Minimal Fillings of Finite Metric Spaces and Dual Problem of Linear Programming

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1. INTROduction: Minimal Fillings of finite metric spaces, a kind of Optimal Connection Problem
2. INTROduction: Linear Programming, Duality
3. Dual LPP for Minimal Fillings: New Possibilities:

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- Explicit formulas for 5- and 6-points spaces, a general algorithm to obtain such formulas.
- Polyhedrons corresponding to trees, Who are they?

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- Polyhedrons corresponding to trees, Who are they?

Remark. All results are obtained in collaboration with Prof. Alexei Tuzhilin. Also results of our post-graduate students, A. Eremin, Z. Ovsyannikov, N. Strelkova are included.

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Recall some necessary definitions from Graph Theory.

Necessary Definitions from Graph Theory

A (*simple*) *graph* is a pair $G = (V, E)$, where $V = \{v_1, \ldots, v_n\}$ is a finite set of *vertices* and a $E = \{e_1, \ldots, e_m\}$ is a finite set of *edges*, and each edge e_i is a two-element subset of V.

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If $e = \{v, v'\}$, then v and v' are *neighbouring*, edge e *connects* them, the edge e and each of the vertices v and v' are *incident*. To be short, the edge of the graph connecting its vertices u and v is denoted by uv.

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A graph is said to be *connected*, if any two its vertices can be connected by a path. We say that $G = (V, E)$ *connects* M, if $M \subset V$. In this case we also say that M is a *boundary of the graph* G. In what follows we always assume that each graph has some fixed boundary which could be empty.

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If we are given with a function $\omega\colon E \to \mathbb{R}$ on the edge set of a graph G, then the pair (G, ω) is referred as a *weighted graph*.

Necessary Definitions from Graph Theory

A *cut of a graph G* is an arbitrary partition of its vertex set into two non-empty non-intersecting subsets. An edge is called a *cut edge of the cut $V = V_1 \sqcup V_2$*, if one of its vertices belongs to V_1, and another vertex belongs to V_2.

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To each family of cuts the so-called *cut matrix* corresponds, whose rows are enumerated by cuts of the family, whose columns are enumerated by the edges of the graph, and whose element standing at the jth position in the ith row equals 1, if the jth edge is a cut edge of the ith cut, and equals 0 otherwise.

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Let G be an arbitrary tree with a boundary M, and let $e \in E$ be an arbitrary edge of the tree G. Elimination of the edge e partitions the tree G into two connected components that are denoted by G_1 and G_2.Put $M_i = M \cap G_i$, $i = 1, 2$. Put $\mathcal{P}_G(e) = \{M_1, M_2\}$.

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In particular, each edge of the tree G generates a cut of the complete graph $K(M)$ with the vertex set M.

Minimal Fillings of Finite Metric Spaces

Let $\mathcal{M} = (M, \rho)$ be a finite metric space, $G = (V, E)$ be a graph connecting M, and $\omega\colon E \to \mathbb{R}_+$ be a non-negative weight function, and $\mathcal{G} = (G, \omega)$ be the corresponding weighted graph. The function ω generates on V the *pseudo-metric d_ω* : the d_ω-distance between two vertices is equal to the least possible weight of the paths in $\mathcal{G}$ connecting these vertices.

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Definition

If for any two points p and q from M the inequality $\rho(p, q) \le d_\omega(p, q)$ holds, then the weighted graph $\mathcal{G}$ is called a *filling* of the space $\mathcal{M}$, and the graph G is referred as the *type* of this filing. The value $\mathrm{mf}(\mathcal{M}) = \inf \omega(\mathcal{G})$, where the infimum is taken over all fillings $\mathcal{G}$ of $\mathcal{M}$ is the *weight of minimal filling*, and each filling $\mathcal{G}$ such that $\omega(\mathcal{G}) = \mathrm{mf}(\mathcal{M})$ is called a *minimal filling*.

Parametric Minimal Filling

Definition

Let $\mathcal{M} = (M, \rho)$ be a finite metric space and $G = (V, E)$ be an arbitrary connected graph connecting M. By $\Omega(\mathcal{M}, G)$ we denote the set of all weight functions $\omega \colon E \to \mathbb{R}$ such that (G, ω) is a filling of the space $\mathcal{M}$. We put

$$\mathrm{mpf}(\mathcal{M}, G) = \inf_{\omega \in \Omega(\mathcal{M}, G)} \omega(G)$$

and we call this value the *weight of minimal parametric filling of the type G for the space $\mathcal{M}$*. If there exists an $\omega \in \Omega(\mathcal{M}, G)$ such that $\omega(G) = \mathrm{mpf}(\mathcal{M}, G)$, then (G, ω) is called a *minimal parametric filling of the type G for the space $\mathcal{M}$*.

