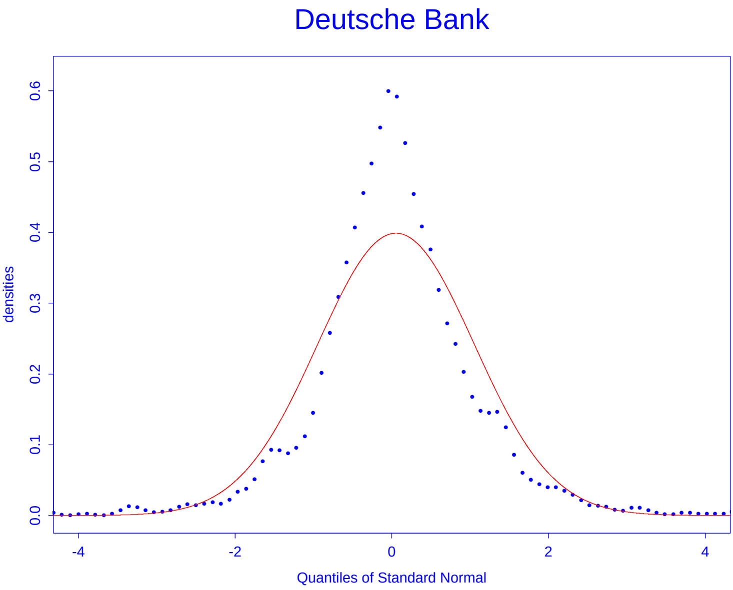
Lévy driven financial models

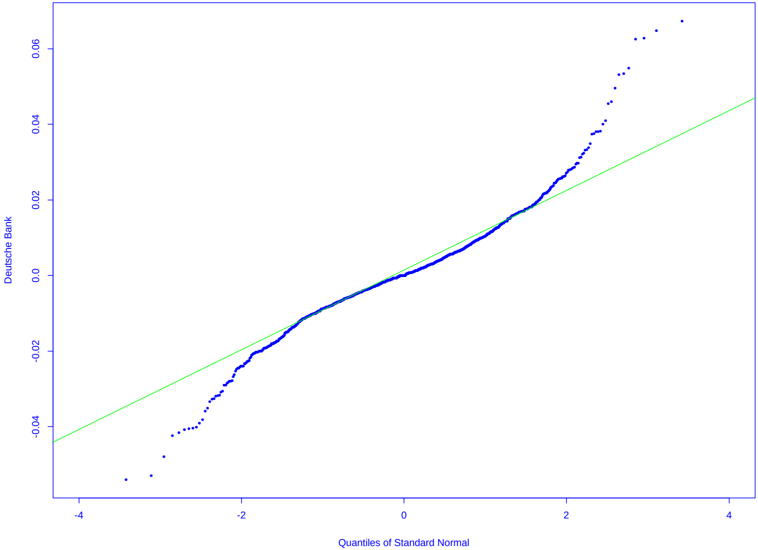
Ernst Eberlein

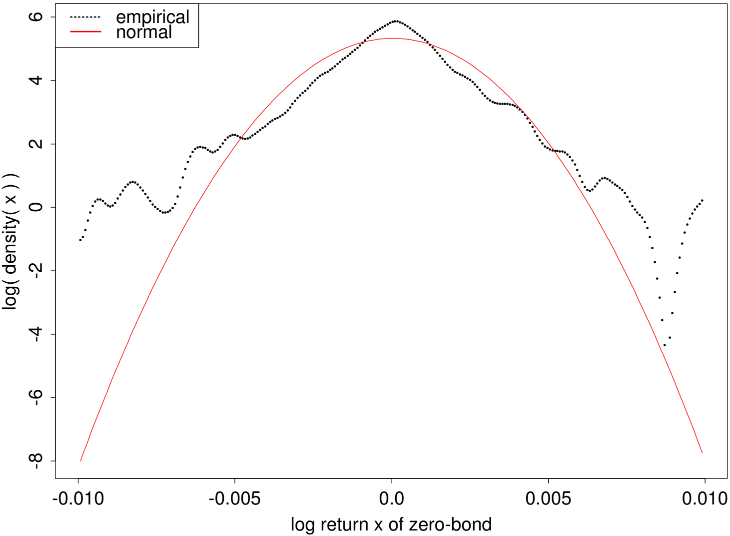
Department of Mathematical Stochastics
and
Freiburg Center for Data Analysis and Modeling
University of Freiburg

Visions in Stochastics
Steklov Mathematical Institute, Moscow   November 1–3, 2010

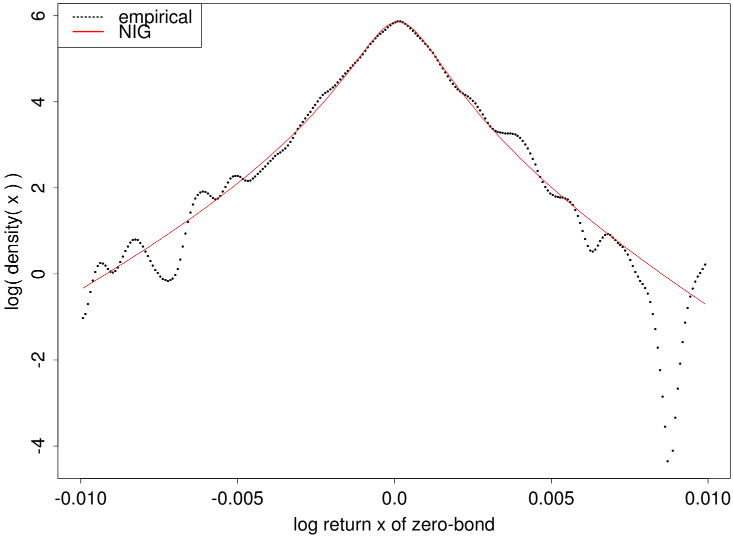

Deutsche Bank
densities
0.0
0.1
0.2
0.3
0.4
0.5
0.6
-4
-2
0
2
4
Quantiles of Standard Normal

QQ plots for Deutsche Bank





empirical
normal
6
4
2
0
-2
-4
-6
-8
log(density(x))
-0.010
-0.005
0.0
0.005
0.010
log return x of zero-bond



empirical
NIG
6
4
2
0
-2
-4
log(density(x))
-0.010
-0.005
0.0
0.005
0.010
log return x of zero-bond

Lévy processes

$L = (L_t)_{t \geq 0}$ process with stationary and independent increments

càdlàg paths: right-continuous with left limits

canonical representation

$$L_t = bt + \sqrt{c} W_t + Z_t + \sum_{s \leq t} \Delta L_s \mathbb{1}_{\{|\Delta L_s| > 1\}}$$

b and $c \geq 0$ real numbers, $(W_t)_{t \geq 0}$ standard Brownian motion

$(Z_t)_{t \geq 0}$ purely discontinuous martingale independent of $(W_t)_{t \geq 0}$

$\Delta L_s = L_s - L_{s-}$ jump at time $s > 0$

Local characteristics of a Lévy process (1)

Triplet of local characteristics of $(L_t)_{t\geq 0}$: (b, c, F)

$$L_t = bt + \sqrt{c}W_t + \int_0^t \int_{\mathbb{R}} x\mathbb{1}_{\{|x|\leq 1\}} \left(\mu^L(\mathrm{d}s, \mathrm{d}x) - \mathrm{d}sF(\mathrm{d}x)\right) + \int_0^t \int_{\mathbb{R}} x\mathbb{1}_{\{|x|>1\}} \mu^L(\mathrm{d}s, \mathrm{d}x)$$

$\nu = \mathcal{L}(L_1)$ is infinitely divisible

$\mathcal{L}(L_1) = \mathcal{L}(L_{1/n}) * \cdots * \mathcal{L}(L_{1/n})$

Local characteristics of a Lévy process (2)

Fourier transform in Lévy –Khintchine form

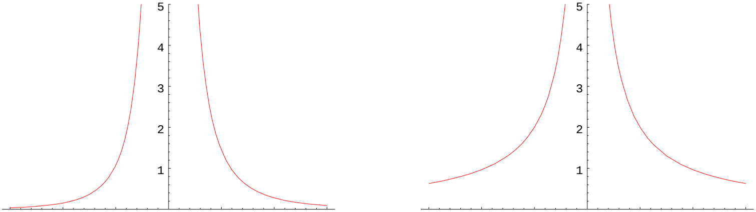
$$E[\exp(\mathrm{i}uL_1)] = \exp\left[\mathrm{i}ub - \frac{1}{2}u^2c + \int_{\mathbb{R}} \left(e^{\mathrm{i}ux} - 1 - \mathrm{i}ux\mathbb{1}_{\{|x|\leq 1\}}\right)F(\mathrm{d}x)\right]$$
$$= \exp(\psi(u))$$

F Lévy measure: $\int_{\mathbb{R}} \min(1, x^2)F(\mathrm{d}x) < \infty$

pricing of derivatives: $E[f(L_T)]$

uses $E[\exp(\mathrm{i}uL_T)] = \exp(\psi(u))^T$

Examples of Lévy measures



The density of the Lévy measure of the normal inverse Gaussian (left) and the α-stable process.

Generalized hyperbolic distributions

(O.E. Barndorff-Nielsen (1977))

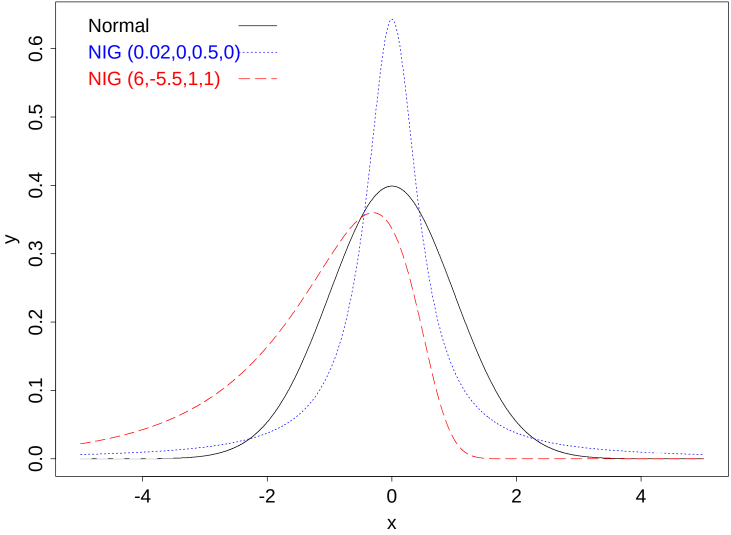
Density: $$d_{GH}(x) = a(\lambda,\alpha,\beta,\delta)\left(\delta^2 + (x-\mu)^2\right)^{(\lambda-1/2)/2} \times K_{\lambda-1/2}\left(a\sqrt{\delta^2+(x-\mu)^2}\right)\exp(\beta(x-\mu))$$

