Phase-lock areas and problems in complex differential equations

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Circle homeomorphisms and rotation number

$S^1 := \mathbb{R}/2\pi\mathbb{Z}$ oriented circle; $f : S^1 \to S^1$ **a positive homeomorphism.**

Definition of the **Poincaré rotation number** $\rho(f) \in \mathbb{R}/\mathbb{Z}$.
H.Poincaré. *Sur les courbes définies par les équations différentialles.* J. Math. Pures App. **I 167** (1885).

$\pi : \mathbb{R} \to S^1 :=$ the **universal covering projection:** $x \mapsto x(\operatorname{mod} 2\pi)$.
$F :=$ a **continuous lifting** of f to $\mathbb{R}$.
Defined up to postcomposition
with translation by $2\pi m$, $m \in \mathbb{Z}$.

$\rho(F)(x) := \lim_{k\to+\infty} \frac{F^k(x)}{2\pi k}$.

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ \downarrow \pi & & \downarrow \pi \\ S^1 & \xrightarrow{f} & S^1 \end{array}$$

The limit exists and is independent of x!
$\rho(f) := \rho(F)(\operatorname{mod} \mathbb{Z})$.

Henri Poincaré (1854–1912)

Rotation numbers: first examples

$\rho(F)(x) := \lim_{k\to+\infty} \frac{F^k(x)}{2\pi k}$.

$\rho(f) := \rho(F)(\text{mod}\, \mathbb{Z})$.

1) $f(x) = x + 2\pi a \quad => \quad \rho(f) = a(\text{mod}\, \mathbb{Z})$ for every $a \in \mathbb{R}$.

2) $f(x)$ has a **fixed point** $<=> \quad \rho(f) = 0(\text{mod}\, \mathbb{Z})$.

3) $f(x)$ has a q-**periodic point** $<=> \quad \rho(f) = \frac{p}{q}(\text{mod}\, \mathbb{Z})$ for some $p \in \mathbb{Z}$.

Given $q, p \in \mathbb{Z}$, $(q, p) = 1$ and a homeomorphism $f : S^1 \to S^1$ with $\rho(f) = \frac{p}{q}$.
Let $O = (A_1, \dots, A_q)$ be its q-periodic orbit cyclically ordered by circle orientation.
Fix a cyclic permutation σ acting on O.
It can be extended to a homeomorphism $g_\sigma : S^1 \to S^1$, $g(O) = O$.

Fact. For every (p, q) and every $s \in \mathbb{Z}$, $(s, q) = 1$, there exists a cyclic permutation σ such that
the composition $g_\sigma \circ f$ has rotation number $\frac{s}{q}$.

Families of circle homeomorphisms and phase-lock areas

A **family** $f = f(x,u)$ of positive circle homeomorphisms $S^1 \to S^1$, $x \mapsto f(x,u)$;
$u :=$ the **parameter**; u lies in a domain $U \subset \mathbb{R}^n$.

The **rotation number function:** $\rho(u) := \rho(f(.,u))$: $U \to S^1 = \mathbb{R}/\mathbb{Z}$.

Main definition.
Phase-lock areas: subsets $\{u \in U \mid \rho(u) = r\} \subset U$ with **non-empty interiors.**

Example. Consider a family $f(x,u)$ of **circle diffeomorphisms.**
Let for some $u_0 \in U$ the diffeomorphism $f_{u_0}(x) = f(x,u_0)$
have a q-periodic point x_0 with $(f^q_{u_0})'(x_0) \neq 1$. Then

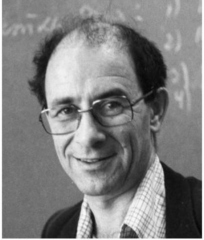
$$\rho(u_0) = \frac{p}{q} \text{ for some } p = p(u_0) \in \mathbb{Z};$$

$$L_{\frac{p}{q}} := \{u \in U \mid \rho(u) = \frac{p}{q}\} \text{ is a } \textbf{phase-lock area.}$$

This example illustrates a **classical fact:** **stability** of the above q-periodic point.

Arnold family of circle diffeomorphisms:
(V.I.Arnold, Geometrical Methods in the Theory of Ordinary Differential Equations. Springer, 1988.)

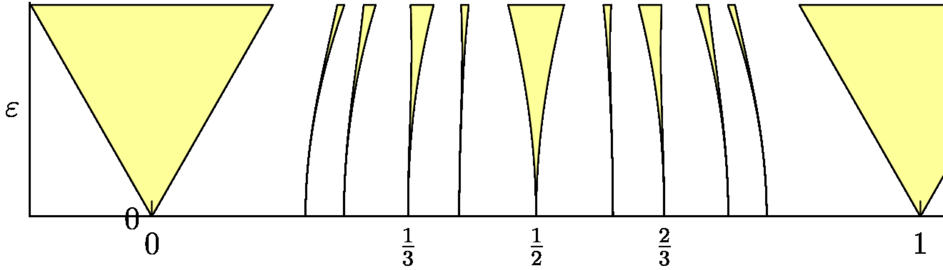
$f_{a,\varepsilon} : x \mapsto x + 2\pi a + \varepsilon \sin x$. Here $u = (a, \varepsilon)$, $0 \leq \varepsilon < 1$. (A)



V.I.Arnold (1937–2010)

Arnold Tongues Effect. *In family (A) the* **phase-lock areas** *exist exactly for* **all the rational** *rotation numbers.* They were named the **Arnold tongues.**

The tongues are connected and start from $(\frac{p}{q}, 0)$ for all $\frac{p}{q} \in \mathbb{Q}$.



The **tongue portrait** is **symmetric** with respect to the lines $\{a = \frac{1}{2}\}$, $\{a = 0\}$, since the **conjugacy of** $f_{a,\varepsilon}$ **by** $x \mapsto -x$ **changes sign of** a.

Arnold family of circle diffeomorphisms:

$$x \mapsto x + 2\pi a + \varepsilon \sin x. \qquad \text{(A)}$$

Arnold Tongues Effect. *In family (A)* **the phase-lock areas** *exist exactly for* **all the rational** *rotation numbers.* They were named the **Arnold tongues.**

1) The tongues are connected and start from any $(\frac{p}{q}, 0)$, $\frac{p}{q} \in \mathbb{Q}$.

2) For any fixed ε, $0 < \varepsilon < 1$ the rotation number as a function of the parameter **a** gives a **Cantor staircase:**

the intersection of the line $\varepsilon = const$

with the **complement to the interiors** of the Arnold tongues

is a **Cantor set**.

This Cantor set has a **positive Lebesgue measure,**

in difference from the **standard** Cantor set.

