z}.$$

Let $\pi_n : M^{n+1} \rightarrow M^n$ be a smooth bundle with the fibre S^1 . Denote by $\xi_{n+1} = \xi_{n+1}(\pi_n)$ the field of vectors tangent to the fiber of this bundle.

Problem. Classify the sequences of smooth manifolds

$$\pi_n : M^{n+1} \rightarrow M^n, n \geq 0,$$

with the fiber S^1 , such that

- for each $n > 1$ there exists an *integer* closed 2-form Ω_n on M^n satisfying the condition

$$\pi_n^* \Omega_n = d\omega_{n+1}, \quad \text{where } \langle \omega_{n+1}, \xi_{n+1} \rangle = \|\xi_{n+1}\|,$$

- for each even n the form Ω_n is nondegenerate.

The following problem is closely related to the previous one and has self-contained interest:

Problem. For the towers

$$M^n \rightarrow M^{n-1} \rightarrow \dots \rightarrow S^1$$

of fibrations described above

calculate the cohomology rings $H^*(M^n; k)$ for $k = \mathbb{Z}$ and $\mathbb{Q}$.

Consider the bundle

$$\hat{\pi}_n : E \rightarrow M^n$$

with the fiber D^2 , such that $\partial E = M^{n+1}$.

In the exact sequence of the pair $(E, \partial E)$ the Gysin homomorphism has the form

$$j_q^n : H^q(M^n) \rightarrow H^{q+2}(M^n) : j_q^n a = [\Omega_n]a.$$

Thus we get the exact sequence

$$0 \leftarrow \ker j_{q-1}^n \leftarrow H^q(M^{n+1}) \leftarrow \operatorname{coker} j_{q-2}^n \leftarrow 0.$$

In the case of rational coefficients we get

$$\dim H^q(M^{n+1}) = (\dim \ker j_{q-1}^n) + (\dim \operatorname{coker} j_{q-2}^n).$$

Denote the Betti number $\dim H^q(M^n, \mathbb{Q})$ by b_q^n .

Thus we have the estimate

$$b_q^{n+1} \leq b_{q-1}^n + b_q^n.$$

D. V. Millionshikov has obtained results on the Betti numbers b_q^n for manifolds M^n defined by the groups L^n . His approach is based on the calculations by L. Goncharova of infinite dimensional Lie algebras cohomologies.

For such manifolds he proved that

$$b_2^n = 3 \text{ for all } n > 5;$$

$$b_3^n = 5 \text{ for all } n > 11.$$

D. V. Millionshchikov used some combinatorial arguments and the Goncharova theorem to sketch the proof of the statement

$$b_q^n = F_{q+2}$$

for n sufficiently large ($n > 3q + 2$),

where F_{q+2} is the $(q + 2)$ -th Fibonacci number.

That is $F_{q+2} = F_{q+1} + F_q$, $q \geq 0$, $F_0 = 0$, $F_1 = 1$.

However no detailed proof of this statement appeared till now.

Recently he suggested to consider the last statement as a conjecture.

Using the computer, Millionschikov calculated Betti numbers b_q^n for $n \leq 30$.

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Addendum.

Massey products.

Let (C, d) — a d.g.a. For $a \in C^p$ put

$$\bar{a} = (-1)^p a.$$

Then we obtain the involution on C , i.e. $\bar{ab} = \bar{a}\bar{b}$, $\bar{\bar{a}} = a$, and

$$dab = (da)b + \bar{a}(db) \text{ for } a \in C^p.$$

Let $T_k^0 = T_k^0(C)$ — the algebra of upper triangular $(k \times k)$ -matrices over C with zeros on the diagonal. For $A = (a_{ij}) \in T_k^0$ put $dA = (da_{ij})$ and $\bar{A} = (\bar{a}_{ij})$.

Let $J^k = (J_{ij}^k) \in T_k^0$, such that $J_{ij}^k = 0$, if $(i, j) \neq (1, k)$, and $J_{1k}^k = 1$.

Lemma. Let $A = (a_{ij}) \in T_{n+1}^0$, such that $a_{i,i+1} \in C^{k_i}$ and

$$dA = \bar{A}A - cJ^{n+1}$$

for some $c \in C$. Then

- $da_{i,i+1} = 0$,
- $c \in C^m$, where $m = k_1 + \cdots + k_n - n + 2$,
- $dc = 0$.

Show that $dc = 0$. We have:

$$d\bar{A} = -d\bar{A} = -A\bar{A} - cJ^{n+1}.$$

Using that $J^{n+1}A = AJ^{n+1} = 0$, we obtain

$$d\bar{A}A = (d\bar{A})A + \bar{A}(dA) = -A\bar{A}A + A\bar{A}A = 0.$$

So

$$(dc)J^{n+1} = d(\bar{A}A) - ddA = 0.$$

Definition. Take n homogeneous elements $a_1, \dots, a_n$ in C , which are cocycles, i.e. $da_i = 0$, $i = 1, \dots, n$.

Assume that there exists a matrix $A \in T_{n+1}^0$ such that:

- $a_{i,i+1} = a_i$
- A satisfies the equation

$$dA = \bar{A}A - cJ^{n+1}$$

for some $c \in C$.