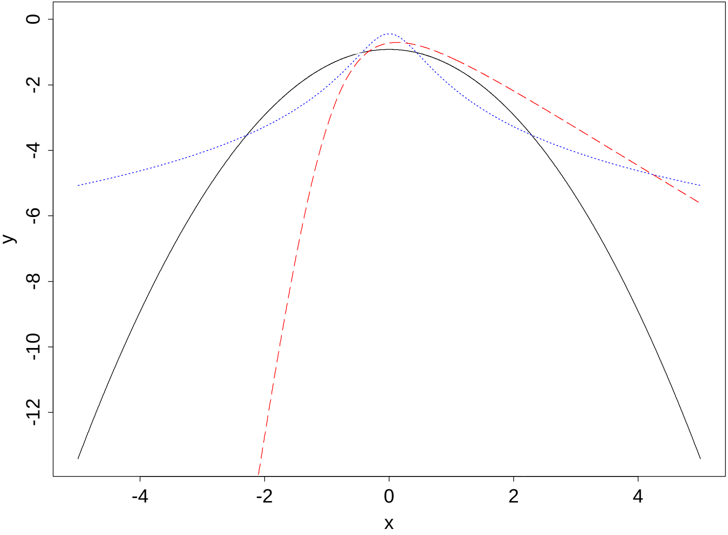
$$a(\lambda,\alpha,\beta,\delta) = \frac{\left(\alpha^2-\beta^2\right)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^{\lambda}K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}$$

K_λ modified Bessel function of the third kind with index λ

Parameters:

$\lambda \in \mathbb{R}$	Class parameter	$\mu \in \mathbb{R}$	Location
$\alpha > 0$	Shape	$\delta > 0$	Scale parameter
β with $0 \le \lvert\beta\rvert < \alpha$	Skewness		(Volatility)


Normal
NIG (0.02,0,0.5,0)
NIG (6,-5.5,1,1)
y
0.0
0.1
0.2
0.3
0.4
0.5
0.6
-4
-2
0
2
4
x



0
-2
-4
-6
-8
-10
-12
y
-4
-2
0
2
4
x

Special cases

Hyperbolic

Normal inverse Gaussian (NIG)

Normal reciprocal inverse Gaussian (NRIG)

Variance gamma

Student t (limiting case)

Cauchy (limiting case)

Skewed Laplace

Normal (limiting case)

Generalized inverse Gaussian (limiting case)

Hyperbolic distribution ($\lambda = 1$)

$$d_H(x) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta K_1\left(\delta\sqrt{\alpha^2 - \beta^2}\right)} \exp\left(-\alpha\sqrt{\delta^2 + (x-\mu)^2} + \beta(x-\mu)\right)$$

Eb., Keller (1995); Eb., Keller, Prause (1998)

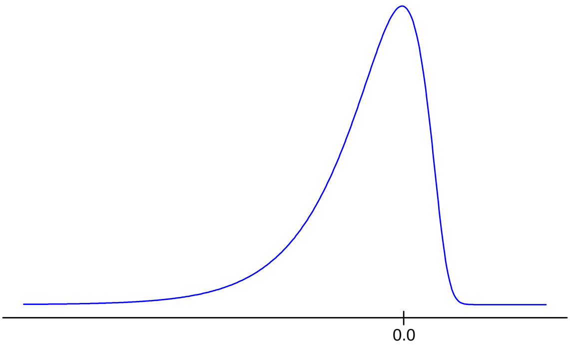
Normal inverse Gaussian (NIG) ($\lambda = -1/2$)

$$d_{NIG}(x) = \frac{\alpha\delta}{\pi} \exp\left(\delta\sqrt{\alpha^2 - \beta^2} + \beta(x-\mu)\right) \frac{K_1(\alpha g_\delta(x-\mu))}{g_\delta(x-\mu)}$$

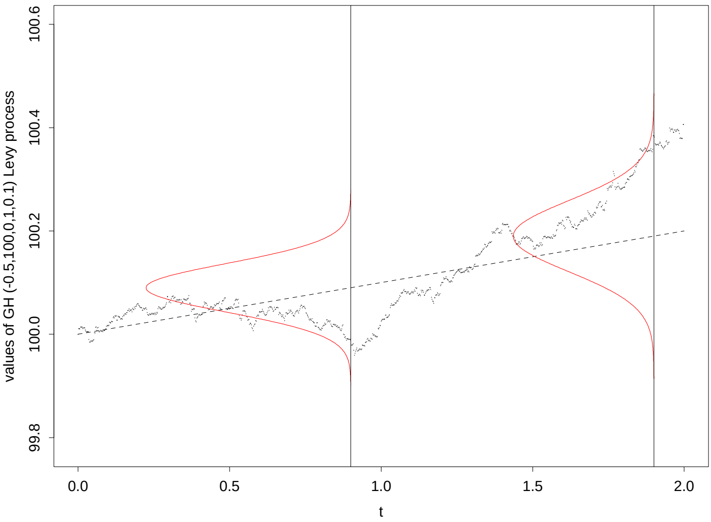
where $g_\delta(x) = \sqrt{\delta^2 + x^2}$

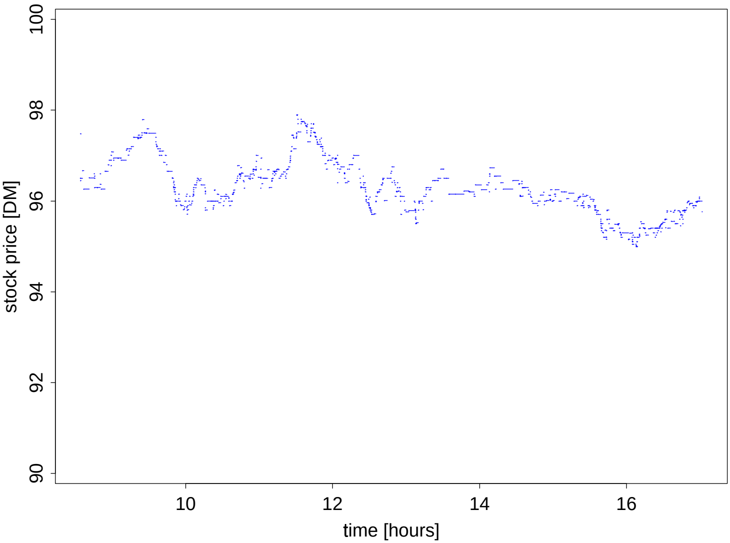
O.E. Barndorff-Nielsen (1998)

Credit profit and loss distribution



GH Levy process with marginal densities




stock price [DM]
100
98
96
94
92
90
10
12
14
16
time [hours]

Financial modeling

Stock prices and indices: geometric Brownian motion (Samuelson 1965)

$$dS_t = \mu S_t \, dt + \sigma S_t dW_t$$

solution

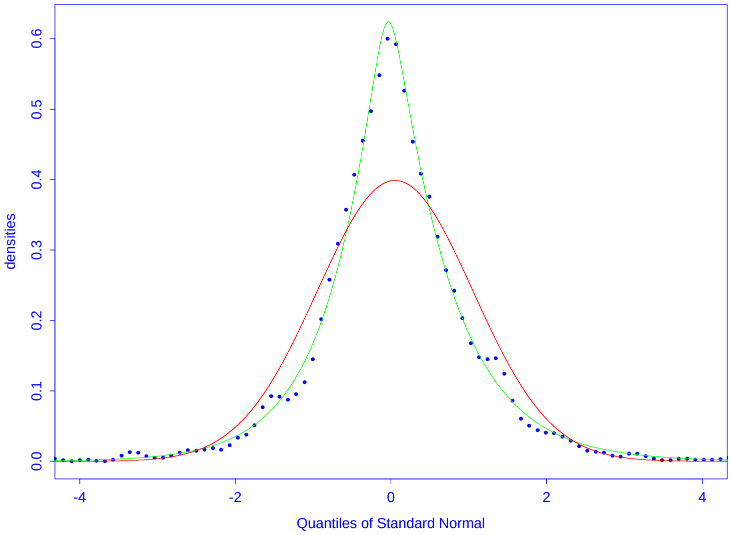
$$S_t = S_0 \exp\left(\sigma W_t + \left(\mu - \frac{\sigma^2}{2}\right) t\right)$$

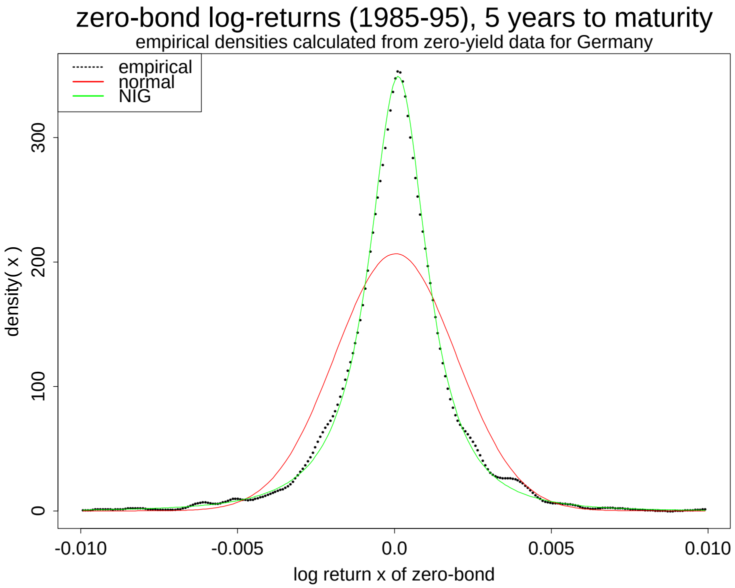
Log returns: $\log S_{t+1} - \log S_t \sim N\left(\mu - \frac{\sigma^2}{2}, \sigma^2\right)$

Correct return distributions: key ingredient

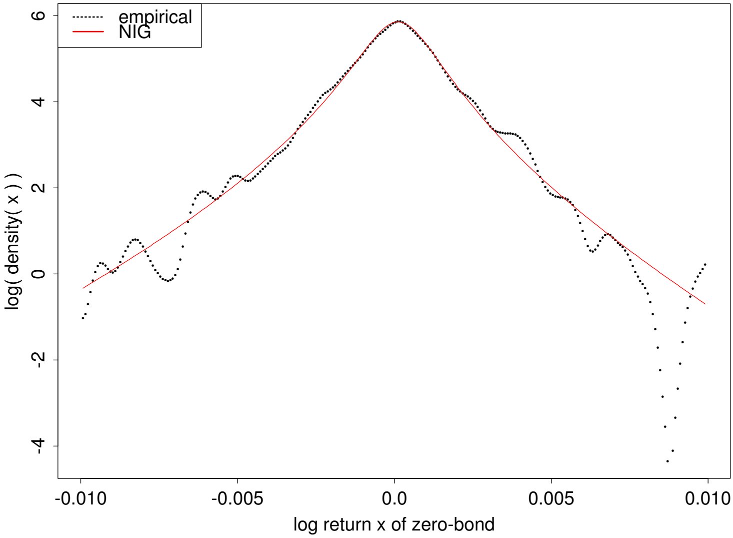
Consistency of the model along different time grids

Deutsche Bank





zero-bond log-returns (1985-95), 5 years to maturity
empirical densities calculated from zero-yield data for Germany
empirical
normal
NIG
density(x)
0
100
200
300
-0.010
-0.005
0.0
0.005
0.010
log return x of zero-bond



empirical
NIG
6
4
2
0
-2
-4
log(density(x))
-0.010
-0.005
0.0
0.005
0.010
log return x of zero-bond