Rotation number of flow on 2-torus

Differential equation on 2-torus $\mathbb{T}^2 := \mathbb{R}^2_{(\phi,\tau)}/2\pi\mathbb{Z}^2$

$$\begin{cases} \dot\phi = f(\phi,\tau) \\ \dot\tau = 1 \end{cases} \qquad f \in C^1, \quad f(\phi+2\pi,\tau) = f(\phi,\tau+2\pi) = f(\phi,\tau).$$

Solution $\phi = \phi(\tau) \in \mathbb{R}$. Uniquely defined by $\phi_0 = \phi(0)$.
If $\phi(\tau)$ is a solution, then $\phi(\tau+2\pi)$ is also a solution.
Rotation number of flow:

$$\rho := \lim_{k\to+\infty} \frac{\phi(2\pi k)}{2\pi k} \in \mathbb{R}.$$

Properties:
1) Independent on choice of the initial condition $\phi_0 = \phi(0)$.
2) ρ **mod** $\mathbb{Z}$**,** the rotation number **of flow mod** $\mathbb{Z}$**,** equals
the rotation number of the **Poincaré map** $h : S^1 \to S^1$ of the transversal circle

$$S^1 = \mathbb{R}_\phi/2\pi\mathbb{Z} = S^1_\phi \times \{0\} \subset \mathbb{T}^2_{\phi,\tau} :$$

the flow map in time 2π **:** $\quad h : S^1 \to S^1, \quad \phi(0) \mapsto \phi(2\pi).$

$$\begin{cases}\dot{\phi} = f(\phi, \tau; u) \\ \dot{\tau} = 1\end{cases} \quad ; \quad u \in U \subset \mathbb{R}^n \text{ is the parameter.} \qquad \text{(B)}$$

The rotation number function $\rho : U \to \mathbb{R}$**:**

$$\rho(u) := \lim_{k \to +\infty} \frac{\phi(2\pi k; u)}{2\pi k} \in \mathbb{R}.$$

Phase-lock areas: subsets $\{u \in U \mid \rho(u) = r\} \subset U$ with **non-empty interiors.**

Example. Let for $u = u_0$ (B) have an attracting (or repelling) **periodic orbit** ($<=>$ h have attracting (repelling) q-periodic point). Then

Period of the orbit $= 2\pi q, \ q \in \mathbb{Z}$; $\quad \rho(u_0) = \frac{p}{q} \in \mathbb{Q}$ for some $p \in \mathbb{Z}$;

$L_{\frac{p}{q}} := \{u \in U \mid \rho(u) = \frac{p}{q}\}$ is a **phase-lock area.**

Goal: study phase-lock areas in a model of **Josephson effect (superconductivity).**

Superconductivity

Phenomenon, when the **electric resistance** becomes **exactly zero.**
Occurs in some metals at temperature $\mathbb{T} < \mathbb{T}_{crit}$.
Occurs, for example, in pure elements:

Aluminium, $T_{crit} = 1.175 \pm 0.002$ K;

Lead (Plumbum), $T_{crit} = 7.196 \pm 0.006$ K.

In intermetallic alloys:
$\mathbf{HgBa_2Ca_2Cu_3O_8}$**,** $T_{crit} = 133$ K.

The resistance **does not change continuously to zero;**
it **jumps to zero,** once $\mathbb{T}$ becomes **less than** $\mathbb{T}_{crit}$.

The critical temperature $\mathbb{T}_{crit}$ depends on the metal.

See the table of critical temperatures $\mathbb{T}_{crit}$ in

V.V.Schmidt. Introduction to the physics of semiconductors. - Moscow, MCCME, 2000.

Josephson effect

Born in UK in 1940.

Brian Josephson. Nobel Prize in physics (1973)

B.Josephson. *"Possible new effects in superconductive tunneling".* Phys. Let., **1 (1962)**, 251–253.

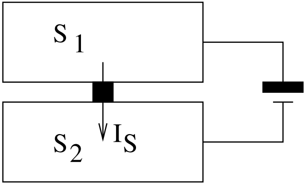
"We here present an approach to the calculation of tunneling currents between two metals that is sufficiently general to deal with the case, when both metals are superconducting. In that case new effects are predicted, due to the possibility that electron pairs may tunnel through the barrier leaving the quasi-particle distribution unchanged."

Nobel Lecture, December 12, 1973. *"The discovery of tunnelling supercurrents".*

Occurs in some metals at temperature $\mathbb{T} < \mathbb{T}_{crit}$.

The Josephson effect

Let two superconductors S_1, S_2 be separated by a very narrow dielectric, thickness $\leq 10^{-5}$cm ($<<$ distance in Cooper pair).
There exists a **supercurrent I_S through the dielectric.**



Supercurrent is carried by coherent **Cooper pairs** of electrons.

Josephson effect

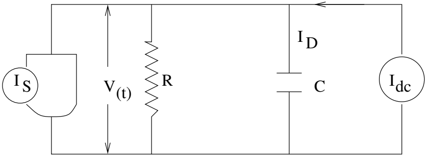
Quantum mechanics. State of S_j: wave function $\Psi_j = |\Psi_j| e^{i\chi_j}$;

χ_j is the *phase*, $\phi := \chi_1 - \chi_2$.

The first Josephson relation

$I_S = I_c \sin\phi$, $I_c \equiv const$.

Equivalent electrical circuit of Josephson junction



$V(t)$:= the **tension;** R:= **resistance;** C:= **capacitor**.

Kirchoff law: $I_{dc} = I_S + I_D + \frac{V(t)}{R}$.

Josephson voltage relation: $V(t) = \frac{\hbar}{2e}\dot{\phi}$

I_D := **Displacement current.** $I_D = C\dot{V} = \frac{\hbar}{2e} C \frac{d^2\phi}{dt^2}$.

General mathematical model

This scheme is described by the following equation,
which is implied by **Kirchoff law** and **Josephson relations:**

$$\varepsilon_1 \frac{d^2\phi}{dt^2} + \varepsilon_2 \frac{d\phi}{dt} + \sin\phi = I_c^{-1} I_{dc} := f(t),$$
$$\varepsilon_1 = \frac{\hbar}{2e} C I_c^{-1}, \ \varepsilon_2 = \frac{\hbar}{2e} \frac{1}{R} I_c^{-1},$$

$$\hbar \simeq 1.054x10^{-34} \text{ joule-second / rad}, \quad e \simeq 1.60x10^{-19} \text{ Coulomb.}$$

In physical works this equation is called the **Langevin equation.**

Our main, "overdamped" case: $\varepsilon_1 = 0$, $\varepsilon_2 = 1$.

$$\frac{d\phi}{dt} = -\sin\phi + f(t).$$

The classical Sine-Gordon equation:

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial^2 x} + \sin\phi = 0$$

Stationary Sine-Gordon equation with perturbation $f(t)$ for $\varepsilon_1 = 1$, $\varepsilon_2 = 0$:

$$\frac{d^2\phi}{dt^2} + \sin\phi = f(t).$$

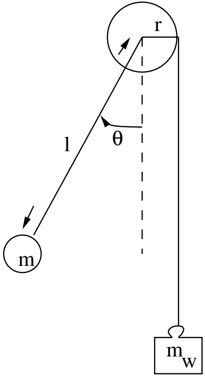


Figure: Mechanical model equivalent to model of Josephson junction: pendulum forced by a weight.