In this case it is told that the *Massey product* $\langle a_1, \dots, a_n \rangle$ of the cocycles $a_1, \dots, a_n$ is defined and equals cocycle c .

Examples $n = 2$.

$$\begin{pmatrix} 0 & \bar{a}_1 & \bar{a}_{13} \\ & 0 & \bar{a}_2 \\ & & 0 \end{pmatrix} \begin{pmatrix} 0 & a_1 & a_{13} \\ & 0 & a_2 \\ & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \bar{a}_1 a_2 \\ & 0 & 0 \\ & & 0 \end{pmatrix}.$$

So

$$c \begin{pmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \bar{a}_1 a_2 \\ & 0 & 0 \\ & & 0 \end{pmatrix} - \begin{pmatrix} 0 & da_1 & da_{13} \\ & 0 & da_2 \\ & & 0 \end{pmatrix}$$

and

$$da_1 = da_2 = 0, \quad c = \langle a_1, a_2 \rangle = \bar{a}_1 a_2 - da_{13}$$

for some a_{13} .

Examples $n = 3$.

$$\begin{pmatrix} 0 & \bar{a}_1 & \bar{a}_{13} & \bar{a}_{14} \\ & 0 & \bar{a}_2 & \bar{a}_{24} \\ & & 0 & \bar{a}_3 \\ & & & 0 \end{pmatrix} \begin{pmatrix} 0 & a_1 & a_{13} & a_{14} \\ & 0 & a_2 & a_{24} \\ & & 0 & a_3 \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \bar{a}_1 a_2 & \bar{a}_1 a_{24} + \bar{a}_{13} a_3 \\ & 0 & 0 & \bar{a}_2 a_3 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix}$$

So

$$c \begin{pmatrix} 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \bar{a}_1 a_2 & \bar{a}_1 a_{24} + \bar{a}_{13} a_3 \\ & 0 & 0 & \bar{a}_2 a_3 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} - \begin{pmatrix} 0 & da_1 & da_{13} & da_{14} \\ & 0 & da_2 & da_{24} \\ & & 0 & da_3 \\ & & & 0 \end{pmatrix}.$$

So, $da_i = 0$, $i = 1, 2, 3$, $\bar{a}_1 a_2 = da_{13}$, $\bar{a}_2 a_3 = da_{24}$,
and $\langle a_1, a_2, a_3 \rangle = c = \bar{a}_1 a_{24} + \bar{a}_{13} a_3$, $\deg c = k_1 + k_2 + k_3 - 1$.

Examples $n = 3$.

$H^*(M^3) = H^*(\Lambda(\omega_1, \omega_2, \omega_3), d)$, where $d\omega_1 = d\omega_2 = 0$, $d\omega_3 = \omega_2 \wedge \omega_1$. The generators of $H^*(M^3)$:

$$[\omega_1], [\omega_2], [\omega_3 \wedge \omega_1], [\omega_3 \wedge \omega_2], [\omega_3 \wedge \omega_2 \wedge \omega_1].$$

So, for

$$a_1 = \omega_1, a_2 = \omega_2, a_3 = \omega_1 \Rightarrow a_{13} = \omega_3, a_{24} = -\omega_3$$

$$\text{and } \langle \omega_1, \omega_2, \omega_1 \rangle = -2\omega_3 \wedge \omega_1,$$

$$a_1 = \omega_1, a_2 = \omega_1, a_3 = \omega_2 \Rightarrow a_{13} = 0, a_{24} = \omega_3$$

$$\text{and } \langle \omega_1, \omega_1, \omega_2 \rangle = \omega_3 \wedge \omega_1,$$

$$a_1 = \omega_1, a_2 = \omega_2, a_3 = \omega_2 \Rightarrow a_{13} = \omega_3, a_{24} = 0$$

$$\text{and } \langle \omega_1, \omega_2, \omega_2 \rangle = -\omega_3 \wedge \omega_2.$$

The matrix equation

$$dA = \bar{A}A - cJ^{n+1}$$

for $n \geq 4$ gives the following relations:

$$da_{i,i+1} = da_i = 0,$$

$$da_{i,k} = \sum_{q=i+1}^{k-1} \bar{a}_{i,q}a_{q,k}, \quad i+2 \leq k \leq n$$

and

$$\langle a_1, \dots, a_n \rangle = c = \sum_{q=2}^n \bar{a}_{1,q}a_{q,n+1} - da_{1,n+1},$$

where $dc = 0$. So

$$da_{i,i+2} = \bar{a}_{i,i+1}a_{i+1,i+2} = \bar{a}_i a_{i+1},$$

$$da_{i,i+3} = \bar{a}_{i,i+1}a_{i+1,i+3} + \bar{a}_{i,i+2}a_{i+2,i+3} = \langle a_i, a_{i+1}, a_{i+2} \rangle.$$

Example $n = 4$.