Exponential Lévy model

$$S_t = S_0 \exp(L_t)$$

$L = (L_t)_{t \geq 0}$ Lévy process with $\mathcal{L}(L_1) = \nu$

Along a time grid with span 1: exact log returns

Alternative description by a stochastic differential equation

$$\mathrm{d}S_t = S_{t-}\left(\mathrm{d}L_t + \frac{c}{2}\,\mathrm{d}t + \int_{\mathbb{R}} (e^x - 1 - x)\mu^L(\mathrm{d}t, \mathrm{d}x)\right)$$

can be written as

$$\mathrm{d}S_t = S_{t-}\mathrm{d}\widetilde{L}_t$$

where $(\widetilde{L}_t)_{t \geq 0}$ is a Lévy process with jumps > -1

Martingality of exponential Lévy models

$$S_t = S_0 \exp(L_t)$$

Pricing of derivatives: martingale model

Necessary assumption: $E[S_t] = S_0 E[\exp(L_t)] < \infty$

This excludes a priori the class of stable processes in general

$E[\exp(L_t)] < \infty \quad \Rightarrow \quad E[L_t] < \infty$

consequently $L_t = bt + \sqrt{c}W_t + \int_0^t \int_{\mathbb{R}} x(\mu^L(\mathrm{d}s, \mathrm{d}x) - \mathrm{d}sF(\mathrm{d}x))$

$S_t = S_0 \exp(L_t)$ is a martingale if $b = -\frac{c}{2} - \int_{\mathbb{R}} (e^x - 1 - x) F(\mathrm{d}x)$

Use either Itô's formula or verify that $M_t = \dfrac{\exp(L_t)}{E[\exp(L_t)]}$ is a martingale

Equivalent martingale measures (EMMs)

$(e^{-rt}S_t)_{t\geq 0}$ has to be a martingale

In general large set of EMMs: market is incomplete

Characterization of the set of all EMMs (Eberlein and Jacod (1997))

Characterization of those EMMs under which L is again a Lévy process

The range of call option prices under all EMMs spans the whole no-arbitrage interval (Eberlein and Jacod (1997))

$$\left((S_0 - Ke^{-rT})^+, S_0\right)$$

Criteria to choose an EMM: Esscher transform, minimal distance MM, minimal entropy MM, utility functions, ...

$\rightarrow$ whole industry

Pricing of derivatives

$f(L_T)$ payoff of the option at maturity T

$f(x) = (x - K)^+$ European call option

$f(x) = (K - x)^+$ European put option

Similarly: digitals, quantos, asset-or-nothing, power options, ...

Given a specific martingale measure (calibration to market data)

$$V = E\left[e^{-rT} f(X_T)\right]$$

Explicit formula for European call

$$V = S_0 \int_\gamma^\infty d_{GH}^{*T}(x; \theta+1)\, \mathrm{d}x - e^{-rT} K \int_\gamma^\infty d_{GH}^{*T}(x; \theta)\, \mathrm{d}x$$

where $\gamma = \ln(K/S_0)$ and d_{GH}^{*T} *GH*-density under risk-neutral measure

Supremum and infimum processes

Let $X = (X_t)_{0 \leq t \leq T}$ be a stochastic process. Denote by

$$\overline{X}_t = \sup_{0 \leq u \leq t} X_u \quad \text{and} \quad \underline{X}_t = \inf_{0 \leq u \leq t} X_u$$

the supremum and infimum process of X respectively. Since the exponential function is monotone and increasing

$$\overline{S}_T = \sup_{0 \leq t \leq T} S_t = \sup_{0 \leq t \leq T} \left(S_0 \mathrm{e}^{L_t} \right) = S_0 \mathrm{e}^{\sup_{0 \leq t \leq T} L_t} = S_0 \mathrm{e}^{\overline{L}_T}. \tag{1}$$

Similarly

$$\underline{S}_T = S_0 \mathrm{e}^{\underline{L}_T}. \tag{2}$$

Valuation formulas – payoff functional

We want to price an option with payoff $\Phi(S_t, 0 \leq t \leq T)$, where Φ is a measurable, non-negative functional.

Separation of payoff function from the underlying process:

Example

Fixed strike lookback option

$$(\overline{S}_T - K)^+ = (S_0 \, \mathrm{e}^{\overline{L}_T} - K)^+ = (\mathrm{e}^{\overline{L}_T + \log S_0} - K)$$

1. The *payoff function* is an arbitrary function $f : \mathbb{R} \to \mathbb{R}_+$; for example $f(x) = (\mathrm{e}^x - K)^+$ or $f(x) = \mathbb{1}_{\{\mathrm{e}^x > B\}}$, for $K, B \in \mathbb{R}_+$.
2. The *underlying process* denoted by X, can be the log-asset price process or the supremum/infimum or an average of the log-asset price process (e.g. $X = L$ or $X = \overline{L}$).

Valuation formulas

Consider the option price as a function of S_0 or better of $s = -\log S_0$

X driving process ($X = L, \overline{L}, \underline{L}$, etc.)

$\Rightarrow$ $$\Phi(S_0\, \mathrm{e}^{L_t}, 0 \leq t \leq T) = f(X_T - s)$$

Time-0 price of the option (assuming $r \equiv 0$)

$$\mathbb{V}_f(X; s) = E\big[\Phi(S_t, 0 \leq t \leq T)\big] = E[f(X_T - s)]$$

Valuation formulas based on Fourier and Laplace transforms

Carr and Madan (1999) plain vanilla options

Raible (2000) general payoffs, Lebesgue densities

Borovkov and Novikov (2002) plain vanilla and lookback options

In these approaches: Some sort of continuity assumption (payoff or random variable)

Valuation formulas – assumptions

M_{X_T} moment generating function of X_T

$g(x) = \mathrm{e}^{-Rx} f(x)$ (for some $R \in \mathbb{R}$) dampened payoff function

$L^1_{bc}(\mathbb{R})$ bounded, continuous functions in $L^1(\mathbb{R})$

Assumptions

(C1) $g \in L^1_{bc}(\mathbb{R})$

(C2) $M_{X_T}(R)$ *exists*

(C3) $\widehat{g} \in L^1(\mathbb{R})$

Valuation formulas

Theorem

Assume that (C1)–(C3) are in force. Then, the price $\mathbb{V}_f(X; s)$ of an option on $S = (S_t)_{0 \leq t \leq T}$ with payoff $f(X_T)$ is given by

$$\mathbb{V}_f(X; s) = \frac{\mathrm{e}^{-Rs}}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{ius} \varphi_{X_T}(-u - iR) \widehat{f}(u + iR) \mathrm{d}u,$$

where φ_{X_T} denotes the extended characteristic function of X_T and $\widehat{f}$ denotes the Fourier transform of f.

Non-path-dependent options

European option on an asset with price process $S_t = e^{L_t}$

Examples: call, put, digitals, asset-or-nothing, double digitals, self-quanto options

$\longrightarrow \quad X_T \equiv L_T, \quad$ i.e. we need φ_{L_T}

Generalized hyperbolic model (GH model): Eberlein, Keller (1995), Eberlein, Keller, Prause (1998), Eberlein (2001)

$$\varphi_{L_1}(u) = e^{iu\mu}\left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iu)^2}\right)^{\lambda/2} \frac{K_\lambda\left(\delta\sqrt{\alpha^2 - (\beta + iu)^2}\,\right)}{K_\lambda\left(\delta\sqrt{\alpha^2 - \beta^2}\right)}$$

$$I_2 = (-\alpha - \beta, \alpha - \beta)$$

$$\varphi_{L_T}(u) = \left(\varphi_{L_1}(u)\right)^T$$

similar: NIG, CGMY, Meixner

Non-path-dependent options II

Stochastic volatility Lévy models: Carr, Geman, Madan, Yor (2003)
Eberlein, Kallsen, Kristen (2003)

Stochastic clock $Y_t = \int_0^t y_s \, \mathrm{d}s \quad (y_s > 0)$

e.g. CIR process

$$\mathrm{d}y_t = K(\eta - y_t)\,\mathrm{d}t + \lambda y_t^{1/2}\,\mathrm{d}W_t$$

Define for a pure jump Lévy process $X = (X_t)_{t \geq 0}$

$$H_t = X_{Y_t} \quad (0 \leq t \leq T)$$

Then

$$\varphi_{H_t}(u) = \frac{\varphi_{Y_t}(-i\varphi_{X_t}(u))}{(\varphi_{Y_t}(-iu\varphi_{X_t}(-i)))^{iu}}$$

Classification of option types

Lévy model $S_t = S_0 e^{L_t}$

payoff	payoff function	distributional properties
$(S_T - K)^+$ call	$f(x) = (e^x - K)^+$	P_{L_T} usually has a density
$\mathbb{1}_{\{S_T > B\}}$ digital	$f(x) = \mathbb{1}_{\{e^x > B\}}$	–″–
$(\overline{S}_T - K)^+$ lookback	$f(x) = (e^x - K)^+$	density of $P_{\overline{L}_T}$?
$\mathbb{1}_{\{\overline{S}_T > B\}}$ digital barrier = one touch	$f(x) = \mathbb{1}_{\{e^x > B\}}$	–″–

Valuation formula for the last case

Payoff function f maybe discontinuous
P_{X_T} does not necessarily possess a Lebesgue density

Assumption

(D1) $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$

(D2) $M_{X_T}(R)$ *exists*

Theorem

Assume (D1)–(D2) then

$$\mathbb{V}_f(X; s) = \lim_{A \to \infty} \frac{e^{-Rs}}{2\pi} \int_{-A}^{A} \mathrm{e}^{-ius} \varphi_{X_T}(u - iR) \widehat{f}(iR - u) \, \mathrm{d}u \tag{3}$$

if $\mathbb{V}_f(X; \cdot)$ *is of bounded variation in a neighborhood of* s *and* $\mathbb{V}_f(X; \cdot)$ *is continuous at* s*.*

Sensitivities – Greeks

$$\mathbb{W}_f(X;S_0) = \frac{1}{2\pi}\int_{\mathbb{R}} S_0^{R-\mathrm{i}u} M_{X_T}(R-\mathrm{i}u)\widehat{f}(u+\mathrm{i}R)\,\mathrm{d}u$$

Delta of an option

$$\Delta_f(X;S_0) = \frac{\partial \mathbb{W}(X;S_0)}{\partial S_0} = \frac{1}{2\pi}\int_{\mathbb{R}} S_0^{R-1-\mathrm{i}u} M_{X_T}(R-\mathrm{i}u)\frac{\widehat{f}(u+\mathrm{i}R)}{(R-\mathrm{i}u)^{-1}}\,\mathrm{d}u$$

Gamma of an option

$$\Gamma_f(X;S_0) = \frac{\partial^2 \mathbb{W}_f(X;S_0)}{\partial^2 S_0} = \frac{1}{2\pi}\int_{\mathbb{R}} S_0^{R-2-\mathrm{i}u} \frac{M_{X_T}(R-\mathrm{i}u)\widehat{f}(u+\mathrm{i}R)}{(R-1-\mathrm{i}u)^{-1}(R-\mathrm{i}u)^{-1}}\,\mathrm{d}u$$

