Linear maps
preserving matrix invariants

Alexander Guterman

Moscow State University

Dedekind, 1880

G is a group, $|G| = n < \infty$

G	x_1	$\dots$	x_i	$\dots$	x_n
x_n			$\vdots$		
$\vdots$			$\vdots$		
x_j	$\cdots$	$\cdots$	$x_k = x_i \cdot x_j$		
$\vdots$					
x_1					

Cayley

table K_G

$P = \det(K_G)$ is homogeneous, $\deg P = n$.

Theorem. G is abelian $\Rightarrow$

$$\det K_G = (a_1^1 x_1 + \ldots + a_n^1 x_n) \cdots (a_1^n x_1 + \ldots + a_n^n x_n)$$

$G = (\mathbb{Z}_3, +)$

Cayley table

G		0 x	1 y	2 z
2	z	z	x	y
1	y	y	z	x
0	x	x	y	z

$$\det(K_{\mathbb{Z}_3}) = x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x+\varepsilon y+\varepsilon^2 z)(x+\varepsilon^2 y+\varepsilon z)$$

$\varepsilon = e^{\frac{2\pi}{3}i}$

Character table

	(0)	(1)	(2)
χ_1	1	1	1
χ_2	1	ε	ε^2
χ_3	1	ε^2	ε

The noncommutative case

1. Dedekind: S_3, $\mathbb{Q}_8$
2. Frobenius, 1896: G is ANY finite group:

Theorem. $\det(K_G) = P_1^{n_1} \cdots P_k^{n_k}$, P_i is irreducible, $\deg(P_i) = n_i$, $i = 1, \ldots, k$.

$$\chi_j(x_i) = \frac{\partial P_i}{\partial x_j}(0, \ldots, 0, 1, 0, \ldots, 0)$$

j-th position

Theorem. [Frobenius, 1896]

$$T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$$

— linear, bijective

$$\det(T(A)) = \det A \qquad \forall A \in M_n(\mathbb{C})$$

$$\Downarrow$$

$$\exists P, Q \in GL_n(\mathbb{C}), \det(PQ) = 1 :$$

$$T(A) = PAQ \quad \forall A \in M_n(\mathbb{C})$$

or

$$T(A) = PA^tQ \quad \forall A \in M_n(\mathbb{C})$$

Definition $T : M_{m\,n}(\mathbb{F}) \to M_{m\,n}(\mathbb{F})$ is standard iff

$\exists P \in GL_m(\mathbb{F}),\ Q \in GL_n(\mathbb{F})$:

$$T(A) = PAQ \quad \forall A \in M_{m,n}(\mathbb{F})$$

or $m = n$ and

$$T(A) = PA^tQ \quad \forall A \in M_{m,n}(\mathbb{F})$$

Let $X \in M_{m,n}(\mathbb{C})$. Then $C_r(X) \in M_{\binom{m}{r},\binom{n}{r}}(\mathbb{C})$ consists from r-minors of X ordered lexicographically by rows and columns.

Theorem. [Schur, 1925] Let $T : M_{m\,n}\mathbb{C}) \to M_{m\,n}(\mathbb{C})$ be bijective and linear, r, $2 \le r \le \min\{m, n\}$, be given. $\exists$ bijective linear $S : M_{\binom{m}{r},\binom{n}{r}}(\mathbb{C}) \to M_{\binom{m}{r},\binom{n}{r}}(\mathbb{C})$ s.t.

$$C_r(T(X)) = S(C_r(X)) \ \ \forall \ \in M_{m,n}(\mathbb{C})$$

iff T is standard.

Theorem. [Dieudonné, 1949]

$\Omega_n(\mathbb{F})$ is the set of singular matrices

$T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ — linear, bijective, $T(\Omega_n(\mathbb{F})) \subseteq \Omega_n(\mathbb{F})$

$$\Downarrow$$

$$\exists P, Q \in GL_n(\mathbb{F})$$

$$T(A) = PAQ \quad \forall A \in M_n(\mathbb{F})$$

or

$$T(A) = PA^tQ \quad \forall A \in M_n(\mathbb{F})$$

E.B. Dynkin, Maximal subgroups of classical groups // The Proceedings of the Moscow Mathematical Society, **1** (1952) 39-166.

$$St_n(\mathbb{F}) \subseteq Fix(S) \subseteq GL_{n^2}(\mathbb{F})$$

The quantity of Linear Preservers for a given matrix invariant is a measure of its complexity. Indeed, to compute the invariant for a given matrix, we reduce it to a certain good form, where computations are easy.

$$\det(A) = \sum_{\sigma \in S_n} (-1)^n a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

- Computations of det require $\sim O(n^3)$ operations

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

- Computations of per require

$\sim (n-1) \cdot (2^n - 1)$ multiplicative operations (Raiser formula).

The explanation:

There are just few linear preservers of permanent in comparison with the determinant. Indeed,

Theorem. [Marcus, May] Linear transformation T is permanent preserver iff

$T(A) = P_1 D_1 A D_2 P_2 \quad \forall A \in M_n(\mathbb{F})$, or

$T(A) = P_1 D_1 A^t D_2 P_2 \quad \forall A \in M_n(\mathbb{F})$

here D_i are invertible diagonal matrices, $i = 1, 2, \det(D_1 D_2) = 1$

P_i are permutation matrices, $i = 1, 2$

• Central simple algebras

Let A be a central simple algebra of dimension $k = n^2$ over $\mathbb{F}$.

Definition The norm $N(a)$ of an element $a \in A$ is the determinant of the left multiplication operator $x \to ax$

Example $A = M_n(\mathbb{F})$. Then $N(a) = \left(\det(a)\right)^n$.

Does the norm determines a central simple algebra up to an automorphism?

Indeed, the norms of central simple algebras are equivalent iff these algebras are either isomorphic or anti-isomorphic.

The proof is based on the Frobenius theorem.

- Group theory

Question Is it possible that two non-isomorphic finite groups have the same group determinant?

Theorem. [E. Formanek, D. Sibley] A group determinant determines the group up to an automorphism

Proof is based on an extension of Dieudonne singularity preserver theorem to the direct products of matrix algebras.