a_1	a_2	a_3	a_{13}	a_{24}	$\langle a_1, a_2, a_3 \rangle$
ω_1	ω_2	ω_1	ω_3	$-\omega_3$	$2\omega_3 \wedge \omega_1 = d\omega_4$
ω_2	ω_1	ω_1	$-\omega_3$	0	$\omega_3 \wedge \omega_1 = \frac{1}{2}d\omega_4$
ω_1	ω_2	ω_2	ω_3	0	$\omega_3 \wedge \omega_2$

We have

$$\langle a_1, a_2, a_3, a_4 \rangle = \bar{a}_{12}a_{25} + \bar{a}_{13}a_{35} + \bar{a}_{14}a_{45}.$$

For $\langle \omega_2, \omega_1, \omega_1, \omega_1 \rangle$:

$$da_{25} = \langle \omega_1, \omega_1, \omega_1 \rangle = 0 \Rightarrow a_{25} = 0$$

$$da_{35} = -\omega_1 \wedge \omega_1 = 0 \Rightarrow a_{35} = 0$$

$$da_{14} = \langle \omega_2, \omega_1, \omega_1 \rangle = \frac{1}{2}d\omega_4 \Rightarrow a_{14} = \frac{1}{2}\omega_4.$$

So, we obtained:

$$\langle \omega_2, \omega_1, \omega_1, \omega_1 \rangle = -\frac{1}{2}\omega_4 \wedge \omega_1 \neq 0.$$

For $\langle \omega_1, \omega_2, \omega_2, \omega_2 \rangle$:

$$a_{25} = 0, \quad a_{35} = 0$$

and

$$da_{14} = \langle \omega_1, \omega_2, \omega_2 \rangle = \omega_3 \wedge \omega_2.$$

We can't find such a_{14} and therefore the Massey product $\langle \omega_1, \omega_2, \omega_2, \omega_2 \rangle$ is not well defined in $H^*(M^4)$.

Infinite-dimensional algebra of vector fields of the line.

Introduce:

the group $L_\infty = \lim_{\leftarrow} L_n$

Lie algebra $\mathcal{L}_\infty = \lim_{\leftarrow} \mathcal{L}_n$

and algebra of operators $\mathcal{A}_\infty = \lim_{\leftarrow} \mathcal{A}_n$.

Let $l_1 = \{x^{k+1} \frac{d}{dx}, k \geq 1\}$ be the well known Lie algebra of vector fields on the line.

We have $\mathcal{L}_\infty \cong l_1$.

Theorem.(L. V. Goncharova, 1973)

$$\dim H_k^q(l_1) = \begin{cases} 1, & \text{if } k = \frac{3q^2 \pm q}{2}, \\ 0, & \text{otherwise} \end{cases}$$

Thus, $\dim H^q(l_1) = 2$ for $q \geq 1$.

The cohomological product in $H^*(l_1)$ is trivial.

It was V. M. Buchstaber (1978) who raised the problem whether $H^*(l_1)$ is *generated, with respect to Massey products*, by $H^1(l_1)$.

The Heisenberg group.

Let us fix a decomposition $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$, $n = k + l$, and a bilinear mapping $\mathcal{B} : \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^l$.

We define a multiplication on $\mathbb{R}^n$ by the formula

$$(v_1, w_1) \cdot (v_2, w_2) = (v_1 + v_2, w_1 + w_2 + \mathcal{B}(v_1, v_2))$$

where $v_i \in \mathbb{R}^k$, $w_i \in \mathbb{R}^l$, $i = 1, 2$.

Note the relation

$$\mathcal{B}(v_1, v_2) = \mathcal{A}(v_1)v_2,$$

where $\mathcal{A}$ is the *linear* mapping $\mathbb{R}^k \rightarrow \text{Hom}(\mathbb{R}^k, \mathbb{R}^l)$.

Thus we obtain a group structure on $\mathbb{R}^n$, which is noncommutative for nonsymmetric mapping $\mathcal{B}$.

A linear change of coordinates

$$B = (B_1, B_2) \in GL(k, \mathbb{R}) \times GL(l, \mathbb{R}) \subset GL(n, \mathbb{R})$$

gives a new multiplication

$$(v_1, w_1) * (v_2, w_2) = (v_1 + v_2, w_1 + w_2 + B_2^{-1} \mathcal{B}(B_1 v_1, B_1 v_2)).$$

For the scalar matrix τE we get

$$(v_1, w_1) *_{\tau} (v_2, w_2) = (v_1 + v_2, w_1 + w_2 + \tau \mathcal{B}(v_1, v_2))$$

and this gives a deformation into the standard addition.

Note: this is a bilinear deformation.

To obtain the well-known Heisenberg group take $k = 2$, $l = 1$ and for $v_i = (x_i, y_i)$ put

$$\mathcal{B} : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^1 : \mathcal{B}(v_1, v_2) = \begin{pmatrix} x_1 & y_1 \end{pmatrix} \begin{pmatrix} 0 & \tau \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \tau x_1 y_2.$$

The Heisenberg multiplication on $\mathbb{R}^3$:

$$(x_1, y_1, w_1) \cdot (x_2, y_2, w_2) = (x_1 + x_2, y_1 + y_2, w_1 + w_2 + \tau x_1 y_2).$$