Described by the equation

$$m_w rg = M_I \frac{d^2\theta}{dt^2} + D_f \frac{d\theta}{dt} + mgl \sin\theta.$$

$M_I :=$ the inertia moment; $D_f :=$ the damping coefficient.

$$\varepsilon_1 \frac{d^2\phi}{dt^2} + \varepsilon_2 \frac{d\phi}{dt} + \sin\phi = I_c^{-1} I_{dc} := f(t),$$

After appropriate rescaling of the time, we obtain

$$\epsilon \frac{d^2\varphi}{d\tau_1^2} + \frac{d\varphi}{d\tau_1} + \sin\varphi = I_c^{-1} I_{dc},$$

$$\epsilon = \frac{\hbar}{2e}\frac{C}{I_c}\left(\frac{2e}{\hbar} R\, I_c\right)^2 = \frac{2e}{\hbar}(C\, R)(R\, I_c).$$

Overdamped case: $|\varepsilon| << 1$.

Denote $I_c^{-1} I_{dc} = f(t)$. We obtain

$$\frac{d\phi}{dt} = -\sin\phi + f(t)$$

Denote $I_c^{-1} I_{dc} = f(t)$. We obtain

$$\frac{d\phi}{dt} = -\sin\phi + f(t)$$

We suppose that

- the function $f(t)$ is **periodic with period T;**
- it depends on parameters, one of them is the period T: $f(t) = f(t; u, T)$, $u \in U$.

V.M.Buchstaber, O.V.Karpov, S.I.Tertychnyi (2004): translation of the model with periodic f as a family of dynamical systems on two-torus

$$\mathbb{T}^2_{(\phi,\tau)} = \mathbb{R}^2 / 2\pi\mathbb{Z}^2, \ \tau := \omega t, \ \omega := \frac{2\pi}{T}, \ g(\tau) := f(\omega^{-1}\tau) :$$

$$\begin{cases} \dot{\phi} = \frac{1}{\omega}(-\sin\phi + g(\tau)) \\ \dot{\tau} = 1 \end{cases} . \qquad (1)$$

$$\mathbb{T}^2_{(\phi,\tau)} = \mathbb{R}^2/2\pi\mathbb{Z}^2, \ \tau := \omega t, \ \omega := \frac{2\pi}{T}, \ g(\tau; u, \omega) := f(\omega^{-1}\tau; u, T) :$$

$$\begin{cases} \dot{\phi} = \frac{1}{\omega}(-\sin\phi + g(\tau)) \\ \dot{\tau} = 1 \end{cases} . \quad (1)$$

Consider $\phi = \phi(\tau)$.

The rotation number of flow:

$$\rho(u;\omega) := \lim_{n\to+\infty} \frac{\phi(2\pi n)}{2\pi n}.$$

Rotation number $\simeq$ **average voltage** over a long time interval;
up to a known constant factor.

Our main problem: Consider the space U of parameters of the T-periodic functions $f(t) = f(t; u, T)$.
Describe the **rotation number** ρ of the flow (1) **with fixed** $\omega = \frac{2\pi}{T}$ as a **function on the space U.**

Phase-lock areas in model of Josephson effect

$$\frac{d\phi}{dt} = -\sin\phi + f(t, u, T). \qquad (1)$$

Quantization effect (Buchstaber, Karpov, Tertychnyi, 2010):
Phase-lock areas exist only for **integer rotation values**

Equation (1) is transformed to **Riccati equation** on $\Phi(t) = e^{i\phi(t)}$:

$$\dot{\Phi} = Q_{2,t}(\Phi), \ \ Q_{2,t} \text{ is a quadratic polynomial,} \qquad (1)\text{bis}$$

- The map $h : \Phi(0) \mapsto \Phi(2\pi)$ is a **Möbius circle transformation:**

$=$ the restriction to the unit circle $S^1 \subset \overline{\mathbb{C}}_\Phi$ of a **conformal automorphism** of the Riemann sphere $\overline{\mathbb{C}}$ with **invariant unit circle.**

Quantization effect $<=$ Well-known **fact** for **Möbius circle transformations** h:
- either h has a **fixed point on the circle;**
- or h is analytically **conjugated to a rotation.**

Many works concern systems (1) with

$$f(t) = f(t; (B, A); \omega) = B + A\cos\omega t;\ \ u = (B, A) \in \mathbb{R}^2.$$

$$\frac{d\phi}{dt} = -\sin\phi + B + A\cos\omega t. \tag{2}$$

Equation (2) occurs in other domains of mathematics,
see Foote, R.L.; Levi, M.; Tabachnikov, S. *Tractrices, bicycle tire tracks, hatchet planimeters, and a 100-year-old conjecture,* Amer. Math. Monthly, **103** (2013), 199–216:

in the investigation of some systems with non-holonomic connections by geometric methods.

In model of the so-called Prytz planimeter and in cinematics of bicycle moving

Analogous equation describes the observed direction to a given point at infinity while moving along a geodesic in the hyperbolic plane.

$$\frac{d\phi}{dt} = -\sin\phi + B + A\cos\omega t; \quad \omega \text{ is fixed.}$$

$$\begin{cases} \dot{\phi} = -\sin\phi + B + A\cos(\tau) \\ \dot{\tau} = \omega \end{cases}, \quad (\phi, \tau) \in \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2; \ \tau := \omega t. \qquad (3)$$

A subfamily of (3) occurred in the work by Yu.S.Ilyashenko and J.Guckenheimer "The duck and the devil: canards on the staircase", MMJ 2001, from the slow-fast system point of view. They have obtained results on its limit cycles, as $\omega \to 0$.

V.M.Buchstaber, O.V.Karpov and S.I.Tertychnyi found the relation of the above-mentioned model of Josephson junction with system (3).