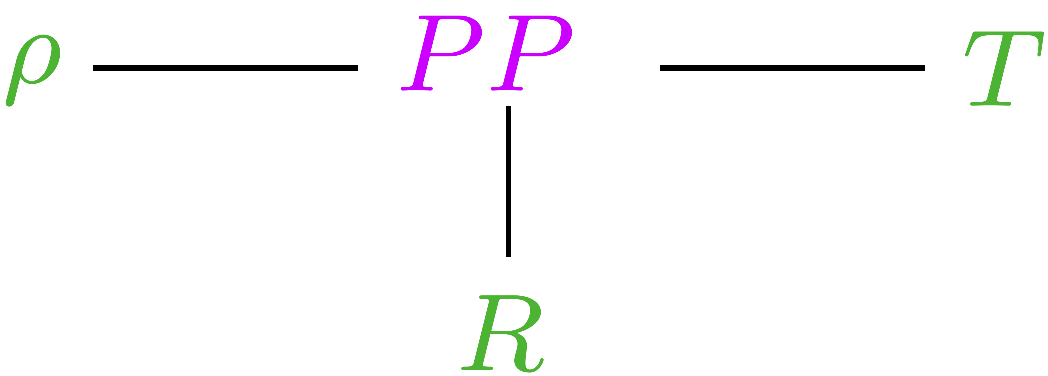
Preserve Problems

$\rho : M_n(R) \to S$ is a certain matrix invariant

$$T : M_n(R) \to M_n(R)$$

$$\rho(T(A)) = \rho(A) \quad \forall A \in M_n(R)$$

$$T = ?$$



Let $\mathbb{F}$ be a field

$\emptyset \neq S \subseteq M_n(\mathbb{F})$	$T(S) \subseteq S$
$\rho : M_n(\mathbb{F}) \to \mathbb{F}$ $\forall A \in M_n(\mathbb{F})$	$\rho(T(A)) = \rho(A)$
$\sim: M_n(\mathbb{F})^2 \to \{0, 1\}$	$A \sim B \Rightarrow T(A) \sim T(B)$ $\forall A, B \in M_n(\mathbb{F})$
P – property in $M_n(\mathbb{F})$	$A \in P \Rightarrow T(A) \in P$

$$T = ?$$

The standard solution in linear case

There are $P, Q \in GL_n(\mathbb{F})$:

$$T(X) = PXQ \quad \forall X \in M_n(\mathbb{F})$$

or

$$T(X) = PXQ \quad \forall X^t \in M_n(\mathbb{F})$$

Basic methods to investigate PPs

1. Matrix theory
2. Theory of classical groups
3. Projective geometry
4. Algebraic geometry
5. Differential geometry
6. Dualisations
7. Tensor calculus
8. Functional identities
9. Model theory

Matrices of finite order

Theorem. (S. Pierce)

$\text{char}\,(\mathbb{F}) = 0$, $T : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ is bijective, linear, preserves zeros of $p(x) = x^k - 1$. Then $\exists\ S \in GL_n(\mathbb{F})$:

$T(X) = \alpha S X S^{-1} \quad \forall X \in \mathcal{M}_n(\mathbb{F})$

or $T(X) = \alpha S X^t S^{-1} \quad \forall X \in \mathcal{M}_n(\mathbb{F})$ α is a root of 1 of degree k in $\mathbb{F}$.

Idempotent matrices

Theorem. [L.B. Beasley]

$T : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ is bijective, linear, preserves zeros of $p(x) = x^2 - x$. Then $\exists$ $S \in GL_n(\mathbb{F})$:

$T(X) = SXS^{-1} \quad \forall X \in \mathcal{M}_n(\mathbb{F})$

or $T(X) = SX^tS^{-1} \quad \forall X \in \mathcal{M}_n(\mathbb{F})$.

Nilpotent matrices

Theorem. [P. Botta]

$T : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ is linear, bijective, preserves zeros of $p(x) = x^k$. Then $\exists$ $S \in GL_n(\mathbb{F})$, $B \in \mathcal{M}_n(\mathbb{F})$, $\alpha \in \mathbb{F}$:

$T(X) = \alpha S X S^{-1} + \mathrm{tr}\,(X)\, B \quad \forall X \in \mathcal{M}_n(\mathbb{F})$

or $T(X) = \alpha S X^t S^{-1} + \mathrm{tr}\,(X)\, B \quad \forall X \in \mathcal{M}_n(\mathbb{F})$.

Definition First order sentences in the language of fields are those mathematical statements which can be written down using only

(a) Variables denoted by $x, y, \ldots$ varying over the elements of the field;
(b) The distinguished elements "0" and "1";
(c) The quantifiers "for all" $(\forall)$ and "there exists" $(\exists)$;
(d) The relation symbol "$=$";
(e) The function symbols "$+$" and "$\cdot$";
(f) Logical connectives: $\neg$ (negation), $\wedge$ (and), $\vee$ (or), $\rightarrow$ (implies), and $\leftrightarrow$ (equivalent).
(g) The separation symbols: left square bracket "[" and right square bracket "]".

Definition Two fields $\mathbb{F}_1$ and $\mathbb{F}_2$ are elementarily equivalent if and only if the set of all first order statements that are true in $\mathbb{F}_1$ is the same as the set of all first order statements that are true in $\mathbb{F}_2$.

Theorem. [transfer principle] Two algebraically closed fields $\mathbb{F}_1$ and $\mathbb{F}_2$ are elementarily equivalent if and only if $char(\mathbb{F}_1) = char(\mathbb{F}_2)$. Consequently, if a first order property holds in one algebraically closed field it holds in each algebraically closed field of the same characteristic.

Definition $A \in M_n(\mathbb{F})$ is of finite order if $\exists$ integer $k > 0$: $A^k = I$.

Not first order condition since k is unbounded.

Definition $A \in M_n(\mathbb{F})$ is nilpotent if $\exists$ integer $k > 0$: $A^k = 0$.

First order condition since $k \leq n$ from LA.

Hence,

Theorem. Let $\mathbb{F}$ be an algebraically closed field of 0 characteristic, $T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be a bijective linear map. T preserves the set of nilpotent matrices if and only if $\exists\ 0 \neq c \in \mathbb{F}$ and $P, B \in M_n(\mathbb{F})$ with P invertible such that T is of the form

$$X \mapsto cPXP^{-1} + (\operatorname{tr} X)B \quad \text{or} \quad X \mapsto cPX^tP^{-1} + (\operatorname{tr} X)B.$$

Theorem. [Howard, 1980]

$\mathbb{F}$ is a.c., char $(\mathbb{F}) = 0$, $n \geq 3$,

$$T : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$$

is bijective linear, $p(x) \in \mathbb{F}[x]$ has at least 2 different roots.