Standard risk measure: Value at Risk

$$P[X_t < u_\alpha] \; = \; \alpha$$

α-quantile of return distribution

$$\mathrm{VaR}(\alpha) \; = \; S_0 - S_0 \exp(u_\alpha)$$

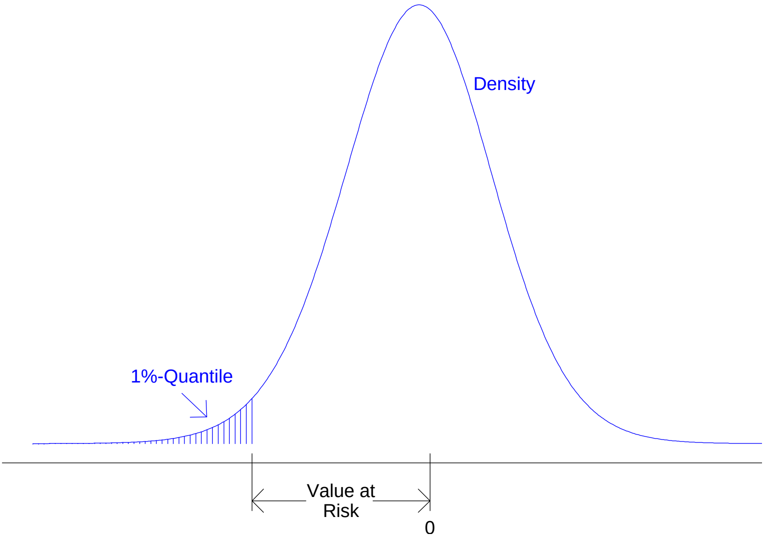
Functional value at risk

$$\alpha \; \longrightarrow \; \mathrm{VaR}(\alpha)$$

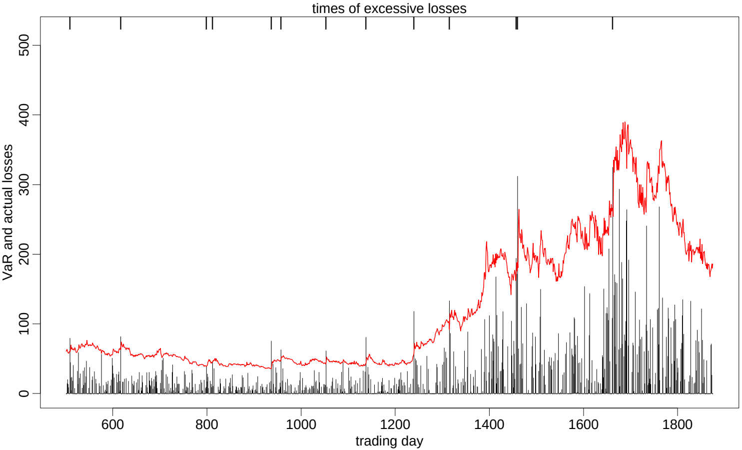
Improvement: Shortfall measure

$$\mathrm{Shortfall}(\alpha, t) \; = E[S_0 - S_0 \exp(X_t) \mid X_t < u_\alpha]$$

Profit-and-Loss Distribution



implied volatility



Interest rate models

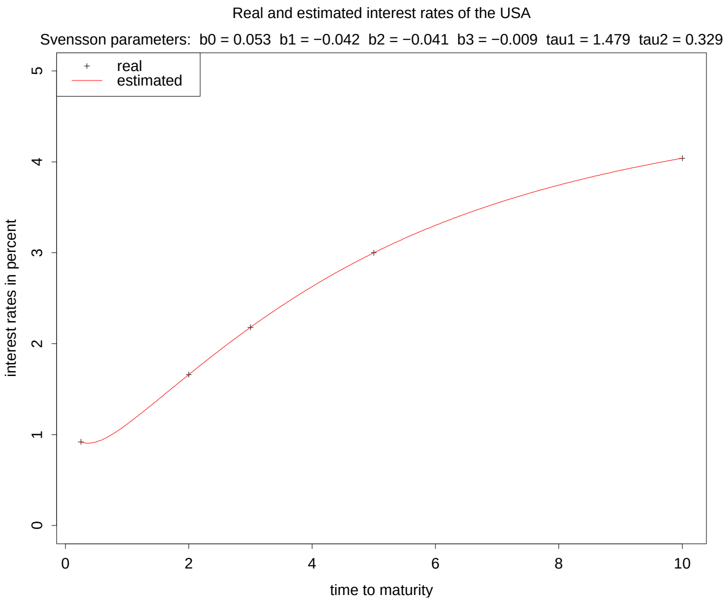
Interest rate models should be able to reproduce

- the observable term structures of interest rates,
- market prices of interest rate derivatives (caps, floors, swaptions)

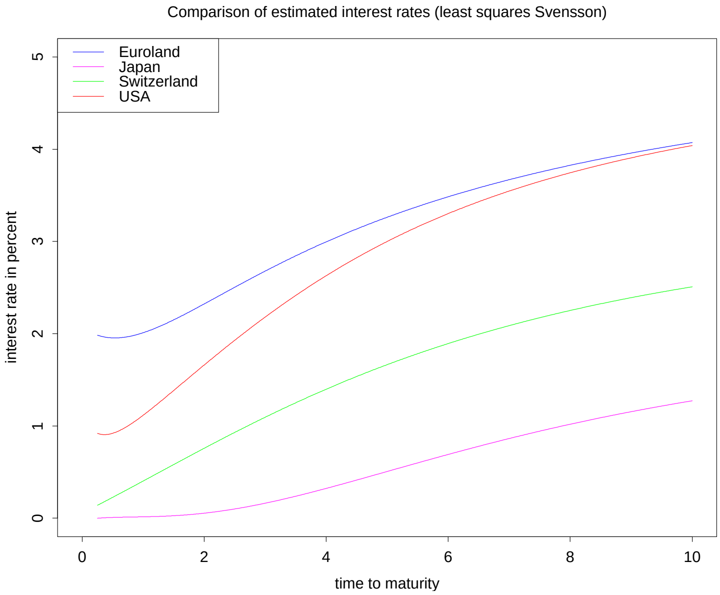
but they should also be

- analytically tractable.

Idea: Use an HJM-type model driven by a (possibly non-homogeneous) Lévy process



Termstructure, February 17, 2004



Termstructure, February 17, 2004

Short rate dynamics

Merton (1970)	$dr_t = \theta\, dt + \sigma\, dB_t$
Vasiček (1977)	$dr_t = k(\theta - r_t)\, dt + \sigma\, dB_t$
Dothan (1978)	$dr_t = ar_t dt + \sigma r_t\, dB_t$
Brennan-Schwartz (1979)	$dr_t = (\theta(t) - ar_t)\, dt + \sigma r_t\, dB_t$
Constantinides-Ingersoll (1984)	$dr_t = \sigma r_t^{3/2}\, dB_t$
Cox-Ingersoll-Ross (1985)	$dr_t = k(\theta - r_t)\, dt + \sigma\sqrt{r_t}\, dB_t$
Ho-Lee (1986)	$dr_t = \theta(t)\, dt + \sigma\, dB_t$
Black-Derman-Toy (1990)	$dr_t = r_t\left(\theta(t) - a\ln r_t + \frac{1}{2}\sigma^2(t)\right)\, dt + \sigma(t) r_t\, dB_t$
Hull-White (1990) (extended CIR)	$dr_t = k\left(\theta(t) - r_t\right)\, dt + \sigma(t)\sqrt{r_t}\, dB_t$
Sandmann-Sondermann (1993)	$dr_t = (1 - e^{-r_t})\left[\left(\theta(t) - \frac{1}{2}(1 - e^{-r_t})\sigma^2\right) dt + \sigma\, dB_t\right]$

Classical modeling of the dynamics of term structures

$B(t,T)$ price at time $t \in [0,T]$ of a default-free zero coupon bond with maturity $T \in [0,T^*]$ $(B(T,T)=1)$

$f(t,T)$ instantaneous forward rate: $B(t,T) = \exp\left(-\int_t^T f(t,u)\,\mathrm{d}u\right)$

Heath, Jarrow, Morton (HJM) framework

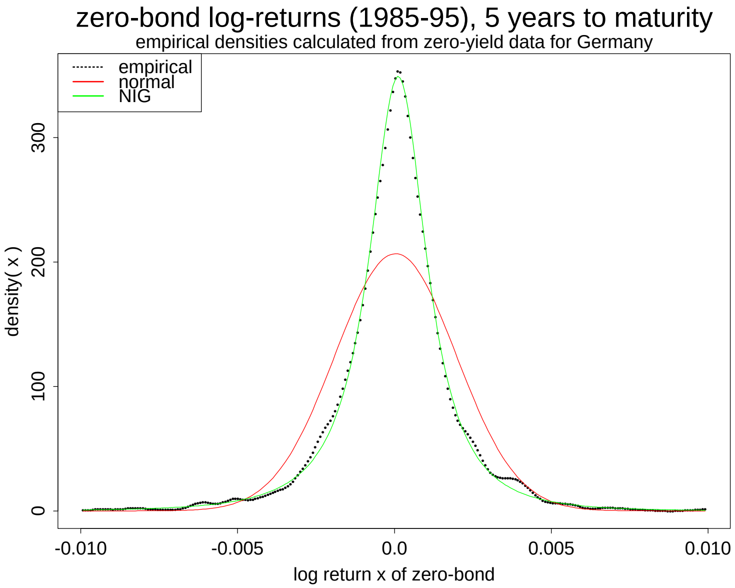
$$\mathrm{d}f(t,T) = \alpha(t,T)\,\mathrm{d}t + \sigma(t,T)^\top \mathrm{d}W_t$$

$(W_t)_{t\geq 0}$ d-dimensional Brownian motion
$\sigma(t,T)$ volatility structure (e.g. Vasiček)

Under the risk-neutral measure

$$B(t,T) = B(0,T)\exp\left[\int_0^t r(s)\,\mathrm{d}s - \frac{1}{2}\int_0^t |\sigma^*(s,T)|^2\,\mathrm{d}s + \int_0^t \sigma^*(s,T)^\top \mathrm{d}W_s\right]$$

where $r(t) = f(t,t)$ short rate


zero-bond log-returns (1985-95), 5 years to maturity
empirical densities calculated from zero-yield data for Germany
empirical
normal
NIG
density(x)
0
100
200
300
-0.010
-0.005
0.0
0.005
0.010
log return x of zero-bond

The driving process

$L = (L^1, \dots, L^d)$ is a d-dimensional time-inhomogeneous Lévy process, i.e. L has independent increments and the law of L_t is given by the characteristic function

$$\mathbb{E}[\exp(i\langle u, L_t\rangle)] = \exp \int_0^t \theta_s(iu)\,\mathrm{d}s \qquad \text{with}$$

$$\theta_s(z) = \langle z, b_s\rangle + \frac{1}{2}\langle z, c_s z\rangle + \int_{\mathbb{R}^d} \left(e^{\langle z,x\rangle} - 1 - \langle z, x\rangle\right) F_s(\mathrm{d}x)$$

where $b_t \in \mathbb{R}^d$, c_t is a symmetric nonnegative-definite $d \times d$-matrix and F_t is a Lévy measure