V.A.Kleptsyn, O.L.Romaskevich, I.V.Schurov proved results **on smallness of gaps** between the phase-lock areas **for small** ω using methods of **slow-fast systems:** paper "Josephson effect and slow-fast systems", Nanostr., Mat. Fiz. Mod. 2013.

$$\frac{d\phi}{dt} = -\sin\phi + B + A\cos\omega t; \quad \omega \text{ is fixed.}$$

$$\begin{cases} \dot{\phi} = -\sin\phi + B + A\cos(\tau) \\ \dot{\tau} = \omega \end{cases}, \quad (\phi, \tau) \in \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2; \ \tau := \omega t. \quad (4)$$

$L_r := \{(B, A) \in \mathbb{R}^2 \mid \rho(B, A) = r\}$ **is a phase-lock area** $<=> Int(L_r) \neq \emptyset$.

The **Shapiro step:** the *intersection of the phase-lock area* L_r *with a line* $\{A = A_0\}$.

ρ is the **average voltage** on the Josephson junction over a long time interval.

Averaging the measured voltage along a wider Shapiro step yields better precision.

Transversal Regularity Theorem (V.Buchstaber, A.Glutsyuk).

a) **For every family of functions** $f(t, u, T)$**,** T**-periodic in** t**,** $u \in U$**,** $T \in \mathbb{R}_+$**,** the function $\rho : U \times \mathbb{R}_+ \to \mathbb{R}$ gives a **regular analytic fibration over** $\mathbb{R} \setminus \mathbb{Z}$**.**

b) For $f(t, U, T) = B + A\cos\omega t$, $\omega = 2\pi T^{-1}$,
the **rotation number function** $\rho : \mathbb{R}^2_{(B,A)} \to \mathbb{R}$ with fixed ω gives
a **regular analytic fibration over** $\mathbb{R} \setminus \mathbb{Z}$ by curves
that are 1-to-1 analytically projected onto the vertical A-axis.

This is **implied by the following facts:**

- the return map $h : S^1_\phi \to S^1_\phi$ is a **Möbius circle diffeomorphism**.
- it is **elliptic** (conjugated to a rotation), if and only if $\rho \notin \mathbb{Z}$;
- in this case $h : \overline{\mathbb{C}}_\Phi \to \overline{\mathbb{C}}_\Phi$ has a **fixed point** $w \in \{|\Phi| < 1\}$, $\Phi = e^{i\phi}$, and $\rho = \arg h'(w)$, where $h'(w) = \frac{dh}{d\Phi}(w)$;
- w and $h'(w)$ **depend analytically** on the parameters $(B, A, \omega) \in \rho^{-1}(\mathbb{R} \setminus \mathbb{Z})$.

Boundaries of phase-lock areas

1). There exist functions $\psi_{r,\pm}(A)$ **analytic in** $A \in \mathbb{R}$ such that the **boundary** ∂L_r **is the union of their graphs:**

$$\partial L_r = \partial_+ L_r \cup \partial_- L_r, \ \partial_\pm L_r := \{B = \psi_{r,\pm}(A)\}.$$

$\psi_{r,\pm}(A)$ have asymptotics of **Bessel type** $r\omega \pm J_r(-\frac{A}{\omega}) + O(\frac{\ln|A|}{A})$**,** as $A \to \infty$.

Observed by S.Shapiro, A.Janus, S.Holly (1964);
V.M.Buchstaber. O.V.Karpov, S.I.Tertychnyi (2005).

Proved by A.V.Klimenko and O.L.Romaskevich (2014).

Explanation of graph property. **Symmetry:** $(\phi, \tau) \mapsto (\pi - \phi, -\tau)$.

$=>$ If the return map h is **parabolic** (has **one fixed point** in the cirle),

then the $\phi = \pm\frac{\pi}{2}(mod 2\pi\mathbb{Z})$ **is a fixed point** and **vice versa.**

$$=> \quad \partial L_r = \cup_\pm \partial L_\pm, \ \partial L_\pm = \{(B, A) \mid h(\pm\frac{\pi}{2}) = \pm\frac{\pi}{2}\}.$$

4) Each L_r is an infinite chain (garland) of domains going to infinity, separated by points of intersection $\partial_+ L_r \cap \partial_- L_r$.

Observed numerically by Buchstaber, Karpov, Tertychnyi.
Follows from results of Klimenko and Romaskevich.

The separation points with $A \neq 0$ are called **constriction points (constrictions).**

The separation points of L_r with $A = 0$ exist for $r \neq 0$, are called **growth points** and their abscissas B_r satisfy the equation $B_r^2 - r^2\omega^2 = 1$.

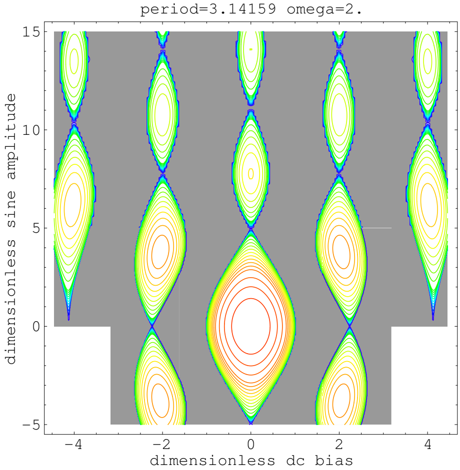
The phase-lock area L_0 has no growth points;
it intersects the B-axis by the segment $[-1, 1]$.
It always contains the square with vertices $(0, \pm 1)$, $(\pm 1, 0)$.

We present pictures of phase-lock areas for different values of ω.
These pictures are **symmetric** with respect to both coordinate axes:
$(B, A) \mapsto (-B, A)$; $(B, A) \mapsto (B, -A)$.

Taking into account these symmetries, we present only upper parts of the pictures.

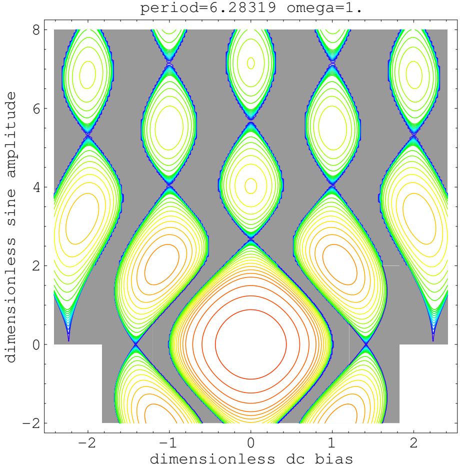
Phase-lock areas for $f(t) = B + A\cos\omega t$, $\omega = 2$

Each phase-lock area L_r is an infinite chain (garland) of domains going to infinity, separated by points of intersection $\partial_+ L_r \cap \partial_- L_r$.
The separation points with $A \neq 0$ are called **constriction points (constrictions).**



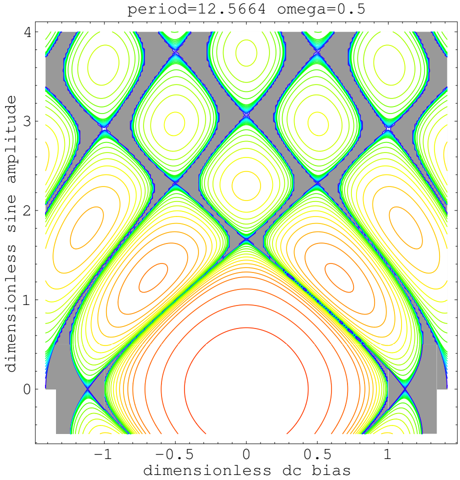
Phase-lock areas for $f(t) = B + A\cos\omega t$, $\omega = 1$

- infinitely many **constrictions** in every phase-lock area.