Then $\exists$ $S \in GL_n(\mathbb{F})$:

$T(X) = SXS^{-1} \quad \forall X \in \mathcal{M}_n(\mathbb{F})$

or $T(X) = SX^tS^{-1} \quad \forall X \in \mathcal{M}_n(\mathbb{F})$

or, if $\exists$ $k \geq 2$, $l \geq 0$, $g(x) \in \mathbb{F}[x]$: $f(x) = x^l g(x^k)$,

$$T(X) = \alpha SXS^{-1} \text{ or } T(X) = \alpha SX^tS^{-1},$$

where α is a root of 1 of degree k in $\mathbb{F}$.

Theorem. [Watkins, 1976]

$\mathbb{F}$ — a.c., char $(\mathbb{F}) = 0$, $n \geq 3$, $T : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ is linear, strongly preserves zeros of $p(x, y) = xy - yx$. Then

either Im (T) is a commutative subspace in $\mathcal{M}_n(\mathbb{F})$,

or $\exists$ $S \in GL_n(\mathbb{F})$, $f : \mathcal{M}_n(\mathbb{F}) \to \mathbb{F}$, $\alpha \in \mathbb{F}$:

$$T(X) = \alpha S X S^{-1} + f(X)\, I \quad \forall X \in \mathcal{M}_n(\mathbb{F})$$

or

$$T(X) = \alpha S X^t S^{-1} + f(X)\, I \quad \forall X \in \mathcal{M}_n(\mathbb{F})$$

Theorem. [Wong]

$T : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ linear, bijective, preserves zeros of $p(x) = xy$.

Then $\exists$ $S \in GL_n(\mathbb{F})$, $\alpha \in \mathbb{F}$, $\alpha \neq 0$:

$$T(X) = \alpha S X S^{-1} \quad \forall X \in \mathcal{M}_n(\mathbb{F}).$$

Theorem. [Chebotar]

char $(\mathbb{F}) \neq 2, 3$, $n \geq 20$, $T : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ bijective, linear, preserves zeros of $p(x) = xy - yx^*$.

Then $\exists$ $S \in GL_n(\mathbb{F})$, $SS^* = S^*S$, $\alpha \in \mathbb{F}$, $\alpha \neq 0$, $\alpha = \alpha^*$:

$$T(X) = \alpha SXS^{-1} \quad \forall X \in \mathcal{M}_n(\mathbb{F}).$$

(Functional identities)

Monotone transformations

Minus order relation

Let $\mathcal{S}$ be a semigroup, $\mathcal{I}(\mathcal{S})$ be the set of idempotents in $\mathcal{S}$.

Wagner order on $\mathcal{I}(\mathcal{S})$: let $f, e \in \mathcal{I}(\mathcal{S})$.

Then $e \bar{\leqslant} f$ iff $ef = fe = e$.

$a \in \mathcal{S}$ is *(von Neumann) regular* in $\mathcal{S}$ if $a \in a\mathcal{S}a$. A solution of $axa = a$ is called an *inner inverse* and is denoted by a^-.

Hartwig-Nambooripad order on regular elements: let $a, b \in \mathcal{S}$ be regular. Then $a \overset{-}{\leqslant} b$ iff $\exists\ a^-$:

$aa^- = ba^-$ and $a^-a = a^-b$.

Can we tackle this order using matricial tools on $M_n(\mathbb{F})$?

Rank-subtractivity: $A, B \in M_n(\mathbb{F})$.

Then $A \overset{-}{\leqslant} B$ iff $\mathsf{rk}\,(B - A) = \mathsf{rk}\,B - \mathsf{rk}\,A$.

Let $\mathcal{S}$ be a semigroup.

Definition Involution $*$ on $\mathcal{S}$ is a bijection $a \to a^*$ $\forall a \in \mathcal{S}$:

1) $(a^*)^* = a$,

2) $(ab)^* = b^* a^* \quad \forall a, b \in \mathcal{S}$.

$*$ is a proper involution if

$$\underbrace{a^* a = a^* b = b^* b = b^* a}$$

$$\Downarrow$$

$$a = b$$

We consider only semigroup with the proper involution, $*$-semigrou

Examples: Boolean rings, groups, proper $*$-rings, in particular, $M_n(R)$, $M_n(\mathbb{C})$.

Definition For $a, b \in \mathcal{S}$ a Drazin Star Partial Order is the following relation:

$$a \overset{*}{\leqslant} b \quad \text{iff} \quad \begin{cases} a^*a = a^*b \\ aa^* = ab^* \end{cases}$$

Theorem. [M.P. Drazin] *If $\mathcal{S}$ is a proper $*$-semigroup then*

$$\overset{*}{\leqslant} \quad \textit{is} \quad \begin{cases} reflexive \\ anti-symmetric \\ transitive \end{cases}$$

Matrix partial orderings are important due to their statistical applications, $\mathcal{S} = M_n(\mathbb{F})$

Let $M_n(\mathcal{S})(A)$ denotes the linear span of columns of a matrix $A \in M_{m\,n}(\mathbb{F})$.

Left $*$-order and right $*$-order:

Definition [J. Baksalary, S. Mitra, LAA, 1991] For $A, B \in M_{m\,n}(\mathbb{C})$ we say that $A * \leqslant B$ iff $A^*A = A^*B$ and $M_n(\mathcal{S})(A) \subseteq M_n(\mathcal{S})(B)$.

Definition [J. Baksalary, S. Mitra] For $A, B \in M_{m\,n}(\mathbb{C})$ we say that $A \leqslant * B$ iff $AA^* = BA^*$ and $M_n(\mathcal{S})(A^*) \subseteq M_n(\mathcal{S})(B^*)$.

Definition [J. Baksalary, J. Hauke] For $A, B \in M_{m\,n}(\mathbb{F})$ we say that $A \overset{\diamond}{\leqslant} B$, iff

$$\begin{cases} \text{Im}\,(\mathbf{A}) \subseteq \text{Im}\,(\mathbf{B}) \\ \text{Im}\,(\mathbf{A}^*) \subseteq \text{Im}\,(\mathbf{B}^*) \\ \mathbf{A}\mathbf{A}^*\mathbf{A} = \mathbf{A}\mathbf{B}^*\mathbf{A} \end{cases}$$

This relation is called a diamond order.