Integrability: $$\int_0^{T^*} \left(|b_s| + |c_s| + \int_{\{|x|\le 1\}} x^2 F_s(\mathrm{d}x)\right) \mathrm{d}s < \infty$$

$$\int_0^{T^*} \int_{\{|x|>1\}} \exp(ux) F_s(\mathrm{d}x)\,\mathrm{d}s < \infty \quad \text{for } |u| \le M$$

Description in terms of modern stochastic analysis

$L = (L_t)$ is a special semimartingale with canonical representation

$$L_t = \int_0^t b_s \,\mathrm{d}s + \int_0^t c_s^{1/2} \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^d} x(\mu^L - \nu)(\mathrm{d}s, \mathrm{d}x)$$

and characteristics

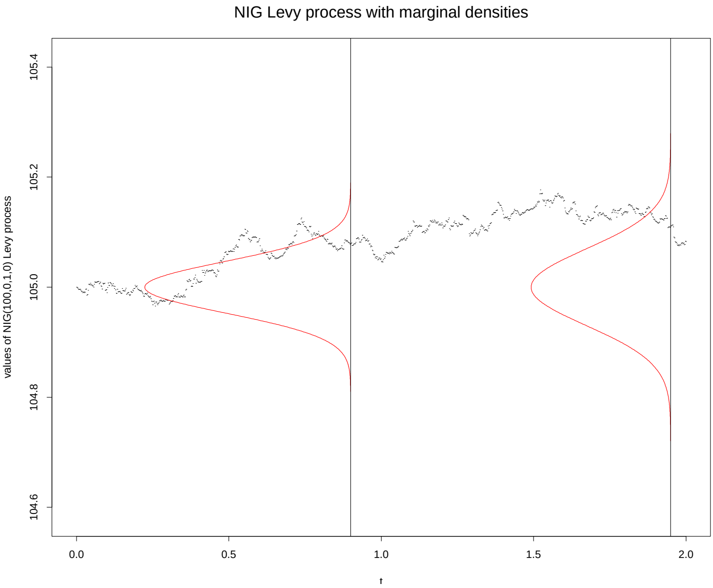
$$A_t = \int_0^t b_s \,\mathrm{d}s, \qquad C_t = \int_0^t c_s \,\mathrm{d}s, \qquad \nu(\mathrm{d}s, \mathrm{d}x) = F_s(\mathrm{d}x)\,\mathrm{d}s$$

$W = (W_t)$ is a standard d-dimensional Brownian motion,

μ^L the random measure of jumps of L and ν is the compensator of μ^L

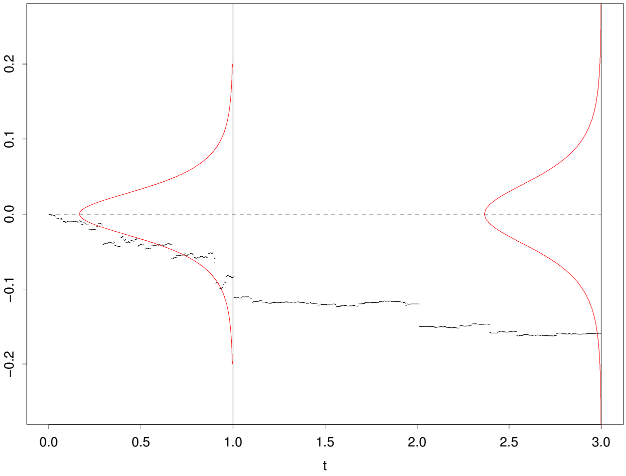
L is also called a process with independent increments and absolutely continuous characteristics (PIIAC)

Simulation of a GH Lévy motion



Simulation of a Lévy process

NIG(10,0,0.100,0) on [0,1]
NIG(10,0,0.025,0) on [1,3]



Lévy forward rate approach

Eberlein, Raible (1999), Eberlein, Özkan (2003),
Eberlein, Jacod, Raible (2005), Eberlein, Kluge (2006)

$$\mathrm{d}f(t,T) = \partial_2 A(t,T)\,\mathrm{d}t - \partial_2 \Sigma(t,T)\,\mathrm{d}L_t \qquad (0 \le t \le T \le T^*)$$

Σ and A are deterministic functions with values in $\mathbb{R}^d$ and $\mathbb{R}$ respectively whose paths are continuously differentiable in the second variable.

The volatility structure is bounded $0 \le \Sigma^i(t,T) \le M \quad (i \in \{1,\ldots,d\})$.

Furthermore, $\Sigma(t,T) \neq 0$ for $t < T$ and $\Sigma(T,T) = 0$ for $T \in [0,T^*]$.

The drift condition $A(t,T) = \theta_s(\Sigma(t,T))$ holds.

Implications

Savings account and default-free zero coupon bond prices are given by

$$B_t = \frac{1}{B(0,t)} \exp\left(\int_0^t \theta_s(\Sigma(s,t))\,\mathrm{d}s - \int_0^t \Sigma(s,t)\,\mathrm{d}L_s \right) \quad \text{and}$$

$$B(t,T) = B(0,T) B_t \exp\left(-\int_0^t \theta_s(\Sigma(s,T))\,\mathrm{d}s + \int_0^t \Sigma(s,T)\,\mathrm{d}L_s \right).$$

Bond prices, once discounted by the savings account, are martingales.

In case $d = 1$, the martingale measure is unique (see Eberlein, Jacod, and Raible (2004)).

Key tool

$L = (L^1, \dots, L^d)$ d-dimensional time-inhomogeneous Lévy process

$$\mathbb{E}[\exp(i\langle u, L_t\rangle)] = \exp \int_0^t \theta_s(iu)\, \mathrm{d}s \qquad \text{where}$$

$$\theta_s(z) = \langle z, b_s\rangle + \frac{1}{2}\langle z, c_s z\rangle + \int_{\mathbb{R}^d} \left(e^{\langle z,x\rangle} - 1 - \langle z, x\rangle\right) F_s(\mathrm{d}x)$$

in case L is a (time-homogeneous) Lévy process, $\theta_s = \theta$ is the cumulant (log-moment generating function) of L_1.

Proposition Eberlein, Raible (1999)

Suppose $f : \mathbb{R}_+ \to \mathbb{C}^d$ is a continuous function such that $|\mathcal{R}(f^i(x))| \leq M$ for all $i \in \{1, \dots, d\}$ and $x \in \mathbb{R}_+$, then

$$\mathbb{E}\left[\exp\left(\int_0^t f(s) dL_s\right)\right] = \exp\left(\int_0^t \theta_s(f(s)) ds\right)$$

Take $f(s) = \sum(s, T)$ for some $T \in [0, T^*]$

Pricing of European options

$$B(t,T) = B(0,T)\exp\left[\int_0^t (r(s) + \theta_s(\Sigma(s,T)))\,\mathrm{d}s + \int_0^t \Sigma(s,T)\mathrm{d}L_s\right]$$

where $r(t) = f(t,t)$ short rate

$V(0,t,T,w)$ time-0-price of a European option with maturity t and payoff $w(B(t,T),K)$

$$V(0,t,T,w) = \mathbb{E}_{\mathbb{P}^*}[B_t^{-1} w(B(t,T),K)]$$

Volatility structures

$$\Sigma(t,T) = \frac{\widehat{\sigma}}{a}(1 - \exp(-a(T-t))) \qquad \text{(Vasiček)}$$

$$\Sigma(t,T) = \widehat{\sigma}(T-t) \qquad \text{(Ho–Lee)}$$

Fast algorithms for Caps, Floors, Swaptions, Digitals, Range options

Forward measure associated with data $T \leq T^*$

Density $\dfrac{d\mathbb{P}_T}{d\mathbb{P}^*} = \dfrac{1}{B_T B(0,T)}$ or $\mathbb{E}_{\mathbb{P}^*}\left[\dfrac{d\mathbb{P}_T}{d\mathbb{P}^*} \mid \mathcal{F}_t\right] = \dfrac{B(t,T)}{B_t B(0,T)}$

For the case of the Lévy term structure model this equals

$$\exp\left(\int_0^t \Sigma(s,T)\,dL_s - \int_0^t \theta_s(\Sigma(s,T))\,ds\right)$$

Compensator of μ^L under $\mathbb{P}_T$: $\nu^T(dt,dx) = e^{\langle \Sigma(t,T),x\rangle}\nu(dt,dx)$

Standard Brownian motion under $\mathbb{P}_T$: $W_t^T = W_t - \int_0^t c_s^{1/2}\Sigma(s,T)\,ds$

Pricing formula for caps

(Eberlein, Kluge (2006))

$$w(B(t,T),K) = (B(t,T) - K)^+$$

Call with strike K and maturity t on a bond that matures at T

$$\begin{aligned} C(0,t,T,K) &= \mathbb{E}_{\mathbb{P}^*}[B_t^{-1}(B(t,T)-K)^+] \\ &= B(0,t)\mathbb{E}_{\mathbb{P}_t}[(B(t,T)-K)^+] \end{aligned}$$

Assume $X = \int_0^t (\Sigma(s,T) - \Sigma(s,t))dL_s$ has a Lebesgue density, then

$$\begin{aligned} C(0,t,T,K) &= \frac{1}{2\pi} KB(0,t)\exp(R\xi) \\ &\quad \times \int_{-\infty}^{\infty} e^{iu\xi}(R+iu)^{-1}(R+1+iu)^{-1} M_t^X(-R-iu)\,du \end{aligned}$$

where ξ is a constant and $R < -1$.