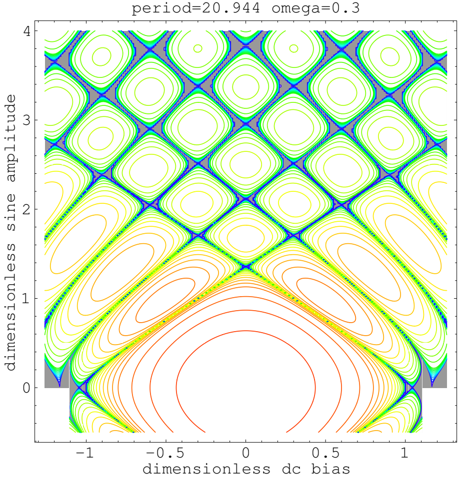
Phase-lock areas for $f(t) = B + A\cos\omega t, \omega = 0.5$

- infinitely many **constrictions** in every phase-lock areas.



Phase-lock areas for $f(t) = B + A\cos\omega t$, $\omega = 0.3$

- infinitely many **constrictions** in every phase-lock areas.



- the **portrait** of phase-lock areas is **symmetric:**
the transformations $(\phi, \tau) \mapsto (\phi, \tau + \pi)$ and $(\phi, \tau) \mapsto (-\phi, \tau + \pi)$ result in differential equation (2) with changed sign at A (respectively, B).

Constrictions:= the separation points in L_r, $r \in \mathbb{Z}$, with $A \neq 0$.

The constrictions with $A > 0$ are ordered by their A-coordinates: $\mathcal{A}_{r,1}, \mathcal{A}_{r,2}, \mathcal{A}_{r,3}, \dots$

Open questions based on numerical simulations

Conjecture 1 (quantization of constrictions): *All the constrictions $\mathcal{A}_{r,k}$ lie in the line $\Lambda_r := \{B = r\omega\}$:=* **the axis** *of the phase-lock area L_r.*

It is based on numerical simulations (Tertychnyi, Filimonov, Kleptsyn, Schurov).

At the moment it is proved that **each constriction $\mathcal{A}_{r,k}$ lies in a line** $\{B = \ell\omega\}$**, where** $0 \leq \ell \leq r$ **and** $\ell \equiv r(mod 2\mathbb{Z})$ (Filimonov, Glutsyuk, Kleptsyn, Schurov).

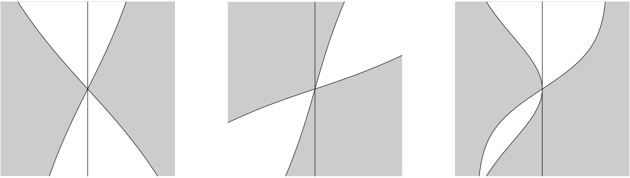
Open questions based on numerical simulations

Conjecture 1 *All constrictions $\mathcal{A}_{r,k}$ in L_r lie in the* **axis** $\Lambda_r := \{B = r\omega\}$.

Conjecture 2. For $k \geq 2$ the k-th component in L_r contains $(A_{r,k-1}, A_{r,k})$.

Definition. **A priori possible types of constrictions:**

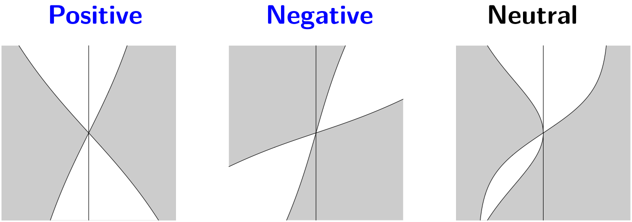
Positive **Negative** **Neutral**



Conjecture 3. Each constriction is positive.

Proposition. Conjecture 3 =>Conjecture 1. Conjecture 2 => Conjecture 1.

Definition. **A priori possible types of constrictions:**



Conjecture 3. Each constriction is positive.

Proposition. Conjecture 3 => Conjecture 2 => Conjecture 1.

Theorem 1 (A.Glutsyuk, J. Dyn. Control Syst. 2019).
Each constriction is either positive, or negative.

Problems on asymptotics of phase-lock area picture

Problem. What happens with the phase-lock area picture, as $\omega \to 0$?

Numerical experiences by V.M.Buchstaber, O.V.Karpov and S.I.Tertychnyi show that as $\omega \to 0$,
- a part of a rescaled phase-lock area portrait approaches a parquet turned by $\frac{\pi}{2}$;
- the main component of the zero phase-lock area tends to the square with vertices $(0, \pm 1)$, $(\pm 1, 0)$.

Example (V.M.Buchstaber, O.V.Karpov, S.I.Tertychnyi, 2012).
For $B = \omega$ and $A = 1$ the differential equation on $\phi(\tau)$ admits an explicit solution: $\phi = \frac{\pi}{2} + \tau$ with rotation number 1.

=> $(\omega, 1) \in L_1$ for every ω. In fact, $(\omega, 1) \in \partial L_1$.
=> this point of the phase-lock areas $L_1 = L_1(\omega)$ tends to the A-axis, as $\omega \to 0$.

Reduction to special double confluent Heun equation.

$$\frac{d\phi}{d\tau} = \frac{1}{\omega}(-\sin\phi + B + A\cos\tau), \tag{5}$$

$$z = e^{i\tau}, \quad \Phi = e^{i\phi}, \ \ell = \frac{B}{\omega}, \ \mu = \frac{A}{2\omega}, \ \lambda = \frac{1}{4\omega^2} - \mu^2,$$

$$\frac{d\Phi}{dz} = z^{-2}((\ell z + \mu(z^2+1))\Phi - \frac{z}{2i\omega}(\Phi^2 - 1)).$$

This is the projectivization of system of linear equations in vector function $(u(z), v(z))$ with $\Phi = \frac{v}{u}$:

$$\begin{cases} v' = \frac{1}{2i\omega z}u \\ u' = z^{-2}(-(\ell z + \mu(1+z^2))u + \frac{z}{2i\omega}v) \end{cases} \tag{6}$$

Reduction of system (5) to a system equivalent to (6) was done in a paper of Buchstaber, Karpov and Tertychnyi in 2010.