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Assertion

Let $\mathcal{M} = (M, \rho)$ be a finite metric space. Then

$$\mathrm{mf}(\mathcal{M}) = \inf\bigl\{\mathrm{mpf}(\mathcal{M}, G)\bigr\},$$

where the infimum is taken over all connected graphs G connecting M.

Fillings of Tree Type

Assertion

Each finite metric space has a minimal filling whose type is a binary tree (possibly, with some degenerate edges, i.e. the edges of weight zero), and a minimal filling whose type is a tree and all weights are positive, and all whose vertices of degree 1 *and* 2 *belong to its boundary.*

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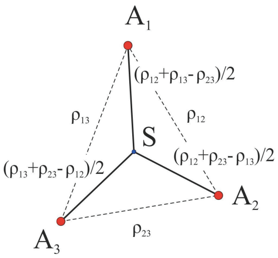

Figure: Minimal filling for a three-point metric space. The weights of edges are alternated sums of distances, the weight of the filling equals half-perimeter of the triangle.

Generalized Fillings: Definition

It turns out to be convenient to expand the class of weighted trees under consideration, namely, permitting arbitrary weights of the edges (not only non-negative). The corresponding objects are called *generalized fillings*, *minimal generalized fillings* and *minimal parametric generalized fillings*. By $\mathrm{mf}_-(\mathcal{M})$ and $\mathrm{mpf}_-(\mathcal{M}, G)$ we denote the corresponding weights.

Example

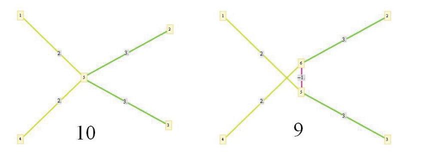

Figure: Minimal parametric filling (left) and minimal generalized parametric filling (right) of the vertex set of the plane rectangle with sides 3 and 4. The type is the same: the moustaches connects the diagonal pairs of the vertices. The interior edge has to be zero in the case of the filling and becomes negative in the generalized case. Here $9 = \mathrm{mpf}_-(\mathcal{M}, G) < \mathrm{mpf}(\mathcal{M}, G) = 10$.

Generalized Fillings

Theorem (Ivanov, Ovsyannikov, Strelkova, Tuzhilin)

For an arbitrary finite metric space $\mathcal{M}$, the set of all its minimal generalized fillings contains its minimal filling, i.e. a generalized minimal filling with nonnegative weight function. Hence, $\mathrm{mf}_{-}(\mathcal{M}) = \mathrm{mf}(\mathcal{M})$.

Tours

Let $\mathcal{M} = (M, \rho)$ be a finite metric space, and G be a tree connecting M. Consider the doubling of a tree G connecting M, i.e. the graph with the same vertex set, but containing each edge of G with multiplicity 2. The resulting graph is Euler's, and each Euler cycle (a cycle passing through each edge exactly once) in it can be decomposed into the union of consecutive *irreducible* boundary paths, i.e. the paths connecting boundary vertices and do not containing other boundary vertices. The corresponding permutation π mapping the beginning vertex of each irreducible boundary path onto its ending one is called a *tour of M with respect to G*.

Example

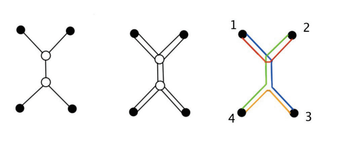

Figure: Binary tree (left), its doubling (middle) and an Euler cycle in the doubling decomposed into the union of irreducible boundary paths. The corresponding tour has the form $\pi = (1, 3, 4, 2)$.

Tours and Perimeters

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The set of all tours on M with respect to G is denoted by $\mathcal{O}(M, G)$. For each tour $\pi \in \mathcal{O}(M, G)$ we put

$$p(\mathcal{M}, G, \pi) = \frac{1}{2} \sum_{x \in M} \rho(x, \pi(x))$$

and we call this value by the *half-perimeter of the space $\mathcal{M}$ with respect to the tour π*. The minimal value of $p(\mathcal{M}, G, \pi)$ over all $\pi \in \mathcal{O}(M, G)$ for all possible trees G (in fact, over all possible cyclic permutations π on M) is called the *half-perimeter of the space $\mathcal{M}$*.