Analogous for the corresponding put and for swaptions

Basic interest rates

$B(t,T)$: price at time $t \in [0, T]$ of a default-free zero coupon bond

$f(t,T)$: instantaneous forward rate

$$B(t,T) = \exp\left(-\int_t^T f(t,u)\,\mathrm{d}u\right)$$

$L(t,T)$: default-free forward Libor rate for the interval T to $T+\delta$

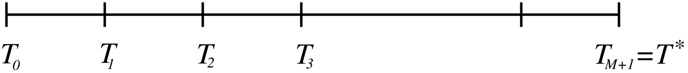
$$L(t,T) := \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right)$$

$F_B(t,T,U)$: forward price process for the two maturities T and U

$$F_B(t,T,U) := \frac{B(t,T)}{B(t,U)}$$

$$\Longrightarrow \quad 1 + \delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)} = F_B(t,T,T+\delta)$$

LIBOR market model



with $\delta = T_{n+1} - T_n$ (fixed accrual period)

$L(t, T)$ forward LIBOR rate for the interval T to $T + \delta$ as of time $t \leq T$

δ-forward LIBOR rate $\quad L(t,T) = \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right)$

For two maturities T, U define the forward process

$$F_B(t,T,U) = \frac{B(t,T)}{B(t,U)}$$

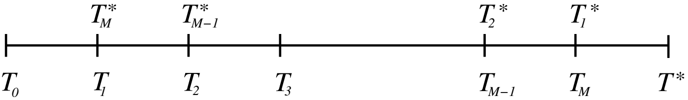
$$\Longrightarrow 1 + \delta L(t,T) = F_B(t,T,T+\delta)$$

Sandmann, Sondermann, Miltersen (1995); Miltersen, Sandmann, Sondermann (1997); Brace, Gatarek, Musiela (1997); Jamshidian (1997)

The Lévy Libor model

(Eberlein, Özkan (2005))

Tenor structure $T_0 < T_1 < \cdots < T_M < T_{M+1} = T^*$
with $T_{i+1} - T_i = \delta$, set $T_i^* = T^* - i\delta$ for $i = 1, \ldots, M$



Assumptions

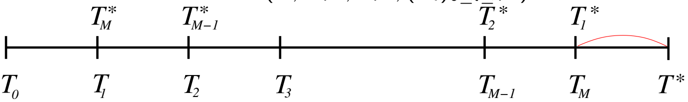
(LR.1): *For any maturity T_i there is a bounded deterministic function $\lambda(\cdot, T_i)$, which represents the volatility of the forward Libor rate process $L(\cdot, T_i)$.*

(LR.2): *We assume a strictly decreasing and strictly positive initial term structure $B(0, T)$ $(T \in]0, T^*])$. Consequently the initial term structure of forward Libor rates is given by*

$$L(0, T) = \frac{1}{\delta} \left(\frac{B(0, T)}{B(0, T + \delta)} - 1 \right)$$

Backward Induction (1)

Given a stochastic basis $(\Omega, \mathcal{F}_{T^*}, \mathbb{P}_{T^*}, (\mathcal{F}_t)_{0 \leq t \leq T^*})$



We postulate that under $\mathbb{P}_{T^*}$

$$L(t, T_1^*) = L(0, T_1^*) \exp\left(\int_0^t \lambda(s, T_1^*) \mathrm{d}L_s^{T^*}\right)$$

where $$L_t^{T^*} = \int_0^t b_s \, \mathrm{d}s + \int_0^t c_s^{1/2} \mathrm{d}W_s^{T^*} + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu^{T^*,L})(\mathrm{d}s, \mathrm{d}x)$$

is a non-homogeneous Lévy process with random measure of jumps μ^L and $\mathbb{P}_{T^*}$-compensator $\nu^{T^*,L}(\mathrm{d}s, \mathrm{d}x) = F_s(\mathrm{d}x)\,\mathrm{d}s$, $F_s(\{0\}) = 0$, where F_s satisfies some integrability conditions

Backward Induction (2)

In order to make $L(t, T_1^*)$ a $\mathbb{P}_{T^*}$-martingale, choose the drift characteristic (b_s) s.t.

$$\int_0^t \lambda(s, T_1^*) b_s \, \mathrm{d}s = -\frac{1}{2} \int_0^t c_s \lambda^2(s, T_1^*) \, \mathrm{d}s - \int_0^t \int_{\mathbb{R}} \left(e^{\lambda(s, T_1^*)x} - 1 - \lambda(s, T_1^*)x \right) \nu^{T^*, L}(\mathrm{d}s, \mathrm{d}x)$$

Transform $L(t, T_1^*)$ in a stochastic exponential

$$L(t, T_1^*) = L(0, T_1^*) \mathcal{E}(H(t, T_1^*))$$

where

$$H(t, T_1^*) = \int_0^t \lambda(s, T_1^*) c_s^{1/2} \mathrm{d}W_s^{T^*} + \int_0^t \int_{\mathbb{R}} \left(e^{\lambda(s, T_1^*)x} - 1 \right) (\mu^L - \nu^{T^*, L})(\mathrm{d}s, \mathrm{d}x)$$

Backward Induction (3)

Recall $F_B(t, T_1^*, T^*) = 1 + \delta L(t, T_1^*)$, therefore,

$$\begin{aligned} dF_B(t, T_1^*, T^*) &= \delta dL(t, T_1^*) \\ &= F_B(t-, T_1^*, T^*)\Bigg(\underbrace{\frac{\delta L(t-, T_1^*)}{1 + \delta L(t-, T_1^*)} \lambda(t, T_1^*)}_{= \alpha(t, T_1^*, T^*)} c_t^{1/2} dW_t^{T^*} \\ &\quad + \int_{\mathbb{R}} \underbrace{\frac{\delta L(t-, T_1^*)}{1 + \delta L(t-, T_1^*)} \left(e^{\lambda(t, T_1^*)x} - 1 \right)}_{= \beta(t, x, T_1^*, T^*) - 1} (\mu^L - \nu^{T^*, L})(dt, dx) \Bigg) \end{aligned}$$

Define the forward martingale measure associated with T_1^*

$$\frac{d\mathbb{P}_{T_1^*}}{d\mathbb{P}_{T^*}} = \mathcal{E}_{T_1^*}(M^1) \qquad \text{where}$$

$$M_t^1 = \int_0^t \alpha(s, T_1^*, T^*) c_s^{1/2} dW_s^{T^*} + \int_0^t \int_{\mathbb{R}} (\beta(s, x, T_1^*, T^*) - 1)\,(\mu^L - \nu^{T^*, L})(ds, dx)$$

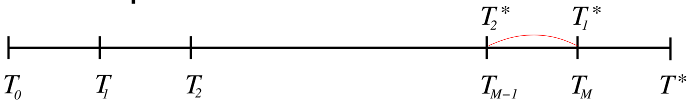
Backward Induction (4)

Then $W_t^{T_1^*} = W_t^{T^*} - \int_0^t \alpha(s, T_1^*, T^*) c_s^{1/2}\, ds$

is the forward Brownian motion for date T_1^* and

$\nu^{T_1^*, L}(dt, dx) = \beta(t, x, T_1^*, T^*) \nu^{T^*, L}(dt, dx)$ is the $\mathbb{P}_{T_1^*}$-compensator for μ^L.

Second step



We postulate that under $\mathbb{P}_{T_1^*}$

$$L(t, T_2^*) = L(0, T_2^*) \exp\left(\int_0^t \lambda(s, T_2^*)\, dL_s^{T_1^*} \right) \quad \text{where}$$

$$L_t^{T_1^*} = \int_0^t b_s^{T_1^*}\, ds + \int_0^t c_s^{1/2}\, dW_s^{T_1^*} + \int_0^t \int_{\mathbb{R}} x(\mu^L - \nu^{T_1^*, L})(ds, dx)$$

Backward Induction (5)

Second measure change

$$\frac{d\mathbb{P}_{T_2^*}}{d\mathbb{P}_{T_1^*}} = \mathcal{E}_{T_2^*}(M^2)$$

where

$$M_t^2 = \int_0^t \alpha(s, T_2^*, T_1^*) c_s^{1/2} \, dW_s^{T_1^*} + \int_0^t \int_{\mathbb{R}} \big(\beta(s, x, T_2^*, T_1^*) - 1\big)(\mu^L - \nu^{T_1^*, L})(ds, dx)$$

This way we get for each time point T_j^* in the tenor structure a Libor rate process which is under the forward martingale measure $\mathbb{P}_{T_{j-1}^*}$ of the form

$$L(t, T_j^*) = L(0, T_j^*) \exp\left(\int_0^t \lambda(s, T_j^*) \, dL_s^{T_{j-1}^*}\right)$$

Forward process model

Postulate

$$1 + \delta L(t, T_1^*) = (1 + \delta L(0, T_1^*)) \exp\left(\int_0^t \lambda(s, T_1^*) \, \mathrm{d}L_s^{T^*} \right)$$

equivalently

$$F_B(t, T_1^*, T^*) = F_B(0, T_1^*, T^*) \exp\left(\int_0^t \lambda(s, T_1^*) \, \mathrm{d}L_s^{T^*} \right)$$

In differential form

$$\mathrm{d}F_B(t, T_1^*, T^*) = F_B(t-, T_1^*, T^*)\Big(\lambda(t, T_1^*) c_t^{1/2} \, \mathrm{d}W_t^{T^*} + \int_{\mathbb{R}} \left(\mathrm{e}^{\lambda(t, T_1^*)x} - 1 \right)(\mu^L - \nu^{T^*,L})(\mathrm{d}t, \mathrm{d}x)\Big)$$

Pricing of caps and floors

Time-T_j-payoff of a cap settled in arrears

$$N\delta(L(T_{j-1}, T_{j-1}) - K)^+$$

N notional amount (set $N = 1$)

K strike rate

Time-t value

$$\begin{aligned} C_t &= \sum_{j=1}^{n} \mathbb{E}_{\mathbb{P}^*}\left[\frac{B_t}{B_{T_j}}\,\delta(L(T_{j-1}, T_{j-1}) - K)^+ \,|\, \mathcal{F}_t\right] \\ &= \sum_{j=1}^{n} B(t, T_j)\mathbb{E}_{\mathbb{P}_{T_j}}\left[\delta(L(T_{j-1}, T_{j-1}) - K)^+ \,|\, \mathcal{F}_t\right] \end{aligned}$$

Analogous for floor

$$N\delta(K - L(T_{j-1}, T_{j-1}))^+$$

Extensions of the basic Lévy market model

Lévy market model
(Eb–Özkan (2005))

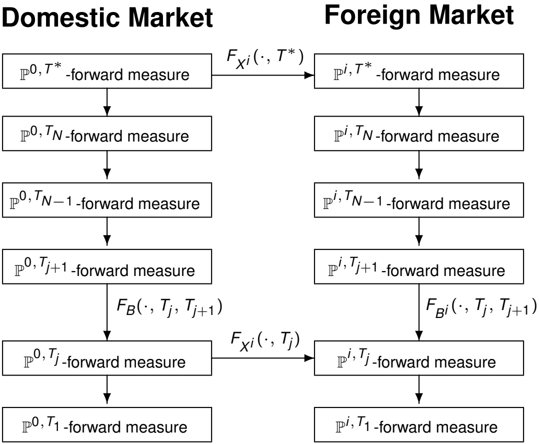
→ Multi-currency setting
(Eb–Koval (2006))

→ Credit risk model
(Eb–Kluge–Schönbucher (2006))

→ Swap rate model
(Eb–Liinev (2006))

→ Duality principle
(Eb–Kluge–Papapantoleon (2006))

Cross-currency Lévy market model



Relationship between domestic and foreign fixed income markets in a discrete-tenor framework.