Reduction to special double confluent Heun equation

(after papers by Buchstaber and Tertychnyi, 2013-2015).
Set

$$E(z) = e^{\mu z} v(z)$$

The system

$$\begin{cases} v' = \frac{1}{2i\omega z} u \\ u' = z^{-2}(-(\ell z + \mu(1+z^2))u + \frac{z}{2i\omega} v) \end{cases} \tag{6}$$

is equivalent to **special double confluent Heun equation:**

$$z^2 E'' + ((\ell+1)z + \mu(1-z^2))E' + (\lambda - \mu(\ell+1)z)E = 0, \tag{7}$$

There exist explicit formulas expressing the solution of the non-linear equation

$$\frac{d\phi}{dt} = -\sin\phi + B + A\cos\omega t$$

via solution of equation (7) (Buchstaber - Tertychnyi).

General 6-parametric family of Heun equations

$$z(z-1)(z-t)E''+$$
$$(c(z-1)(z-t)+dz(z-t)+(a+b+1-c-d)z(z-1))E'+$$
$$+(abz-\nu)E=0. \quad (8)$$

Four Fuchsian singularities: 0, 1, t, ∞.
Parameters: $a, b, c, d; t, \nu$.

Double confluent Heun equation

$$z^2E''+(-z^2+cz+t)E'+(-az+\lambda)E=0$$

is its limit with pairs of confluenting singularities $(0,1)$, (t,∞).

It has only two singularities: 0 and ∞; both are *irregular.*

Geometry of phase-lock areas and special double confluent Heun equations.

Family of **dynamical systems on torus.** Parameters $(B, A; \omega)$.

$$z = e^{\tau}, \quad \ell = \frac{B}{\omega}, \; \mu = \frac{A}{2\omega}, \; \lambda = \frac{1}{4\omega^2} - \mu^2,$$

Family of corresponding **special double confluent Heun equations:**

$$z^2 E'' + ((\ell + 1)z + \mu(1 - z^2))E' + (\lambda - \mu(\ell + 1)z)E = 0. \qquad (9)$$

It is enough to consider the points (B, A) with $\ell = \frac{B}{\omega} \geq 0$ (portrait symmetry).

$(B, A) \in \mathbb{R}^2$ **constriction**, $B > 0 =>$ (9) has **entire solution** (B., T.).
There is explicit **transcendental** equation $\xi_\ell(\lambda, \mu) = 0$ on the parameters (λ, μ) for which (9) has entire solution (B.-T., B.-G.).

The function $\xi_\ell(\lambda, \mu)$ is holomorphic in $(\lambda, \mu) \in \mathbb{C}^2$. Constructed via an infinite product of explicit matrix functions in (λ, μ^2) whose elements are affine functions.

Double coufluent Heun equations with polynomial solutions

V.M.Buchstaber and S.I.Tertychnyi have described those parameter values $(B, A; \omega)$ for which the corresponding Heun equation with opposite sign at $\ell = \frac{B}{\omega}$,

$$z^2 E'' + ((-\ell + 1)z + \mu(1 - z^2))E' + (\lambda + \mu(\ell - 1)z)E = 0, \qquad (10)$$

has a **polynomial solution:** the so-called **generalized simple intersections.**

It was shown that they may exist only for $\ell \in \mathbb{N}$, and for every fixed $\ell \in \mathbb{N}$ the corresponding parameters $(\lambda = \frac{1-A^2}{4\omega^2}, \frac{A}{2\omega})$ form the **zero locus** of a **polynomial** of degree ℓ in (λ, μ^2) that is the **determinant** of a remarkable **three-diagonal matrix.**

It is known that for every given $\ell \in \mathbb{N}$ the **generalized simple intersections** are exactly the **intersections** of the **axis** $\Lambda_\ell := \{B = \omega\ell\}$ **with the boundaries** ∂L_s of phase-lock areas with $s \equiv \ell (\mathrm{mod}\, 2)$, $0 \leq s \leq \ell$ (Buchstaber, Glutsyuk, 2017).

Theorem (A.Glutsyuk, 2019) For every $\ell \in \mathbb{N}$ and every $\omega > 0$ small enough (dependently on ℓ) the axis Λ_ℓ contains exactly ℓ distinct generalized simple intersections. They depend continuously on ω and tend to the A-axis, as $\omega \to 0$: their common abscissa $\omega\ell$ tends to zero.

Linear system associated to model of Josephson effect and constrictions.

$$\ell = \frac{B}{\omega}, \ \mu = \frac{A}{2\omega}.$$

$$\begin{cases} v' = \frac{1}{2i\omega z} u \\ u' = z^{-2}(-(\ell z + \mu(1+z^2))u + \frac{z}{2i\omega} v) \end{cases}. \qquad (6)$$

System (6) has **two irregular singularities of Poincaré rank 1:** 0 and ∞.

$M :=$ **monodromy operator around 0 of system (6).**

It is a linear operator in the two-dimensional space of germs of solutions at $z_0 \neq 0$:

germ $\mapsto$ its analytic extension along a positive loop.

$M :=$ **monodromy operator around 0 of system (6).**

(B, A) lies in the **boundary of a phase-lock area**

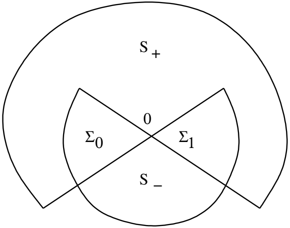
$<=>$ the return map of the non-linear system either $= Id$, or is **parabolic**

$<=>$ M has **equal eigenvalues**.

Theorem (Filimonov-Glutsyuk-Kleptsyn-Schurov),
Funct. Anal. Appl. 48 (2014), No. 4, 272–285:

- (B, A) is a **constriction** $<==>$ system (6) has **trivial monodromy:** $M = Id$.

Stokes operators. General theory (Jurkat, Lutz, Peyerimhoff, Sibuya, Balser)



The results of the above-mentioned authors imply the existence of the following **canonical fundamental solution matrices** $W_{\pm}(z)$ "at zero" in open sectors $S_{\pm}$ covering $\mathbb{C}^*$; $S_+ \cap S_- = \Sigma_0 \cup \Sigma_1$:

$$W_{\pm}(z) = H_{\pm}(z)F(z), \ F(z) = \begin{pmatrix} z^{-\ell}e^{\mu(\frac{1}{z}-z)} & 0 \\ 0 & 1 \end{pmatrix}, \quad H_{\pm}(0) = Id,$$

$H_{\pm}$ is **holomorphic** in $S_{\pm}$ and C^{∞} in $S_{\pm} \cup \{0\}$.