Multi-Tours

Let us consider the graph in which every edge of G is taken with the multiplicity k, $k \geq 1$. The resulting graph possesses an Euler cycle consisting of irreducible boundary paths. This Euler cycle generates a *multi-tour of M with respect to G* that can be defined as a bijection $\pi\colon X \to X$, where $X = \sqcup_{i=1}^{k} M$, and π maps the beginning vertices of irreducible boundary paths onto the ending ones. The set of all multi-tours on M with respect to G is denoted by $\mathcal{O}_\mu(M, G)$.

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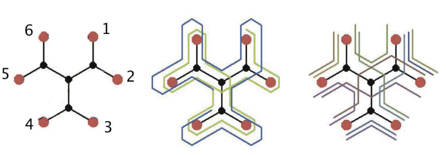

Figure: A part of a multi-tour with multiplicity 2 (left), and the irreducible boundary paths forming this multi-tour (right). The multi-tour $(1, 2, 5, 6, 3, 4, 5, 6, 1, 3, 4, 2)$ starts as a green polygonal line (left) and becomes blue when multiplicity of edges becomes more than 2.

Multi-Tour Formula for Minimal Filling

Let $\mathcal{M} = (M, \rho)$ be a finite metric space, and G be a tree connecting M. As in the case of tours, for each multitour $\pi \in \mathcal{O}_\mu(M, G)$ we put

$$p(\mathcal{M}, G, \pi) = \frac{1}{2k} \sum_{x \in X} \rho\big(x, \pi(x)\big).$$

Theorem (A. Yu. Eremin–2013)

For an arbitrary finite metric space $\mathcal{M} = (M, \rho)$ and an arbitrary binary tree G joining M, the weight of minimal parametric generalized filling can be calculated as follows

$$\mathrm{mpf}\_(\mathcal{M}, G) = \max\big\{p(\mathcal{M}, G, \pi) \mid \pi \in \mathcal{O}_\mu(M, G)\big\}.$$

The weight of minimal filling can be calculated as follows

$$\mathrm{mf}(\mathcal{M}) = \mathrm{mf}_-(\mathcal{M}) = \min_G \max\big\{p(\mathcal{M}, G, \pi) \mid \pi \in \mathcal{O}_\mu(M, G)\big\},$$

where minimum is taken over all binary trees G connecting M.

Minimal Parametric Fillings as Linear Programming

Let $\mathcal{M} = (M, \rho)$ be a finite metric space connected by a (connected) graph $G = (V, E)$, and $\Omega(\mathcal{M}, G)$ be the set consisting of all the weight functions $\omega\colon E \to \mathbb{R}_+$ such that $\mathcal{G} = (G, \omega)$ is a filling of $\mathcal{M}$.

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The set $\Omega(\mathcal{M}, G) \subset \mathbb{R}^E$ is determined by the linear inequalities of two types: $\omega(e) \ge 0$, $e \in E$, and $\sum_{e\in\gamma_{pq}} \omega(e) \ge \rho(p, q)$, where γ_{pq} stands for the unique path in the tree G connecting the boundary vertices p and q. The weight functions of minimal parametric fillings correspond to minima points of the linear function $\sum_{e\in E} \omega(e)$ restricted to the set $\Omega(\mathcal{M}, G)$. Thus, *the problem of minimal parametric filling finding is a linear programming problem*.

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Assertion

Let $\Omega_m(\mathcal{M}, G)$ be the subset of $\Omega(\mathcal{M}, G)$ consisting of the weight functions such that $\mathcal{G}$ is a minimal parametric filling of $\mathcal{M}$. Then set $\Omega(\mathcal{M}, G)$ is closed and convex in the linear space $\mathbb{R}^E$ of all the functions on E, and $\Omega_m(\mathcal{M}, G) \subset \Omega(\mathcal{M}, G)$ is a nonempty convex compact.

General Linear Programming Problem (GLLP)

Consider $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Write vectors from $\mathbb{R}^n$ in the form $x = (x_1, x_2)$, where $x_i \in \mathbb{R}^{n_i}$. Let we be given with a linear function $F(x) = \langle f_1, x_1 \rangle + \langle f_2, x_2 \rangle$, where $f_i \in \mathbb{R}^{n_i}$.

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Problem: Find the least possible value of the *objective function* F on the subset $X \subset \mathbb{R}^n$, where X is defined by a system of linear equations and linear equalities as follows:

$$X = \{x = (x_1, x_2) \mid A_{11}x_1 + A_{12}x_2 \le b_1, \quad A_{21}x_1 + A_{22}x_2 = b_2, \quad x_1 \ge 0\},$$

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The set X is called the *solutions space*. If X is not empty, then X is a convex polyhedral subset of the space $\mathbb{R}^n$. In this case we put $F_* = \inf_{x \in X} F(x)$, and if F_* is finite, then $X_* = \{x \in X : F(x) = F_*\}$. The problem is called *solvable*, if X_* is non-empty. In this case each point $x_* \in X_*$ is called a *solution*.