$(\mathcal{S}, \overset{*}{<})$ is a partial ordered structure

Problem

What are the morphisms of this ordered structure that are monotone?

$$T : \mathcal{S} \to \mathcal{S}$$

$$\forall a, b \in \mathcal{S}, \quad a \overset{*}{<} b \Rightarrow T(a) \overset{*}{<} T(b)$$

Below $\mathcal{S} = M_n(\mathbb{F})$, $\mathbb{F}$ is a field,

$$T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$$

Definition

$$T : M_{m,n}(\mathbb{F}) \to M_{m,n}(\mathbb{F})$$

preserves the order $<$ (or, T is monotone wrt $<$), if

$$A < B \Rightarrow T(A) < T(B)$$

Definition

$$T : M_{m,n}(\mathbb{F}) \to M_{m,n}(\mathbb{F})$$

strongly preserves the order $<$ (strongly monotone wrt $<$), if

$$A < B \Leftrightarrow T(A) < T(B)$$

P. G. Ovchinnikov:

Theorem. Let H be a Hilbert space, $\dim H \geq 3$, $B(H)$ be the algebra of bounded linear operators on H, $T : \mathcal{I}(B(H)) \to \mathcal{I}(B(H))$ be a poset automorphism. Then either $T(P) = APA^{-1}$ $\forall\ P \in \mathcal{I}(B(H))$ or $T(P) = AP^*A^{-1}$ $\forall\ P \in \mathcal{I}(B(H))$. Here A is a semi-linear bijection $H \to H$ if $\dim H < \infty$, and continuous invertible linear or conjugate linear operator, otherwise.

P. G. Ovchinnikov:

Corollary $\mathcal{P}$ is the set of idempotents in $M_n(\mathbb{C})$, $n \geq 3$. $T : \mathcal{P} \to \mathcal{P}$ is a bijection strongly monotone wrt $\bar{\leqslant}$. Then $\exists$ a semi-linear bijection $L : \mathbb{C}^n \to \mathbb{C}^n$ such that

$$T(X) = LXL^{-1} \text{ or } T(X) = LX^*L^{-1}$$

The questions arising

- Can we work with the transformation on the whole $M_n(\mathbb{F})$?

- Can we classify just monotone transformations, which are not strongly monotone?

- Can we work with some other order relations?

Linear case

Matrix deformation approach

[Guterman]

Definition For a given binary matrix relation

$$\sim\ : M_n(\mathbb{F}) \times M_n(\mathbb{F}) \to \{0, 1\}$$

we consider a deformation which is a subset

$$L_{\mathbb{F}}(\sim) \subseteq M_n(\mathbb{F}),$$

$$L_{\mathbb{F}}(\sim) := \{X \in M_n(\mathbb{F}) | \exists 0 \neq R, S \in M_n(\mathbb{F}) : \\ \forall \lambda \in \mathbb{F} \qquad R \sim (\lambda X + S)\}.$$

WHY DO WE NEED THIS NOTION?

The properties

Lemma. $\sim_1, \sim_2$ are binary relations on $M_n(\mathbb{F})$ and for all $A, B \in M_n(\mathbb{F})$

$$A \sim_1 B \Rightarrow A \sim_2 B$$

Then $L_{\mathbb{F}}(\sim_1) \subseteq L_{\mathbb{F}}(\sim_2)$.

Lemma. $T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is linear and bijective;

T preserves $\sim$

$(\forall A, B \in M_n(\mathbb{F})$ if $A \sim B$ then $T(A) \sim T(B))$

Then

$$T(L_{\mathbb{F}}(\sim)) \subseteq L_{\mathbb{F}}(\sim)$$

.

Why $L_{\mathbb{F}}(\sim)$ is better than $\sim$?

Theorem. $\mathbb{F}$ is a field of complex or real numbers. Then $\Omega_n(\mathbb{F}) \subseteq L_{\mathbb{F}}(\overset{*}{<})$.

the set of singular matrices

Proof. Based on the properties of the singular value decomposition.

Definition [R. Hartwig, K. Nambooripad]

The Minus-order: $A \stackrel{-}{\leqslant} B$ if $\operatorname{rk}(B-A) = \operatorname{rk} B - \operatorname{rk} A$.

Corollary There is a following set inclusion:

$$\Omega_n(\mathbb{F}) \subseteq L_{\mathbb{F}}(\stackrel{*}{<}) \subseteq L_{\mathbb{F}}(\stackrel{-}{\leqslant}) \subseteq \Omega_n(\mathbb{F})$$

Theorem 19

Lemma 1

direct computations

$$\Downarrow$$

$$L_{\mathbb{F}}(\stackrel{*}{<}) = \Omega_n(\mathbb{F})$$

Proposition. Let $T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ be a linear and bijective transformation which is monotone with respect to the Drazin star partial order. Then T is a singularity preserver i.e., $T(\Omega_n(\mathbb{F})) \subseteq \Omega_n(\mathbb{F})$.

Corollary $\left(\text{Proposition} + \begin{array}{l}\text{Dieudonné}\\ \text{Theorem}\end{array}\right)$

All linear maps which are monotone w.r.t. the Drazin star partial order are standard!

What are the standard linear transformations which leave the star-order invariant?

Theorem. Bijective linear $T : M_{mn}(\mathbb{F}) \to M_{mn}(\mathbb{F})$ monotone w.r.t. $\overset{*}{\leqslant}$ is of the form

$T(X) = \alpha PXQ$ or,

if $m = n$, $T(X) = \alpha PX^tQ,$

$P, Q \in GL_n(\mathbb{F})$ are unitary, $\alpha \in \mathbb{F}^*$.

Definition [J. Baksalary, J. Hauke] Let $A, B \in M_{m\,n}(\mathbb{F})$ we say that $A \overset{\sigma}{\leqslant} B$, if $A \bar{\lessgtr} B$ and $\sigma(A) \subseteq \sigma(B)$.

Definition [J. Gross]

For $A, B \in M_{m\,n}(\mathbb{F})$ it is said that $A \overset{\sigma_1}{\leqslant} B$, if

$$A \bar{\lessgtr} B \quad \text{and} \quad \sigma_1(A) \le \sigma_1(B).$$

Here $\sigma(A)$ and $\sigma_1(A)$ denote nonzero singular values (the square roots of the eigenvalues of AA^*) and, respectively, maximal singular value of complex or real matrices.