Pricing cross-currency derivatives (1)

Cross-currency swaps

Floating-for-floating cross-currency $(i;\ell;0)$ swap

Libor rate $L^i(T_{j-1},T_{j-1})$ of currency i is received at each date T_j

Libor rate $L^\ell(T_{j-1},T_{j-1})$ of currency ℓ is paid

Payments are made in units of the domestic currency

Thus the cashflow at time point T_j is (notional = 1)

$$\delta(L^i(T_{j-1},T_{j-1}) - L^\ell(T_{j-1},T_{j-1}))$$

Pricing cross-currency derivatives (2)

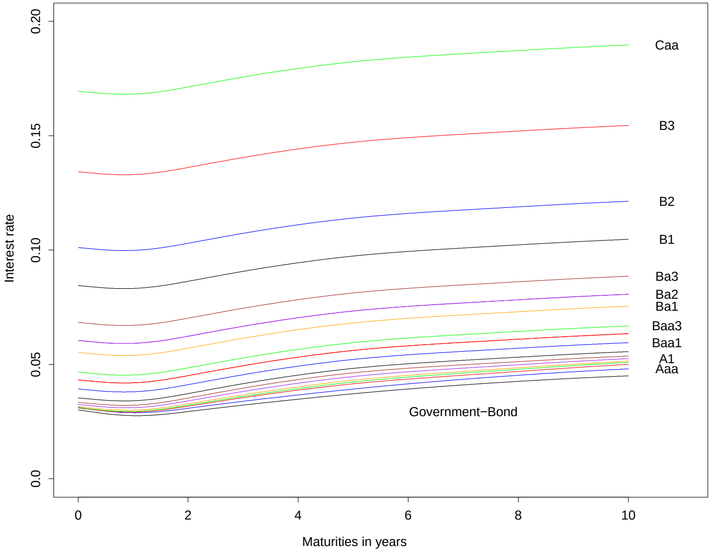
The time-0-value of a floating-for-floating $(i; \ell; 0)$ cross-currency forward swap in units of the domestic currency is

$$CCFS_{[i;\ell;0]}(0) = B^0(0,T_j)\left(\sum_{j=1}^{N+1}\frac{B^i(0,T_{j-1})}{B^i(0,T_j)}\exp\left(\mathcal{D}^i(0,T_{j-1},T_j)\right) - \sum_{j=1}^{N+1}\frac{B^\ell(0,T_{j-1})}{B^\ell(0,T_j)}\exp\left(\mathcal{D}^\ell(0,T_{j-1},T_j)\right)\right)$$

where

$$\mathcal{D}^i(0,T_{j-1},T_j)$$
$$= -\int_0^{T_{j-1}} \lambda^i(s,T_{j-1})^\top c_s \zeta^i(s,T_j,T_{j+1})\,ds$$
$$- \int_0^{T_{j-1}}\int_{\mathbb{R}^d}\left(\exp\left(\lambda^i(s,T_{j-1})^\top x\right)-1\right)\left(\overline{\zeta}_i(s,x,T_j,T_{j+1})-1\right)\nu_{0,T_j}(ds,dx)$$

Lévy credit risk model



The Lévy Libor model with default risk

(Eberlein, Kluge, Schönbucher 2006)

$B^0(t,T_k)$: time-t price of a defaultable zero coupon bond with zero recovery and maturity T_k

τ: time of default

$\overline{B}(t,T_k)$: pre-default value of the defaultable bond

$$\Longrightarrow \quad B^0(t,T_k) = 1\!\!1_{\{\tau>t\}}\overline{B}(t,T_k), \quad \overline{B}(T_k,T_k) = 1 \quad (k = 1,\dots,n)$$

Terminal value of the defaultable bond

$$B^0(T_k,T_k) = 1\!\!1_{\{\tau>T_k\}}\overline{B}(T_k,T_k) = 1\!\!1_{\{\tau>T_k\}}$$

The Lévy Libor model with default risk (2)

- The *defaultable forward Libor rates* for the interval $[T_k, T_{k+1}]$ are given by
$$\overline{L}(t,T_k) := \frac{1}{\delta_k}\left(\frac{\overline{B}(t,T_k)}{\overline{B}(t,T_{k+1})} - 1\right).$$
- The *forward Libor spreads* are given by
$$S(t,T_k) := \overline{L}(t,T_k) - L(t,T_k).$$
- The *default risk factors* or *forward survival processes* are given by
$$D(t,T_k) := \frac{\overline{B}(t,T_k)}{B(t,T_k)}.$$
- The discrete-tenor *forward default intensities* are given by
$$H(t,T_k) := \frac{1}{\delta_k}\left(\frac{D(t,T_k)}{D(t,T_{k+1})} - 1\right) = \frac{S(t,T_k)}{1+\delta L(t,T_k)}.$$

Pricing contingent claims with defaultable forward measures

X promised payoff at date T_i with zero recovery upon default

π_t^X its price at time $t \in [0, T_i]$

$$\pi_t^X = 1\!\!1_{\{\tau > t\}} B(t, T_i) \mathbb{E}_{\mathbb{Q}_{T_i}}[X 1\!\!1_{\{\tau > T_i\}} | \mathcal{G}_t] \qquad (t \in [0, T_i])$$

The defaultable forward measures $\overline{\mathbb{Q}}_{T_i}$ and $\overline{\mathbb{P}}_{T_i}$ are the appropriate tools.

If X is $\mathcal{G}_{T_i}$-measurable

$$\pi_t^X = 1\!\!1_{\{\tau > t\}} \overline{B}(t, T_i) \mathbb{E}_{\overline{\mathbb{Q}}_{T_i}}[X | \mathcal{G}_t] = B^0(t, T_i) \mathbb{E}_{\overline{\mathbb{Q}}_{T_i}}[X | \mathcal{G}_t].$$

If X is $\mathcal{F}_{T_i}$-measurable

$$\pi_t^X = 1\!\!1_{\{\tau > t\}} \overline{B}(t, T_i) \mathbb{E}_{\overline{\mathbb{P}}_{T_i}}[X | \mathcal{F}_t] = B^0(t, T_i) \mathbb{E}_{\overline{\mathbb{P}}_{T_i}}[X | \mathcal{F}_t].$$

Default payments

Denote by $e_k^X(t)$ the time-t value of receiving an amount of X at T_{k+1}

$\Leftrightarrow$ default occured in period $(T_k, T_{k+1}]$

Lemma

If X is $\mathcal{F}_{T_k}$-measurable, then for $t \leq T_k$

$$e_k^X(t) = \mathbb{1}_{\{\tau > t\}} \overline{B}(t, T_{k+1}) \delta_k \mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}} [X H(T_k, T_k) | \mathcal{F}_t]$$

Pricing of defaultable coupon bonds

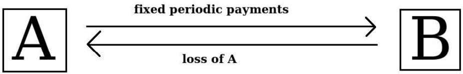
Fixed coupon of c to be paid at dates $T_1, \ldots, T_m$

$$B^{\pi}_{\text{fixed}}(0) = \overline{B}(0, T_m) + \sum_{k=0}^{m-1} \overline{B}(0, T_{k+1}) \left(c + \pi(1+c)\delta_k \mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}} [H(T_k, T_k)] \right).$$

Floating coupon bond that pays Libor plus a constant spread x
Promised payoff at the date T_{k+1}: $\quad \delta_k(L(T_k, T_k) + x)$

$$\begin{aligned} B^{\pi}_{\text{floating}}(0) \; = \; & \overline{B}(0, T_m) + \sum_{k=0}^{m-1} \delta_k \overline{B}(0, T_{k+1}) \Big(x + \mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}} [L(T_k, T_k)] \\ & + \pi(1 + \delta_k x) \mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}} [H(T_k, T_k)] \\ & + \pi \delta_k \mathbb{E}_{\overline{\mathbb{P}}_{T_{k+1}}} [H(T_k, T_k) L(T_k, T_k)] \Big). \end{aligned}$$

Credit default swaps (CDS)



Standard default swap: Default of a coupon bond

A receives: $1 - \pi(1 + c)$ (fixed coupon)

$1 - \pi(1 + \delta_k(L(T_k, T_k) + x))$ (floating coupon)

Time-0 value of the fee payments: $s \sum_{k=1}^{m} \overline{B}(0, T_{k-1})$
s default swap rate

$$s_{\text{fixed}} = \frac{1 - \pi(1 + c)}{\sum_{k=1}^{m} \overline{B}(0, T_{k-1})} \sum_{k=1}^{m} \left(\overline{B}(0, T_k) \delta_{k-1} \mathbb{E}_{\overline{\mathbb{P}}_{T_k}} [H(T_{k-1}, T_{k-1})] \right)$$

$$s_{\text{floating}} = \frac{1}{\sum_{k=1}^{m} \overline{B}(0, T_{k-1})} \sum_{k=1}^{m} \Bigg(\overline{B}(0, T_k) \delta_{k-1} \Big((1 - \pi(1 + \delta_{k-1} x)) \times \mathbb{E}_{\overline{\mathbb{P}}_{T_k}} [H(T_{k-1}, T_{k-1})] - \pi \delta_{k-1} \mathbb{E}_{\overline{\mathbb{P}}_{T_k}} [H(T_{k-1}, T_{k-1}) L(T_{k-1}, T_{k-1})] \Big) \Bigg)$$

The Theme

Call option in FX market: Euro/Dollar

Gives you the right to buy Euros paying in Dollars.
At the same time a right to sell Dollars getting Euros.

Payoff $(S_T - K)^+$ (S_t) exchange rate, K strike

$$(S_T - K)^+ = KS_T\Big(\frac{1}{K} - \frac{1}{S_T}\Big)^+$$
$$= KS_T(K' - S'_T)^+$$

$\uparrow$

Dollar/Euro rate

Call price = $K\cdot$ put price (in the dual market)

$\longrightarrow$ duality principle

Exponential semimartingale models

Let $\mathcal{B} = (\Omega, \mathcal{F}, \mathbf{F}, P)$ be a stochastic basis, where $\mathcal{F} = \mathcal{F}_T$ and $\mathbf{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$. We model the price process of a financial asset as an exponential semimartingale

$$S_t = \mathrm{e}^{H_t}, \quad 0 \leq t \leq T.$$

$H = (H_t)_{0 \leq t \leq T}$ is a *semimartingale* with canonical representation

$$H = H_0 + B + H^c + h(x) * (\mu^H - \nu) + (x - h(x)) * \mu^H$$

or, in detail

$$H_t = H_0 + B_t + H_t^c + \int_0^t \int_{\mathbb{R}} h(x) \mathrm{d}(\mu^H - \nu) + \int_0^t \int_{\mathbb{R}} (x - h(x)) \mathrm{d}\mu^H,$$

where

- $h = h(x)$ is a truncation function; canonical choice $h(x) = x1_{\{|x|\leq 1\}}$;
- $B = (B_t)_{0\leq t\leq T}$ is a predictable process of bounded variation;
- $H^c = (H^c_t)_{0\leq t\leq T}$ is the continuous martingale part of H;
- $\nu = \nu(\omega; dt, dx)$ is the predictable compensator of the random measure of jumps $\mu^H = \mu^H(\omega; dt, dx)$ of H.

For the processes B, $C = \langle H^c \rangle$, and the measure ν we use the notation

$$\mathbb{T}(H|P) = (B, C, \nu)$$

which will be called the *triplet of predictable characteristics* of the semimartingale H with respect to the measure P.

Assumption: The truncation function satisfies the *antisymmetry* property

$$h(-x) = -h(x).$$

Martingale and dual martingale measures

Assumption ($\mathbb{ES}$)

The process $1_{\{x>1\}} e^x * \nu$ has bounded variation.