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One can search the greatest possible value of F on X instead of the least one, and one problem can be reduced to the other by changing of the sign of the objective function F. The supremum of the values $F(x)$ over X is denoted by F^*, and if F^* is finite, then $X^* = \{x \in X : F(x) = F^*\}$.

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If CLPP is solvable, then its set of solutions X always contains so-called *angular* or *extreme points* of X. Moreover, in this case the set X_* is a convex polyhedral set all whose angular points is also angular points of X. From geometrical point of view, these angular points are vertices of the polyhedral set.

Canonical Linear Programming Problem (CLLP)

CLLP corresponds to the case $n_2 = m_1 = 0$, i.e., $X = \{x \in \mathbb{R}^n \mid x \geq 0, \quad Ax = b\}$.

If CLPP is solvable, then its set of solutions X always contains so-called *angular* or *extreme points* of X. Moreover, in this case the set X_* is a convex polyhedral set all whose angular points is also angular points of X. From geometrical point of view, these angular points are vertices of the polyhedral set.

Angular points can be easily described (here for brevity $n = n_1$, $b = b_2$, $A = A_{21}$). By A_j we denote the jth column of the matrix A.

Assertion

Let $r = m > 0$ be the rank of A. A point $x = (x^1, \dots, x^n) \in \mathbb{R}^n$ is angular, if and only if there exist linear independent columns $A_{j_1}, \dots, A_{j_r}$ of the matrix A, such that

$$A_{j_1} x^{j_1} + \cdots + A_{j_r} x^{j_r} = b,$$

and $x^{j_k} \geq 0$, $k = 1, \dots, r$, and the remaining x^j equal 0.

Duality in Linear Programming

The *dual problem* to GLLP is stated as follows: Find the greatest possible value of the linear function $H(\lambda) = -\langle b_1, \lambda_1 \rangle - \langle b_2, \lambda_2 \rangle$, where the variable vectors $\lambda_i \in \mathbb{R}^{m_i}$ form the vector $\lambda = (\lambda_1, \lambda_2)$ belonging to the polyhedral domain $\Lambda \subset \mathbb{R}^m$, $m = m_1 + m_2$, that is defined by the following system of linear equations and inequalities:

$$\Lambda = \big\{\lambda = (\lambda_1, \lambda_2) \mid A_{11}^T \lambda_1 + A_{21}^T \lambda_2 \le -f_1, \quad A_{12}^T \lambda_1 + A_{22}^T \lambda_2 = -f_2, \quad \lambda_1 \ge 0\big\}.$$

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As it is well-known, the dual problem to the dual problem is equivalent to the initial one, therefore usually one speaks about *mutual duality*.

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In accordance with *Duality Principle*, mutually dual problems of Linear Programming are solvable or non solvable simultaneously, and if the problems are solvable, then $F_* = H^*$, and $F(x) = F_* = H^* = H(\lambda)$, iff $x \in X_*$ and $\lambda \in \Lambda^*$.

Generalised Minimal Fillings Problem as GLPP

Let (M, ρ) be an arbitrary finite metric space, $M = \{p_1, \ldots, p_n\}$, and put $d_{ij} = \rho(p_i, p_j)$. Let $G = (V, E)$ be some tree connecting M. Describe weight functions $\omega\colon E \to \mathbb{R}$ that make the tree G to a generalized filling of the finite metric space (M, ρ).

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For each pair p_i, p_j of the boundary vertices there exists unique path $\gamma(i,j)$ in $G = (V, E)$ connecting them. By definition, a weighted tree (G, ω) is a generalized filling of (M, ρ), if and only if

$$\sum_{e \in \gamma(i,j)} \omega(e) \geq d_{ij}, \quad \text{for all } 1 \leq i < j \leq n. \tag{1}$$

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$$\sum_{e\in\gamma(i,j)}\omega(e)\ge d_{ij},\quad \text{for all } 1\le i<j\le n. \tag{1}$$

The weight of this filling equals $\sum_{e\in E}\omega(e)$ Thus, to find a minimal parametric filling of the type G, it is necessary to find the least value of the linear function $F(\omega)=\sum_{e\in E}\omega(e)$ on the convex polyhedral subset Ω_G of the space $\mathbb{R}^{|E|}$ defined by the system of linear inequalities (1).