Bijective monotone maps

P. Šemrl:

Theorem. $\mathcal{P}_n \subset M_n(\mathbb{F})$ is a set of all idempotents. $|\mathbb{F}| \geq 3$, $n \geq 3$,

$$T : \mathcal{P}_n \to \mathcal{P}_n$$

is a bijection monotone wrt $\bar{\leqslant}$. Then $\exists \varphi : \mathbb{F} \to \mathbb{F}$ — automorphism and $A \in GL_n(\mathbb{F})$:

$$T(X) = AX^{\varphi}A^{-1} \quad \forall X \in \mathcal{P}$$

or

$$T(X) = A(X^{\varphi})^t A^{-1} \quad \forall X \in \mathcal{P}$$

$X^{\varphi} = [\varphi(x_{ij})]$ for $X = [x_{ij}]$

- Can a semigroup became a group ?

- Does bijectivity follow from monotonicity?

- What happens in the non-linear case?

Additive monotone maps

Definition $\preceq_1$ on $M_{m\,n}(\mathbb{F})$ is weaker than $\preceq_2$, if for all $A, B \in M_{m\,n}(\mathbb{F})$

$$A \preceq_2 B \Rightarrow A \preceq_1 B.$$

In this case $\preceq_2$ is stronger than $\preceq_1$.

Examples.

$$\overset{*}{\leqslant} \;\Rightarrow\; *\!\leqslant,\ \leqslant\!*$$

$$\overset{*}{\leqslant} \;\Rightarrow\; \overset{\sigma_1}{\leqslant},\ \overset{\sigma}{\leqslant}$$

$$\overset{*}{\leqslant} \;\Rightarrow\; \overset{\diamond}{\leqslant}$$

$$*\!\leqslant,\ \leqslant\!* \;\Rightarrow\; \bar{\leqslant}$$

$$\overset{\sigma_1}{\leqslant},\ \overset{\sigma}{\leqslant} \;\Rightarrow\; \bar{\leqslant}$$

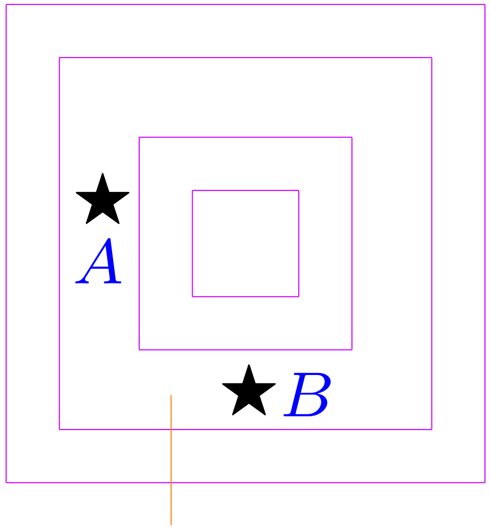
$$\overset{\diamond}{\leqslant} \;\not\Rightarrow\; \bar{\leqslant}$$

$$\bar{\leqslant} \;\not\Rightarrow\; \overset{\diamond}{\leqslant}$$

Definition A partial order $\preceq$ on $M_{m\,n}(\mathbb{F})$ is called unitary invariant, if for arbitrary matrices $A, B \in M_{m\,n}(\mathbb{F})$ the inequality $A \preceq B$ is equivalent to $UAV \preceq UBV$ for all $U \in U_n(\mathbb{F})$, $V \in U_m(\mathbb{F})$.

Examples. All aforesaid order relations are unitary invariant.

The partial order relations on $M_{m\,n}(\mathbb{F})$, we have defined, behave well with respect to the rank function on matrices, namely:



r-th component which consists of matrices of the fixed rank equal to r

$\forall A, B \in M_{m\,n}(\mathbb{F})$

(i) if $A \preceq B$, then $\operatorname{rk} A \leq \operatorname{rk} B$;

(ii) if $A \preceq B$ and $\operatorname{rk} A = \operatorname{rk} B$, then $A = B$.

Definition We say that an order relation $\preceq$ on $M_{m\,n}(\mathbb{F})$ is regular, if it satisfies (i), (ii) and also

(iii) $\preceq$ is unitary invariant

(iv) $\preceq$ is weaker than Drazin order

Regular orders and corresponding monotone transformations

Let T be fixed.

We find and fix some matrix $Z \in M_{m,n}(\mathbb{F})$ such that the following two conditions hold simultaneously:

a) $\operatorname{rk} Z = 1$ and

b) for all $X \in M_{m,n}(\mathbb{F})$, which satisfy the condition $\operatorname{rk} X = 1$, we have

$$\operatorname{rk} T(X) \leq \operatorname{rk} T(Z).$$

Let $Z = \zeta U_Z E_{1,1} V_Z$ be a singular value decomposition of Z.

We define $\widehat{T}_Z : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ by

$$\widehat{T}_Z(X) = T(\zeta U_Z X V_Z) \text{ for all } X \in M_n(\mathbb{F})$$

Then

a) $\forall A$, $\operatorname{rk} A = 1 \Rightarrow \operatorname{rk} \widehat{T}_Z(A) \leq \operatorname{rk} \widehat{T}_Z(E_{1,1})$.

b) $\widehat{T}_Z$ is additive and monotone with respect to the order $\preceq$.

Theorem. [Alieva, Guterman] Let $\preceq$ be a regular partial order relation on $M_{m\,n}(\mathbb{F})$. Assume that

$$T : M_{m\,n}(\mathbb{F}) \to M_{m\,n}(\mathbb{F})$$

be an additive monotone map with respect to order $\preceq$. Then T has one of the following forms:

1) $T(X) = PX^{\varphi}Q$ for all $X \in M_{m,n}(\mathbb{F})$,

2) (if $m = n$) $T(X) = P(X^{\varphi})^{t}Q$ for all $X \in M_n(\mathbb{F})$,

3) $T(X) = 0$ for all $X \in M_{m\,n}(\mathbb{F})$,

here $\varphi : \mathbb{F} \to \mathbb{F}$ is a field endomorphism, $X^{\varphi} = [\varphi(x_{i,j})]$, where $X = [x_{i,j}]$,

$P \in GL_m(\mathbb{F}), Q \in GL_n(\mathbb{F})$.

Corollaries If additive T is monotone wrt regular $\preceq$ then T is "bijective" up to φ.

If $\mathbb{F}$ has the property: all non-zero endomorphisms are automorphisms, then T is automatically bijective.

Theorem. Additive transformations over $\mathbb{C}$ monotone wrt any of $\overset{*}{\leqslant}$, $*\!\leqslant$, $\leqslant\!*$, $\overset{\diamond}{\leqslant}$, $\overset{\sigma}{\leqslant}$, $\overset{\sigma_1}{\leqslant}$, then T is automatically bijective.