Here we fix an **analytic branch** of $F(z)$ on S_+.
The branch of $F(z)$ on $S_- :=$ its **counterclockwise analytic extension** to S_-.

$\Sigma_0 =$ the **left** component of $S_+ \cap S_-$; $\Sigma_1 =$ the **right** component.

The results of Yurkat, Lutz, Peyerimhof, Balser, Sibuya imply:

$$W_-(z) = W_+(z)C_0 \text{ on } \Sigma_0; \quad W_+(z)\,\mathrm{diag}(e^{-2\pi i\ell}, 1) = W_-(z)C_1 \text{ on } \Sigma_1,$$

$$C_0 = \begin{pmatrix} 1 & c_0 \\ 0 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 1 & 0 \\ c_1 & 1 \end{pmatrix} \; := \; \textbf{Stokes matrices}.$$

$$\mathrm{diag}(e^{-2\pi i\ell}, 1) = \textbf{monofromy of } F(z) :=\; \text{the } \textbf{formal monodromy.}$$

Classical theorem: $M = C_1^{-1} C_0^{-1}\,\mathrm{diag}(e^{-2\pi i\ell}, 1)$.

i.e., **formal monodromy** $= MC_1C_0$.

Special case: $\ell \in \mathbb{Z}$. Then
the formal monodromy is trivial: $\mathrm{diag}(e^{-2\pi i\ell}, 1) = Id$, $M = C_1^{-1}C_0^{-1}$.

Family of deformed Heun equations and Painlévé equations

A **singular point** of linear differential equation (system) is called **apparent,** if in its neighborhood all solutions are **holomorphic.**

S. Yu. Slavyanov, O. L. Stesik, *Antiquantization of deformed Heun-class equations,* Theoret. and Math. Phys., 186:1 (2016), 118–125.

The authors introduced a family of **deformed Heun equations** consisting of **Heun-class equations** with added one **apparent singularity.**

They have shown that this family realizes a transition from the **quantum Hamiltonian** corresponding to a **confluent Heun equation** to the **classical Hamiltonian** of the corresponding **Painlévé equation.**

They have obtained the complete list of such transitions.

In particular, they have obtained an explicit form of family of **deformed double confluent Heun equations.**

They have shown that their family is transformed to **Painlévé 3 equation.**

New approach to problem on constrictions.

Study **isomonodromic deformations** of linear systems (6).

See **Yulia Bibilo.** *Josephson Effect and Isomonodromic Deformations.* https://arxiv.org/abs/1805.11759

The **conditions** on a point $(B, A; \omega)$ to lie in the union of **boundaries** of the phase-lock areas (or to be a **constriction**)

are **preserved under isomonodromic deformation,** since they are conditions on the **conjugacy class of the monodromy operator.**

See the results presented in the previous slides.

Isomonodromic deformation allows to transform a given point $(B, A; \omega)$ (e.g., constriction) of the union of boundaries of phase-lock areas

to an infinite **sequence** of points (constrictions) $(B_k, A_k; \omega_k)$ of boundaries with constant $\ell = \frac{B_k}{\omega_k} = \frac{B}{\omega}$.

Formal normal forms and monodromy-Stokes data

A.A.Bolibruch, S.Malek, C.Mitschi. *On the generalized Riemann–Hilbert problem with irregular singularities.* Expositiones Mathematicae 24(3) (2006), 235-272.

General linear 2×2 system on $\overline{\mathbb{C}}$
with irregular singularities of type pole of order 2 (Poincaré rank 1) at 0 and ∞:

$$\frac{dY}{dz} = B(z)Y, \ B(z) = \frac{A_2}{z^2} + \frac{A_1}{z} + A_0, \ Y \in \mathbb{C}^2, \ A_j \in \mathrm{Mat}_2(\mathbb{C}). \tag{11}$$

If A_2 has distinct eigenvalues, then there exists a formal invertible matrix power series transforming the germ of (11) at 0 to **formal normal form** of type (11) with **diagonal** matrices $\widetilde{A}_q$, $q = 0, 1, 2$. $\mathrm{Spec}(\widetilde{A}_2) = \mathrm{Spec}(A_2)$. Same at infinity, for A_0.

The **formal monodromy** = the **monodromy of the formal normal form.**

Given a $z_0 \in \mathbb{C} \setminus \{0\}$. $V = \{$**Germs of local solutions** at $z_0\}$.
Monodromy operator and **Stokes operators** act on V.
They form **Monodromy - Stokes representation** acting on V.

Generalized Riemann–Hilbert Problem

$$\frac{dY}{dz} = B(z)Y, \ B(z) = \frac{A_2}{z^2} + \frac{A_1}{z} + A_0, \ Y \in \mathbb{C}^2, \ A_j \in \mathrm{Mat}_2(\mathbb{C}). \quad (11).$$

Generalized Riemann–Hilbert Problem. Which monodromy-Stokes representations and formal normal forms at singularities are realized by (11)?

The results of the above-mentioned paper imply the following

Sufficient conditions of realizability by linear systems.

(i) The **eigenvalues** of each matrix A_2, A_0 (which are prescribed by the formal normal forms at 0 and at infinity) are **distinct.**

(ii) The **Monodromy-Stokes representation** is *irreducible:*
has no non-trivial invariant subspace.

Definition. A family of linear systems (11) with $A_k = A_k(t)$ is **isomonodromic,** if the **Monodromy - Stokes representation** remains **constant** (up to congugacy).

$$\frac{dY}{dz} = \left(\frac{A_2(t)}{z^2} + \frac{A_1(t)}{z} + A_0(t) \right) Y, \ \ Y \in \mathbb{C}^2, \ \ A_j(t) \in \mathrm{Mat}_2(\mathbb{C}). \qquad (12)$$

$t \in \mathcal{D} \subset \mathbb{C}^k$, $A_j(t)$ are **holomorphic.**

Theorem. A family (12) is **isomonodromic,**

<=> there exists a **meromorphic** differential matrix-valued 1-form

$$\Omega = \Omega_z dz + \Omega_t dt$$

such that

$$\Omega|_{\text{fixed } t} = \Omega_z dz = \left(\frac{A_2(t)}{z^2} + \frac{A_1(t)}{z} + A_0(t) \right) dz;$$

$$d\Omega = \Omega \wedge \Omega. \qquad (13)$$

<=> **Flat connection:** $\quad \frac{\partial Y}{\partial z} = \Omega_z Y; \ \ \frac{\partial Y}{\partial t} = \Omega_t Y.$

This theorem follows from results of Jimbo, Miwa, Ueno, Malgrange.