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GLLP: Minimise $F(x) = \langle f_1, x_1\rangle + \langle f_2, x_2\rangle$ on $X \subset \mathbb{R}^n$, where

$$X = \{x = (x_1, x_2) \mid A_{11}x_1 + A_{12}x_2 \le b_1, \quad A_{21}x_1 + A_{22}x_2 = b_2, \quad x_1 \ge 0\},$$

where A_{ij} are fixed $(m_i \times n_j)$ matrices, and $b_i \in \mathbb{R}^{m_i}$.

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In our case $n_1 = 0$, and n_2 is equal to the number $|E|$ of edges of the tree G. The variables forming the vector x_2 are the variables $\omega(e)$. Put $E = \{e_1, \ldots, e_{|E|}\}$, $\omega(e_i) = \omega_i$. Further, all our constrains on the variables ω_i have the form of inequalities, therefore only the matrix A_{12} is non-zero.

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The rows of this matrix are enumerated by the ordered pairs (i, j), $1 \le i < j \le n$, where $|M| = n$. Therefore $m_1 = n(n-1)/2$. The columns of the matrix A_{12} correspond to edges of the tree. Let a_{ij}^k be the element of A_{12} standing at the row (i, j) at the place corresponding to the edge e_k. And $a_{ij}^k = 1$, iff the edge e_k is in the path $\gamma(i, j)$, otherwise $a_{ij}^k = 0$. Thus,

$$\Omega_G = \{x_2 \mid A_{12}x_2 \le b_1, \quad \text{where } x_2 = -(\omega_1, \dots, \omega_{|E|}), \text{ and } b_1 = -(d_{12}, \dots, d_{(n-1)\,n})\},$$

and we minimise the linear function $F(x) = \langle f_2, x_2\rangle$, $f_2 = -(1, \dots, 1)$, on Ω_G.

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Dual Problem: In our case $m_2 = 0$, the components λ_{ij} of λ_1 are enumerated by the pairs (i, j), $1 \le i < j \le n$. Maximise

$$H(\lambda) = H(\lambda_1) = -\langle b_1, \lambda_1 \rangle = \sum_{1 \le i < j \le n} d_{ij}\lambda_{ij}$$

on $\Lambda_G \subset \mathbb{R}^{m_1}$ defined as

$$A_{12}^T\lambda_1 = -f_2, \qquad \lambda_1 \ge 0. \tag{2}$$

Dual Problem to GMF LPP

Maximise

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The dual problem is a CLPP. The matrix A_{12}^T is an $(n_2 \times m_1)$ matrix. Its rows correspond to edges of the tree G, and its columns correspond to ordered pairs (i,j), $i<j$, i.e., to the edges of the complete graph $K(M)$ with the vertex set M.

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Let a_k^{ij} be the element of the kth row of the matrix A_{12}^T standing at the (i,j)th column. Let the edge e_k of the tree G generates the partition $\mathcal{P}_G(e_k)$ of the set M. The element a_k^{ij} equals 1, iff the vertices of the edge (i,j) of the graph $K(M)$ belong to distinct elements of the partition $\mathcal{P}_G(e_k)$.
All the components of the vector $-f_2$ equal 1.

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All the components of the vector $-f_2$ equal 1.

Assertion

The matrix A_{12}^T is the cut matrix of the graph $K(M)$ corresponding to the cuts family generated by the edges of the tree G connecting M.

Dual Problem to GMF LPP

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Let G be an arbitrary binary tree with a boundary M consisting of $n \geq 2$ vertices. Then the rank of the matrix $C(G)$ is maximal, and it equals $2n-3$.

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Let G be an arbitrary binary tree with the boundary M. Each multi-tour π corresponds to a cyclic walk c_π in the complete graph $K(M)$, whose consecutive edges have the form $\pi(j)\pi(j+1)$.

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By each multi-tour π of the tree G construct a vector $w^\pi \in \mathbb{R}^{m_1}$, $m_1 = n(n-1)/2$, whose component w_{ij}^π corresponding to an edge ij of the graph $K(M)$, $i < j$, is equal to the number of occurrences of the edge ij in the walk c_π.

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By each multi-tour π of the tree G construct a vector $w^\pi \in \mathbb{R}^{m_1}$, $m_1 = n(n-1)/2$, whose component w^π_{ij} corresponding to an edge ij of the graph $K(M)$, $i<j$, is equal to the number of occurrences of the edge ij in the walk c_π.