In comparison with linear case: there are additive non-bijective monotone wrt minus-order transformations, in particular, over $\mathbb{C}$

Examples of orders which are not unitary invariant:

Definition A generalized inverse matrix A^- for a fixed matrix $A \in M_n(\mathbb{F})$ is defined to be any solution of the matrix equation $AA^-A = A$. A generalized inverse matrix A_r^-, which in addition satisfies the condition $A_r^- A A_r^- = A_r^-$, is called a reflexive. A group generalized inverse matrix $A^\sharp$ is defined to be a reflexive generalized inverse matrix which commutes with the matrix A.

Definition A matrix A is said to be of index k if $\text{Im}\, A \supsetneq \text{Im}\, A^2 \supsetneq \ldots \supsetneq \text{Im}\, A^k = \text{Im}\, A^{k+1} = \ldots$.

Definition [S.-K. Mitra] Let $A \in M_n(\mathbb{F})$ be a matrix of index 1 and $B \in M_n(\mathbb{F})$ be an arbitrary matrix. We say that $A \overset{\sharp}{\leqslant} B$ iff

$$AA^{\sharp} = BA^{\sharp} = A^{\sharp}B.$$

Definition The core-nilpotent decomposition of a square matrix $A \in M_n(F)$ is the following decomposition: $A = C_A + N_A$, where N_A is nilpotent matrix and C_A is a matrix of index 1, moreover $C_A N_A = N_A C_A = 0$. $\exists!$

Definition [R. Hartwig, S.-K. Mitra]

$$A \overset{\text{cn}}{\leqslant} B, \text{ iff } \quad \begin{cases} C_A \overset{\sharp}{\leqslant} C_B \\ N_A \overset{-}{\leqslant} N_B \end{cases}$$

Non-regular orders

I. Bogdanov, A. Guterman,

M. Efimov, A. Guterman

Lemma. Let $A_1, \dots, A_n \in M_n(\mathbb{F})$. Then TFAE:

1. $0 \overset{\sharp}{<} A_1 \overset{\sharp}{<} \cdots \overset{\sharp}{<} A_n$
2. $0 \overset{\text{cn}}{<} A_1 \overset{\text{cn}}{<} \cdots \overset{\text{cn}}{<} A_n$
3. $\forall i = 1, \dots, n$ A_i are diagonalizable matrices of rank i in the same basis.

Definition Let $A \in M_n(\mathbb{F})$

$$\mathcal{D}(A) := \left\{ B \in M_n(\mathbb{F}) | A, B \begin{array}{c}\text{are simultaneously}\\ \text{diagonalizable}\end{array} \right\}$$

A is not diagonalizable $\Rightarrow D(A) = \emptyset$

Definition $T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ preserves simultaneous diagonalizability if

$T(D(A)) \subseteq D(T(A))$

Corollary T additive, monotone with respect to $\overset{\text{cn}}{\leqslant}$ or $\overset{\text{cn}}{<}$ $\Rightarrow$ T preserves simultaneous diagonalizability.

Theorem. [Omladič, Šemrl] $\mathbb{F} = \mathbb{C}$, $n > 3$, linear $T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ preserves the set of diagonalizable matrices iff $T(A) = cPAP^{-1} + f(A)I$

or $T(A) = cPA^tP^{-1} + f(A)I$ for some $P \in GL_n(\mathbb{F})$, $c \in \mathbb{F}^*$, f − linear functional on $M_n(\mathbb{F})$, $f(I) \neq -c$.

using Motzkin-Taussky Theorem

Theorem. Let $\mathbb{F}$ be a field, $char\,\mathbb{F} \neq 2$, $n \geq 2$ be integer. Then additive $T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is monotone with respect to either $\overset{\sharp}{\leqslant}$ or $\overset{\text{cn}}{\leqslant}$ partial order iff either $T \equiv 0$ or there exist $\alpha \in \mathbb{F}^*$, $P \in GL_n(\mathbb{F})$ and endomorphism $\varphi : \mathbb{F} \to \mathbb{F}$ such that T has one of the following forms:

$$T(X) = \alpha P X^{\varphi} P^{-1} \quad \forall X \in M_n(\mathbb{F})$$

or

$$T(X) = \alpha P (X^{\varphi})^t P^{-1} \quad \forall X \in M_n(\mathbb{F})$$

Example Let $|\mathbb{F}| = n = 2$. Then linear transformation defined on basis by $T(E_{ii}) = E_{ii}$, $T(E_{ij}) = I + E_{ij}$ if $i \neq j$ is monotone with respect to $\overset{\sharp}{\leqslant}$, $\overset{\text{cn}}{\leqslant}$, but non-standard.

What about non-linear transformations?

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Example Let $\mathbf{I}_n^1(\mathbb{F})$ be the set of matrices of index 1,
$M_1 = M_n(\mathbb{F}) \setminus \mathbf{I}_n^1(\mathbb{F})$.
Let $T(A) = A$ for all $A \in \mathbf{I}_n^1(\mathbb{F})$,
$T|_{M_1}$ is an arbitrary bijection.
Then T is bijective, T is monotone with respect to $\overset{\sharp}{\leqslant}$, but T can be non-standard.

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We need some additional assumptions on T or $\mathbb{F}$ or special subset $S \subset M_n(\mathbb{F})$!

Counting functions:

Definition 1. $k_A \colon \mathbb{F} \times \mathbb{N} \to \mathbb{Z}_+$:

for $\lambda \in \mathbb{F}$ and $r \in \mathbb{N}$, $k_A(\lambda, r) =$ number of Jordan blocks of A of the size r corresponding to λ.

If there are no Jordan blocks of A with λ of the size r then $k_A(\lambda, r) = 0$.