Necessary condition for being isomonodromic: $\mathrm{diag}\, A_1(t) \equiv const$.

Equivalent to **constance of eigenvalues of the formal monodromy** $C_0 C_1 M$.

Jimbo-Miwa-Ueno isomonodromic deformation: family of type

$$\frac{dY}{dz} = \left(-\frac{t}{z^2}\mathbf{A}(t) + \frac{1}{z}\mathbf{B}(t) + \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}\right) Y \tag{14}$$

Defined by matrix functions $\mathbf{B}(t) \in \mathrm{Mat}_2(\mathbb{C})$, $G(t) \in GL_2(\mathbb{C})$ such that

$$\mathbf{B}(t) = \begin{pmatrix} -b & * \\ * & 0 \end{pmatrix}, \qquad \widetilde{\mathbf{B}}(t) := G^{-1}(t)\mathbf{B}(t)G(t) = \begin{pmatrix} -b & * \\ * & 0 \end{pmatrix}.$$

Set

$$\mathbf{A}(t) := G(t)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} G^{-1}(t),$$

$$\Omega := \left(-\frac{t}{z^2}\mathbf{A}(t) + \frac{1}{z}\mathbf{B}(t) + \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}\right) dz + \frac{1}{z}\mathbf{A}(t)dt. \tag{15}$$

$$\Omega := \left(-\frac{t}{z^2}\mathbf{A}(t) + \frac{1}{z}\mathbf{B}(t) + \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}\right) dz + \frac{1}{z}\mathbf{A}(t)dt. \qquad (15)$$

$$\mathbf{B}(t) = \begin{pmatrix} -b & * \\ * & 0 \end{pmatrix}, \qquad \widetilde{\mathbf{B}}(t) := G^{-1}(t)\mathbf{B}(t)G(t) = \begin{pmatrix} -b & * \\ * & 0 \end{pmatrix},$$

$$\mathbf{A}(t) := G(t) \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} G^{-1}(t).$$

Set

$$y(t) := -\frac{\mathbf{B}_{12}(t)}{\mathbf{A}_{12}(t)}; \; w(\tau) := \frac{y(\tau^2)}{\tau}.$$

Theorem (Jimbo-Miwa-Ueno). Ω in (17) defines **isomonodromic family,**

$<=>$ the function $w(\tau)$ satisfies **Painlévé 3** equation:

$$w'' = \frac{(w')^2}{w} - \frac{w'}{\tau} + w^3 - 2b\frac{w^2}{\tau} - \frac{1}{w} + (2b-2)\frac{1}{\tau}. \qquad (16)$$

$$w'' = \frac{(w')^2}{w} - \frac{w'}{\tau} + w^3 - 2b\frac{w^2}{\tau} - \frac{1}{w} + (2b-2)\frac{1}{\tau}. \qquad (16)$$

Theorem

Every P3 equation of type (16) admits a one-parameter family of solutions

$$w(\tau) = \frac{d}{d\tau}\log u(\tau), \quad u(\tau) = s^b(C_1 J_b(\tau) + C_2 Y_b(\tau)).$$

Here $J_b(\tau)$ and $Y_b(\tau)$ are Bessel functions: two linearly independent solutions of Bessel equation

$$\frac{d^2y}{d\tau^2} + \frac{1}{\tau}\frac{dy}{d\tau} + (1 - \frac{b^2}{\tau^2})y = 0.$$

See R. Conte. *Painleve property: One century later.*
CRM series in mathematical physics, 1999:

A system (14):

$$\frac{dY}{dz} = \left(-\frac{t}{z^2}\mathbf{A}(t) + \frac{1}{z}\mathbf{B}(t) + \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}\right) Y \qquad (14)$$

with $\mathbf{B}_{12} \neq 0$ is of **Josephson type**
(i.e., obtained from a system coming from model of **Josephson effect** with $\mu \neq 0$ by conjugation by constant diagonal matrix),

$<=>$ $\mathbf{A}_{12} = \mathbf{A}_{21} = 0$
$<=>$ $y = -\frac{\mathbf{B}_{12}}{\mathbf{A}_{12}}$ has a **pole** at t,

$<=>$ $w(\tau) = \frac{y(\tau^2)}{\tau}$ has a **pole** at $\tau = \sqrt{t}$.

Take a **system (14)** coming from a model of Josephson effect, corresponding to a **constriction.** $<=>$ **Trivial Monodromy-Stokes representation.**

Consider its **Jimbo-Miwa-Ueno deformation**
given by **Bessel solution** $w(\tau)$ of P3.
Poles of $w(\tau)$ $=$
zeros of Bessel function $C_1 J_b(\tau) + C_2 Y_b(\tau)$: $\tau_0 = \tau$, $\tau_1, \tau_2, \dots$.
$t_k = \tau_k^2$.

Systems (14) with $t = t_k$ correspond to **constrictions** $(B_k, A_k; \omega_k)$
in model of Josephson effect,

with $\ell = \frac{B_k}{\omega_k} = b$.

Summary

The methods of **real** dynamical systems on torus
and linear **complex** differential equations
have led to solutions of some problems
on phase-lock areas in a model of Josephson effect
and double confluent Heun equations.

The problems of **phase-lock areas** in a model of Josephson effect
have led to the study of an interesting family of dynamical systems
that is transformed to a family of
classical complex linear equations on the Riemann sphere
well-known as **double confluent Heun equations.**

We describe how relation to monodromy
and isomonodromic deformations had arisen.

Altogether, this opens new approaches to solution of physical problems.

V.M. Buchstaber, O.V. Karpov, S.I.Tertychnyi. *Electrodynamic properties of a Josephson junction biased with a sequence of δ-function pulses.* J. Exper. Theoret. Phys., **93** (2001), No.6, 1280–1287.

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A.A. Glutsyuk, V.A. Kleptsyn, D.A. Filimonov, I.V. Schurov. *On the adjacency quantization in an equation modeling the Josephson effect.* Funct. Analysis and its Appl. **45** (2011), No. 3, 192–203.

Buchstaber, V.M.; Tertychnyi, S.I. *Holomorphic solutions of the double confluent Heun equation associated with the RSJ model of the Josephson junction.* Theoret. and Math. Phys., **182:3** (2015), 329–355.

V.M.Buchstaber, A.A.Glutsyuk. *On monodromy eigenfunctions of Heun equations and boundaries of phase-lock areas in a model of overdamped Josephson effect.* Proc. Steklov Inst. of Math., **297** (2017). Volume dedicated to D.V.Anosov.

Glutsyuk, A.A. *On constrictions of phase-lock areas in model of overdamped Josephson effect and transition matrix of the double-donfluent Heun equation.* J. Dyn. Control Syst. 25:3 (2019), 323–349.