Assertion

Let G be an arbitrary binary tree with the boundary M, and let π be its arbitrary multi-tour of multiplicity k. Then the vector $\dfrac{1}{2k}w^\pi$ satisfies System (3), and hence, belongs to the solutions space Λ_G of the dual problem, and the function H at it equals the multi-perimeter of the multi-tour π.

Dual Problem to GMF LPP

Assertion

For any non-negative integer solution λ *of the equations system* $A_{12}^T\lambda_1 = -2kf_2$, *where* k *is a positive integer, there exists a multi-tour* π *of the tree* G *of multiplicity* k, *such that* $\lambda = w^\pi$.

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For any non-negative integer solution λ of the equations system $A_{12}^T\lambda_1 = -2kf_2$, where k is a positive integer, there exists a multi-tour π of the tree G of multiplicity k, such that $\lambda = w^\pi$.

The weight formula for minimal parametric filling obtained by A. Eremin permits to reduce the problem of minimal parametric filling finding to the search of a multi-tour of maximal multi-perimeter, and the multiplicity of such multi-tours is estimated from above by the value $(C_n^2)!$.

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Assertion

Let $\mathcal{M} = (M, \rho)$ be an arbitrary finite metric space, and let G be an arbitrary binary tree connecting M. Then $\mathrm{mf}_-(\mathcal{M}, G)$ is equal to the maximal value of $H = \sum d_{ij}\lambda_{ij}$ on the vertices of the polyhedron Λ_G of the dual problem defined by System (3). If $\lambda \in \Lambda^*$ is a vertex which the maximum is attained at, then $\mathrm{mf}_-(\mathcal{M}, G)$ equals multi-perimeter of the multi-tour that corresponds to the solution λ.

Dual Problem to GMF LPP

Assertion

Let a finite metric space M consists of $n \geq 3$ points, and let G be a binary tree connecting M. The weight of minimal parametric filling of the type G of the space M is attained at a multi-tour of the tree G, whose multiplicity does not exceed 2^{2n-5}.

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Remark: The idea of the proof is to estimate the determinant of $0, 1$ matrix. Apparently the best general estimate on the determinant of a matrix consisting of ones and zeros was obtained by Faddeev, Sominskii. We use a better special H. Bruhn, D Rautenbach estimate but it is not exact also. The sequence 2^{2n-5}, $n \geq 3$, starts as $2, 8, 32, 128, 512, \ldots$. But the formulas obtained by Ivanov, Tuzhilin and by Bednov together with the computational results of the present paper, see below, show that the sequence $\{k_n\}$ of maximal multiplicities of the multi-tours corresponding to the vertices of the polyhedrons Λ_G for the binary trees G with $n \geq 3$ boundary vertices starts as $1, 1, 1, 2, 2, \ldots$. An interesting algebraic problem appears: Is it possible to improve the estimate assuming that the trees under consideration are binary, and the paths connect boundary vertices only.

Four-Point Spaces

Let $n = 4$. Consider the unique binary tree with 4 vertices of degree 1, and let the vertices of its moustaches are the pairs 1, 2 and 3, 4.

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$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

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Then Λ is a straight segment in the 6-dimensional space with the ends (vertices)

$$\frac{1}{2}(1,0,1,1,0,1), \qquad \frac{1}{2}(1,1,0,0,1,1),$$

the values of the objective function at this vertices equal

$$\frac{1}{2}\big(d_{12} + d_{14} + d_{23} + d_{34}\big), \qquad \frac{1}{2}\big(d_{12} + d_{13} + d_{24} + d_{34}\big),$$

respectively, and the weight of the minimal parametric filling of this type equals to the maximum of these two values (two half-perimeters of the corresponding tours).

Five-Point Spaces

Let $n = 5$. Consider unique binary tree with 5 vertices of degree 1, and let the vertices of its moustaches are the pairs of the points of the space M having the numbers 1, 2 and 4, 5, respectively, and the remaining boundary vertex has the number 3.

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The solution space Λ is a 3-dimensional tetrahedron in 10-dimensional space with the vertices

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The values of the objective function at these vertices equal

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$$\frac{1}{2}(d_{12}+d_{15}+d_{23}+d_{34}+d_{45}), \quad \frac{1}{2}(d_{12}+d_{13}+d_{25}+d_{34}+d_{45}),$$
$$\frac{1}{2}(d_{12}+d_{14}+d_{23}+d_{35}+d_{45}), \quad \frac{1}{2}(d_{12}+d_{13}+d_{24}+d_{35}+d_{45}),$$

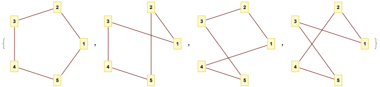

Figure: Tours of five-point spaces appearing in the weight formula.