$K_A \colon \mathbb{F} \to \mathbb{Z}_+$ is the total number of Jordan blocks of A corresponding to λ,

$$K_A(\lambda) = \sum_{r=1}^{\infty} k_A(\lambda, r).$$

Definition 2. Let $\mathbb{F}$ be any field, $A \in M_n(\mathbb{F})$, $A = C_A + N_A$ be the core-nilpotent decomposition of A. The maps $S_A^i : \mathbb{F} \to M_n(\mathbb{F})$, $i = 1, 2, 3$ are

$S_A^1(\lambda)$: if $\lambda = 0$, $S_A^1(0) = N_A$

if $\lambda \neq 0$, $S_A^1(\lambda) = X_\lambda$ is such that $X_\lambda \overset{\sharp}{\leqslant} A$,

$K_{X_\lambda}(\lambda) = K_A(\lambda)$ and $\mathrm{Spec}\,(X_\lambda) = \{\lambda, 0\}$.

$$S_A^2(\lambda) = S_{A+I}^1(\lambda + 1) - S_A^1(\lambda) \text{ for all } \lambda \in \mathbb{F};$$

$$S_A^3(\lambda) = S_A^1(\lambda) - \lambda S_A^2(\lambda) \text{ for all } \lambda \in \mathbb{F}.$$

Theorem 3. *[Efimov, Guterman] These definitions are correct.*

Lemma 4. *Let* $\mathbb{F}$ *be any field,* $A \in M_n(\mathbb{F})$, $\lambda \in \overline{\mathbb{F}}$. *Then* $\exists!$ $X_\lambda \in \mathbf{I}_n^1(\mathbb{F})$, $X_\lambda \overset{\sharp}{\leqslant} A$, $K_{X_\lambda}(\lambda) = K_A(\lambda)$ *and* $\mathsf{Spec}\,(X_\lambda) = \{\lambda, 0\}$.

Properties of these maps:

Theorem 5. *[Efimov, Guterman] Let* $A \in M_n(\mathbb{F})$.

1. *If* $\lambda \notin \mathrm{Spec}\,(A) \subseteq \overline{\mathbb{F}}$ *then* $S_A^i(\lambda) = 0$ *for* $i = 1, 2, 3$.

2. $\mathrm{rk}\,(S_A^2(\lambda)) = \deg_{\chi_A}(z - \lambda)$ *is the multiplicity of* λ *in the characteristic polynomial* χ_A.

3. $S_A^i(\lambda) \perp S_A^j(\mu)$ *for all* $\lambda \neq \mu$, $i, j = 1, 2, 3$.

4. $S_A^i(\lambda) S_A^2(\lambda) = S_A^2(\lambda) S_A^i(\lambda) = S_A^i(\lambda)$ *for all* $\lambda \in \overline{\mathbb{F}}$, $i = 1, 2, 3$.

5. $S_A^2(\lambda)$ *is idempotent for all* $\lambda \in \overline{\mathbb{F}}$.

6. $S_A^3(\lambda)$ *is nilpotent for all* $\lambda \in \overline{\mathbb{F}}$.

7. $A = \sum_{\lambda \in \overline{\mathbb{F}}} S_A^1(\lambda) = \sum_{\lambda \in \overline{\mathbb{F}}} (\lambda S_A^2(\lambda) + S_A^3(\lambda))$, $I = \sum_{\lambda \in \overline{\mathbb{F}}} S_A^2(\lambda)$.

8. For any polynomial $f \in \overline{\mathbb{F}}[t]$ *it holds that*

$$f(A) = \sum_{\lambda \in \overline{\mathbb{F}}} (f(\lambda) S_A^2(\lambda) + \frac{f'(\lambda)}{1!} S_A^3(\lambda) + \ldots + \frac{f^{(n-1)}(\lambda)}{(n-1)!} (S_A^3(\lambda))^{n-1}).$$

9. $\overline{\mathbb{F}}[A] = \{f(A)\}_{f \in \overline{\mathbb{F}}[t]} = \langle \{S_A^2(\lambda), S_A^3(\lambda), \ldots, (S_A^3(\lambda))^{n-1}\}_{\lambda \in \overline{\mathbb{F}}} \rangle$*, and nonzero matrices in* $\{S_A^2(\lambda), S_A^3(\lambda), \ldots, (S_A^3(\lambda))^{n-1}\}_{\lambda \in \overline{\mathbb{F}}}$ *are linearly independent.*

10. If $\lambda \in \mathbb{F}$ *then* $S_A^i(\lambda) \in M_n(\mathbb{F})$, $i = 1, 2, 3$.

11. If A *commutes with some* $B \in M_n(\mathbb{F})$*, then* $S_A^i(\lambda)$ *commutes with* B *for all* $\lambda \in \overline{\mathbb{F}}$ *and* $i = 1, 2, 3$.

12. If $\operatorname{Ind} A = 1$ *and* A *is orthogonal to some* $B \in M_n(\mathbb{F})$ *then*

a) all matrices $S^i_A(\lambda)$ *are orthogonal to* B*,*

b) $S^i_{A+B}(\lambda) = S^i_A(\lambda) + S^i_B(\lambda)$ *for* $\lambda \neq 0$ *and* $i = 1, 2, 3$*.*

c) $S^i_A(\lambda) \perp S^j_B(\mu)$ *for all* $\lambda, \mu \in \mathbb{F} \setminus \{0\}$*,* $i, j = 1, 2, 3$*.*

13. If $A \overset{\sharp}{\leqslant} C$ *for some* $C \in M_n(\mathbb{F})$*, then for all* $\Lambda \subset \overline{\mathbb{F}} \setminus \{0\}$ *we have* $\sum\limits_{\lambda \in \Lambda} S^i_A(\lambda) \overset{\sharp}{\leqslant} \sum\limits_{\lambda \in \Lambda} S^i_C(\lambda)$*,* $i = 1, 2$*. In particular,* $S^i_A(\lambda) \overset{\sharp}{\leqslant} S^i_C(\lambda)$ *for* $\lambda \neq 0$ *and* $i = 1, 2$*.*

Definition 6. The decompositions

$$A = \sum_{\lambda \in \overline{\mathbb{F}}} S_A^1(\lambda) = \sum_{\lambda \in \overline{\mathbb{F}}} (\lambda S_A^2(\lambda) + S_A^3(\lambda))$$

are called spectrally orthogonal decompositions of A.