Six-Point Spaces

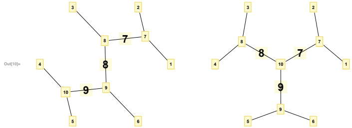

Figure: Binary trees with six boundary vertices.

Six-Point Spaces

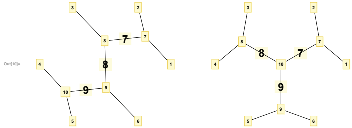

Figure: Binary trees with six boundary vertices.

At first we consider the tree with two moustaches. Then

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Six-Point Spaces

Λ is a 6-dim convex polyhedron in the 15-dim space with 8 vertices

$$\frac{1}{2}(1,0,0,0,1,1,0,0,0,1,0,0,1,0,1), \quad \frac{1}{2}(1,1,0,0,0,0,0,0,1,1,0,0,1,0,1),$$
$$\frac{1}{2}(1,0,1,0,0,1,0,0,0,0,0,1,1,0,1), \quad \frac{1}{2}(1,1,0,0,0,0,1,0,0,0,0,1,1,0,1),$$
$$\frac{1}{2}(1,0,0,1,0,1,0,0,0,1,0,0,0,1,1), \quad \frac{1}{2}(1,1,0,0,0,0,0,1,0,1,0,0,0,1,1),$$
$$\frac{1}{2}(1,0,1,0,0,1,0,0,0,0,1,0,0,1,1), \quad \frac{1}{2}(1,1,0,0,0,0,1,0,0,0,1,0,0,1,1).$$

Six-Point Spaces

Λ is a 6-dim convex polyhedron in the 15-dim space with 8 vertices

$$\frac{1}{2}(1,0,0,0,1,1,0,0,0,1,0,0,1,0,1), \quad \frac{1}{2}(1,1,0,0,0,0,0,0,1,1,0,0,1,0,1),$$
$$\frac{1}{2}(1,0,1,0,0,1,0,0,0,0,0,1,1,0,1), \quad \frac{1}{2}(1,1,0,0,0,0,1,0,0,0,0,1,1,0,1),$$
$$\frac{1}{2}(1,0,0,1,0,1,0,0,0,1,0,0,0,1,1), \quad \frac{1}{2}(1,1,0,0,0,0,0,1,0,1,0,0,0,1,1),$$
$$\frac{1}{2}(1,0,1,0,0,1,0,0,0,0,1,0,0,1,1), \quad \frac{1}{2}(1,1,0,0,0,0,1,0,0,0,1,0,0,1,1).$$

The values of the objective function at this vertices are equal to

$$\frac{1}{2}(d_{12}+d_{16}+d_{23}+d_{34}+d_{45}+d_{56}), \quad \frac{1}{2}(d_{12}+d_{13}+d_{26}+d_{34}+d_{45}+d_{56}),$$
$$\frac{1}{2}(d_{12}+d_{14}+d_{23}+d_{36}+d_{45}+d_{56}), \quad \frac{1}{2}(d_{12}+d_{13}+d_{24}+d_{36}+d_{45}+d_{56}),$$
$$\frac{1}{2}(d_{12}+d_{15}+d_{23}+d_{34}+d_{46}+d_{56}), \quad \frac{1}{2}(d_{12}+d_{13}+d_{25}+d_{34}+d_{46}+d_{56}),$$
$$\frac{1}{2}(d_{12}+d_{14}+d_{23}+d_{35}+d_{46}+d_{56}), \quad \frac{1}{2}(d_{12}+d_{13}+d_{24}+d_{35}+d_{46}+d_{56}),$$

Six-Point Spaces

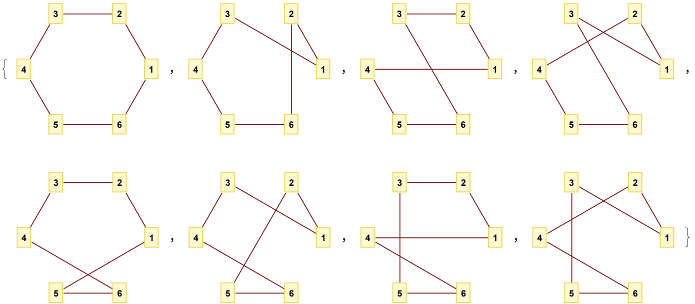

Figure: Tours of the binary tree with six boundary vertices and two moustaches that appear in the formula of the weight of minimal parametric filling of this type.