Theorem 7. *[Efimov, Guterman] Let* $\mathbb{F}$ *be algebraically closed,* $n \geq 3$, $T\colon \mathcal{D}_n(\mathbb{F}) \to \mathcal{D}_n(\mathbb{F})$ *be monotone with respect to* $\overset{\sharp}{\leqslant}$*-order and injective. Then* $\exists$ $P \in GL_n(\mathbb{F})$, $0 \neq f\colon \mathbb{F} \to \mathbb{F}$, *and injective* $\sigma\colon \mathbb{F} \to \mathbb{F}$ *satisfying* $\sigma(0) = 0$ *such that*

$$T(A) = \sum_{\lambda \in \mathbb{F}} \sigma(\lambda) P^{-1} (S_A^2(\lambda))^f P \text{ for all } A \in \mathcal{D}_n(\mathbb{F})$$

or

$$T(A) = \sum_{\lambda \in \mathbb{F}} \sigma(\lambda) P^{-1} [(S_A^2(\lambda))^f]^t P \text{ for all } A \in \mathcal{D}_n(\mathbb{F})$$

Theorem 8. *[Efimov, Guterman] Let* $\mathbb{F}$ *be algebraically closed, let* $n \geq 3$*, and* $T\colon \mathcal{D}_n(\mathbb{F}) \to \mathcal{D}_n(\mathbb{F})$ *be strongly monotone with respect to* $\overset{\sharp}{<}$*-order. Then* T *is injective and the result of previous theorem holds.*

Theorem 9. *[Efimov, Guterman] Let* $\mathbb{F}$ *be algebraically closed,* $M = \{A \in \mathbf{I}_n^1(\mathbb{F}) \mid \sum\limits_{\lambda \in \mathbb{F}} K_A(\lambda) = 1\}$ *be the set of matrices with the unique Jordan block,*
$T\colon \mathbf{I}_n^1(\mathbb{F}) \to \mathbf{I}_n^1(\mathbb{F})$ *be bijective and strongly monotone with respect to* $\overset{\sharp}{<}$*-order with additional assumption*
$T(\lambda I) = \lambda I$ *for all* $\lambda \in \mathbb{F}$*.*
Then for any $A \in \mathbf{I}_n^1(\mathbb{F}) \setminus M$ *there exists* $P_A \in GL_n(\mathbb{F})$ *such that* $T(A) = P_A^{-1} A P_A$*.*

Here T can be any bijection on M!

Definition 10. Let $A, B \in M_n(\mathbb{F})$. The matrices A and B are called pairwise orthogonal, $A \perp B$, if $AB = BA = 0$.

Definition 11. The map $T\colon \mathbf{I}_n^1(\mathbb{F}) \to \mathbf{I}_n^1(\mathbb{F})$ is 0-additive, if for any matrices $A, B \in \mathbf{I}_n^1(\mathbb{F})$ with $A \perp B$ it holds:

(i) $T(A) \perp T(B)$;

(ii) $T(A+B) = T(A) + T(B)$.

Theorem 12. *[Efimov, Guterman] Let $\mathbb{F}$ be algebraically closed and $T\colon \mathbf{I}_n^1(\mathbb{F}) \to \mathbf{I}_n^1(\mathbb{F})$ be bijective. Then T is strongly monotone with respect to $\overset{\sharp}{<}$-order if and only if both T and T^{-1} are 0-additive.*

Remark 13.

1. On $\mathbf{I}_n^1(\mathbb{F})$, in particular, on $\mathcal{D}_n(\mathbb{F})$, $\overset{\sharp}{\leqslant}$- and $\overset{cn}{\leqslant}$-orders are equivalent.

2. No linearity or additivity is assumed in above Theorems.

Theorem 14. *Let* $n \geq 3$, $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$ *is injective and continuous, one of a, b, c is true:*

a) T is monotone with respect to $\overset{\sharp}{\leqslant}$-order;

b) T is monotone with respect to $\overset{cn}{\leqslant}$-order;

c) T is 0-additive map.

Then there are $P \in GL_n(\mathbb{C})$, $\alpha \in \mathbb{C} \setminus \{0\}$ such that

$$\begin{aligned}
T(X) &= \alpha P^{-1} X P && \text{for all } X \in M_n(\mathbb{C}) \text{ or}\\
T(X) &= \alpha P^{-1} X^t P && \text{for all } X \in M_n(\mathbb{C}) \text{ or}\\
T(X) &= \alpha P^{-1} \overline{X} P && \text{for all } X \in M_n(\mathbb{C}) \text{ or}\\
T(X) &= \alpha P^{-1} \overline{X}^t P && \text{for all } X \in M_n(\mathbb{C}).
\end{aligned}$$

Corollary 15. *In the conditions of Theorem*

1. *the map* T *is automatically surjective and* $\mathbb{R}$*-linear.*

2. *assumptions (a) and (b) are equivalent.*

Example 16. Let $\mathbb{F} = \overline{\mathbb{F}}$. Assume $T\colon \mathbf{I}_n^1(\mathbb{F}) \to \mathbf{I}_n^1(\mathbb{F})$ is bijective, $T(\mathcal{M}) = \mathcal{M}$, $T(X) = X$ for all $X \notin \mathcal{M}$. Then T is strongly monotone with respect to $\overset{\sharp}{<}$-order.

$\mathcal{M}$ is the set of index 1 matrices with unique Jordan block.

Example 17. Let $\|\cdot\|$ be a norm in $M_n(\mathbb{C})$ and $\varepsilon > 0$ be such that ε-neighborhood of I in the norm $\|\cdot\|$ does not contain singular matrices. Let $T\colon M_n(\mathbb{C}) \to M_n(\mathbb{C})$:

$$T(X) = \max\{1 - \varepsilon^{-1}\|X - I\|, 0\}I.$$

Then T is non-injective continuous $\overset{\sharp}{\leqslant}$-monotone and is not 0-additive, is not $\mathbb{R}$-linear, does not have the form as in the statement.

Proof. Let $X, Y \in M_n(\mathbb{C})$, $\operatorname{Ind} X = 1$, $X \overset{\sharp}{\leqslant} Y$.

If $X \notin$ ε-neighborhood of I then $T(X) = 0 \overset{\sharp}{\leqslant} T(Y)$.

Otherwise $\operatorname{rk} X = n$. Hence $X = Y$ and $T(X) = T(Y)$.

T is not 0-additive: $T(E_{11}) + T(I - E_{11}) = 0 \neq I = T(I)$.

The following example convinces us that without continuity assumption even the assumptions of bijectivity and strong monotonicity do not guarantee the that T has good form:

Example 18. Let $T\colon M_n(\mathbb{F}) \to M_n(\mathbb{F})$:

$$T(A) = \sum_{\lambda \in \mathbb{F}} (\lambda S^2_A(\lambda) - S^3_A(\lambda)).$$

(In the SOD of A via S^2 and S^3 we changed plus to minus).

Then

(1) T is bijective,

(2) T is strongly $\overset{\sharp}{\leqslant}$- monotone,

(3) on the whole $M_n(\mathbb{F})$ the map T is not additive, so it is not of the form described in Theorem.

One small note to the proof...

$$(PAQ)^{\sharp} =$$

$$= PA(AA^{\sharp}QPA + I - AA^{\sharp})^{-2}Q$$

instead of

$$(PAQ)^{*} = Q^{*}A^{*}P^{*}$$