Six-Point Spaces

Let us pass to the case of the tree with three moustaches. We have:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Six-Point Spaces

The solution space Λ is 6-dim convex polyhedron in the 15-dim space. It has 12 vertices, whose coordinates are

$$\frac{1}{2}(1,0,0,0,1,0,1,0,0,1,1,0,0,0,1), \quad \frac{1}{2}(1,0,1,0,0,0,0,0,1,1,1,0,0,0,1),$$
$$\frac{1}{2}(1,0,0,1,0,0,1,0,0,1,0,1,0,0,1), \quad \frac{1}{2}(1,0,1,0,0,0,0,1,0,1,0,1,0,0,1),$$
$$\frac{1}{4}(2,1,0,0,1,0,1,1,0,2,0,1,1,0,2), \quad \frac{1}{4}(2,0,1,1,0,1,0,0,1,2,0,1,1,0,2),$$
$$\frac{1}{2}(1,0,0,0,1,1,0,0,0,1,0,0,1,0,1), \quad \frac{1}{2}(1,1,0,0,0,0,0,0,1,1,0,0,1,0,1),$$
$$\frac{1}{4}(2,0,1,0,1,1,0,1,0,2,1,0,0,1,2), \quad \frac{1}{4}(2,1,0,1,0,0,1,0,1,2,1,0,0,1,2),$$
$$\frac{1}{2}(1,0,0,1,0,1,0,0,0,1,0,0,0,1,1), \quad \frac{1}{2}(1,1,0,0,0,0,0,1,0,1,0,0,0,1,1).$$

Six-Point Spaces

The values of the objective function at these vertices are

$$\frac{1}{2}(d_{12}+d_{16}+d_{24}+d_{34}+d_{35}+d_{56}), \quad \frac{1}{2}(d_{12}+d_{14}+d_{26}+d_{34}+d_{35}+d_{56}),$$
$$\frac{1}{2}(d_{12}+d_{15}+d_{24}+d_{34}+d_{36}+d_{56}), \quad \frac{1}{2}(d_{12}+d_{14}+d_{25}+d_{34}+d_{36}+d_{56}),$$
$$\frac{1}{4}(2d_{12}+d_{13}+d_{16}+d_{24}+d_{25}+2d_{34}+d_{36}+d_{45}+2d_{56}),$$
$$\frac{1}{4}(2d_{12}+d_{14}+d_{15}+d_{23}+d_{26}+2d_{34}+d_{36}+d_{45}+2d_{56}),$$
$$\frac{1}{2}(d_{12}+d_{16}+d_{23}+d_{34}+d_{45}+d_{56}), \quad \frac{1}{2}(d_{12}+d_{13}+d_{26}+d_{34}+d_{45}+d_{56}),$$
$$\frac{1}{4}(2d_{12}+d_{14}+d_{16}+d_{23}+d_{25}+2d_{34}+d_{35}+d_{46}+2d_{56}),$$
$$\frac{1}{4}(2d_{12}+d_{13}+d_{15}+d_{24}+d_{26}+2d_{34}+d_{35}+d_{45}+2d_{56}),$$
$$\frac{1}{2}(d_{12}+d_{15}+d_{23}+d_{34}+d_{46}+d_{56}), \quad \frac{1}{2}(d_{12}+d_{13}+d_{25}+d_{34}+d_{46}+d_{56}).$$

Notice four vertices, such that the common denominator of their coordinates equals 4. They correspond to multi-tours of multiplicity two.

Six-Point Spaces

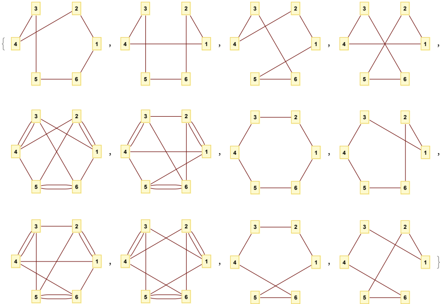

Figure: Multi-tours of the binary tree with six boundary vertices and three moustaches that appear in the formula of the weight of minimal parametric filling of this type.

Six-Point Spaces

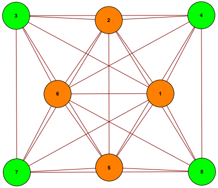

Figure: Graph of the polyhedron for the 6-snake. Degrees are 6 and 7.

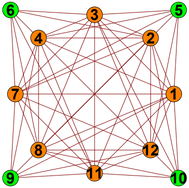

Figure: Graph of the polyhedron for the 6-T-joint. Degrees are 6 and 10.

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Thank you for your Attention!