Then, H is exponentially special and

$$S = e^H \in \mathcal{M}_{\text{loc}}(P) \Leftrightarrow B + \frac{C}{2} + (e^x - 1 - h(x)) * \nu^H = 0.$$

Moreover, we assume that $S \in \mathcal{M}(P)$, therefore $ES_T = 1$. Define on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T})$ a *new* probability measure P' with

$$\frac{dP'}{dP} = S_T.$$

Since $S > 0$ (P-a.s.), we have $P \ll P'$ and

$$\frac{dP}{dP'} = \frac{1}{S_T}.$$

Introduce the process

$$S' = \frac{1}{S}.$$

Then, denoting by H' the *dual* of the semimartingale H, i.e. $H' = -H$, we have

$$S' = \mathrm{e}^{H'}.$$

Proposition

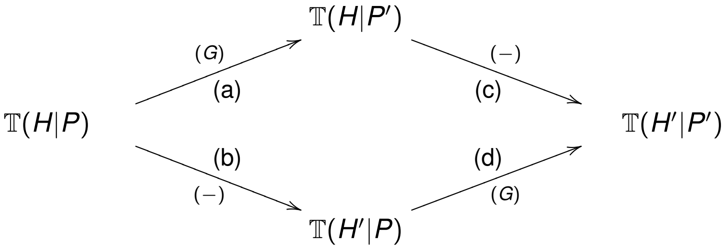
Suppose $S = \mathrm{e}^{H} \in \mathcal{M}(P)$ i.e. S is a P-martingale. Then the process $S' \in \mathcal{M}(P')$ i.e. S' is a P'-martingale.

Theorem

The triplet $\mathbb{T}(H'|P') = (B', C', \nu')$ *can be expressed via the triplet* $\mathbb{T}(H|P) = (B, C, \nu)$ *by the following formulae:*

$$
\begin{aligned}
B' &= -B - C - h(x)(\mathrm{e}^x - 1) * \nu \\
C' &= C \\
1_A(x) * \nu' &= 1_A(-x)\mathrm{e}^x * \nu, \quad A \in \mathcal{B}(\mathbb{R}).
\end{aligned}
$$

Structure of the proof:



$\xrightarrow{(G)}$: Girsanov's theorem, $\xrightarrow{(-)}$: dual of a semimartingale.

European options

Theorem

The prices of standard call and put options satisfy the following duality relations:

$$\mathbb{C}_T(S;K) = K\,\mathbb{P}'_T(K';S')$$

and

$$\mathbb{P}_T(K;S) = K\,\mathbb{C}'_T(S';K').$$

Proof: Using the dual measure

$$\begin{aligned}\mathbb{C}_T(S;K) &= E\left[S_T\frac{(S_T-K)^+}{S_T}\right] = E'\left[\frac{(S_T-K)^+}{S_T}\right] = E'[(1-KS'_T)^+] \\ &= KE'\left[\left(\frac{1}{K}-S'_T\right)^+\right] = KE'[(K'-S'_T)^+],\end{aligned}$$

where $K' = \frac{1}{K}$. □

Floating strike lookback options

Payoff of a call: $\left(S_T - \alpha \inf_{0 \le t \le T} S_t\right)^+$ for an $\alpha \ge 1$

Assume $H' = (H'_t)_{0 \le t \le T}$ satisfies the *reflection principle*

$$\mathrm{Law}\left(\sup_{t \le T} H'_t - H'_T | P'\right) = \mathrm{Law}(-\inf_{t \le T} H'_t | P')$$

(holds for Lévy processes), then

$$\mathbb{C}_T(S; \alpha \inf S) = \alpha \mathbb{P}'_T\left(\frac{1}{\alpha}; \inf S'\right)$$

Value of a *floating strike* lookback call
$\longrightarrow$ value of a *fixed strike* lookback put

Required Reserve Capital (1)

X random variable: outcome (cashflow) of a risky position

For setting capital requirements: non-dynamic

In complete markets: unique pricing kernel given by a probability measure Q

value of the position: $E^Q[X]$

position is acceptable if: $E^Q[X] \geq 0$

company's objective is: maximizing $E^Q[X]$

Real markets: incomplete

Instead of a unique probability measure Q we have to consider a set of probability measures $Q \in \mathcal{M}$

$$E^Q[X] \geq 0 \quad \text{for all } Q \in \mathcal{M} \qquad \text{or} \qquad \inf_{Q \in \mathcal{M}} E^Q[X] \geq 0$$

Required Reserve Capital (2)

Specification of $\mathcal{M}$ (test measures, generalized scenarios)

Axiomatic theory of risk measures: desirable properties

Monotonicity: $X \geq Y \Longrightarrow \varrho(X) \leq \varrho(Y)$

Cash invariance: $\varrho(X + c) = \varrho(X) - c$

Scale invariance: $\varrho(\lambda X) = \lambda \varrho(X), \quad \lambda \geq 0$

Subadditivity: $\varrho(X + Y) \leq \varrho(X) + \varrho(Y)$

Examples: Value at Risk (VaR)
Tail-VaR (expected shortfall)

General risk measure: $\varrho_m(X) = -\int_0^1 q_u(X) m(du)$

Any coherent risk measure has a representation

$$\varrho(X) = - \inf_{Q \in \mathcal{M}} E^Q[X]$$

Required Reserve Capital (3)

Acceptability of a cash flow?

Maybe it exposes the general economy to too much risk of loss

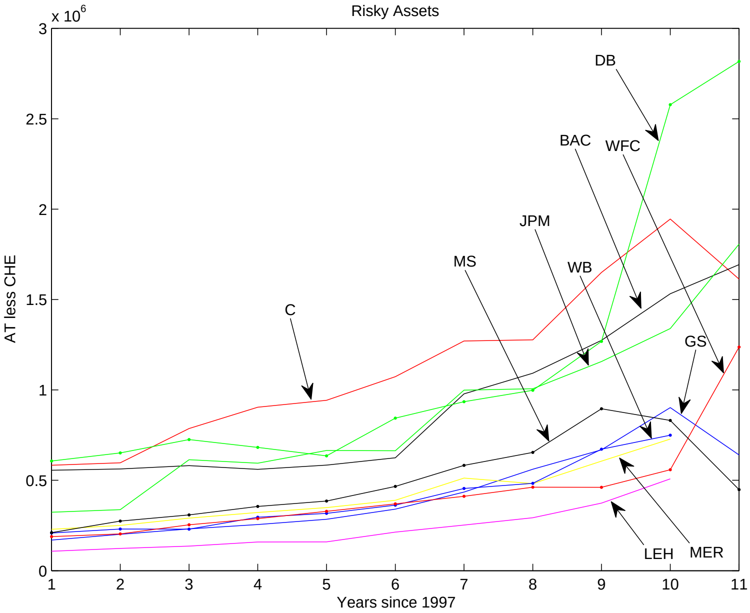
Business set up with limited liability and insufficient capital

$\longrightarrow$ Add capital C such that cash flow $C + X$ is acceptable

$$\inf_{Q \in \mathcal{M}} E^Q[C + X] \geq 0$$

Smallest such capital

$$C = -\inf_{Q \in \mathcal{M}} E^Q[X]$$



Risky Assets
x 10^6
3
2.5
2
1.5
1
0.5
0
AT less CHE
1
2
3
4
5
6
7
8
9
10
11
Years since 1997
DB
BAC
WFC
JPM
MS
WB
C
GS
LEH
MER

Required Reserve Capital (4)

Computation of this required reserve capital

Link between acceptability and concave distortions
(Cherny and Madan (2009))

$\rightarrow$ Concave distortions

Assume acceptability is completely defined by the distribution function of the risk

$\Psi(u)$: concave distribution function on $[0, 1]$

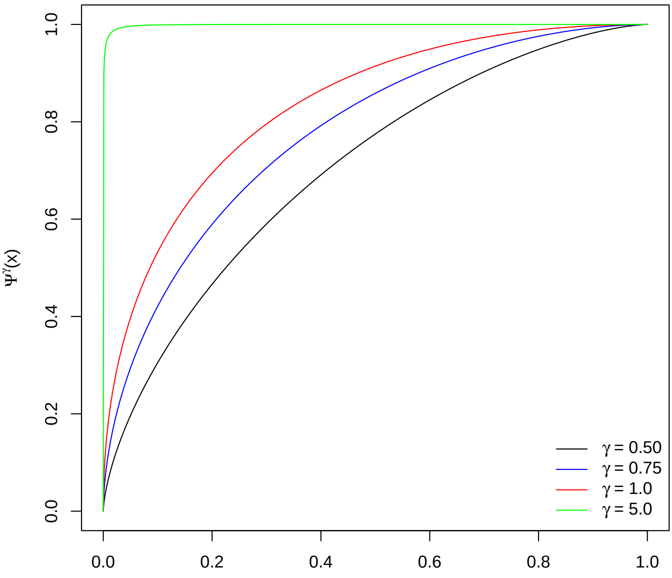
$\Rightarrow$ $\mathcal{M}$ the set of supporting measures is given by all measures Q with density $Z = \frac{dQ}{dP}$ s.t.

$$E^P[(Z - a)^+] \leq \sup_{u \in [0,1]} (\Psi(u) - ua) \quad \text{for all } a \geq 0$$

Acceptability of X with distribution function $F(x)$

$$\int_{-\infty}^{+\infty} x d\Psi(F(x)) \geq 0$$

Distortion



1.0
0.8
0.6
0.4
0.2
0.0
$\Psi^\gamma(x)$
0.0
0.2
0.4
0.6
0.8
1.0
$\gamma = 0.50$
$\gamma = 0.75$
$\gamma = 1.0$
$\gamma = 5.0$

Computation of Required Reserve Capital and the value of the taxpayer put

In Billions of US Dollars

	Reserve Capital Required	Reserve Capital Held	Limited Liability Put Value	Required to Actual Ratio	Adjustment Factor
JPM	698.039	368.149	293.96	1.8961	0.3154
MS	116.273	210.519	29.75	0.5523	0.4113
GS	−83.840	244.425	3.37	−0.3430	0.1796
BAC	246.065	124.905	158.17	1.9700	0.2840
WFC	366.832	72.092	220.14	5.0884	0.2107
C	434.596	325.681	156.21	1.3344	0.3984

References

- M. Beinhofer, E. Eberlein, A. Janssen: Correlations in Lévy interest rate models. Preprint, University of Freiburg (2009)
- E. Eberlein, A. Papapantoleon, A. N. Shiryaev: Esscher transform and the duality principle for multidimensional semimartingales. to appear in Annals of Applied Probability (2009)
- E. Eberlein, D. B. Madan: Sato processes and the valuation of structured products. Quantitative Finance 9 (1) (2009) 27–42
- E. Eberlein: Jump-type Lévy processes. In *Handbook of Financial Time Series*, T. G. Andersen, R. A. Davis, J.-P. Kreiß, T. Mikosch (eds.), Springer (2009), pp. 439–455
- E. Eberlein, K. Glau, A. Papapantoleon: Analysis of Fourier transform valuation formulas and applications. Preprint, University of Freiburg (2008)
- E. Eberlein, R. Frey, E. A. von Hammerstein: Advanced credit portfolio modeling and CDO pricing. In Mathematics – Key Technology for the Future, W. Jäger and H.-J. Krebs (eds.), Springer (2008), pp. 253–280

References (cont.)

- E. Eberlein, A. Papapantoleon, A. N. Shiryaev: On the duality principle in option pricing: semimartingale setting. Finance and Stochastics 12 (2) (2008) 265–292
- E. Eberlein, R. Frey, M. Kalkbrener, L. Overbeck: Mathematics in Financial Risk Management. Jahresbericht der Deutschen Mathematiker Vereinigung 109 (2007) 165–193
- E. Eberlein, W. Kluge: Calibration of Lévy term structure models. In *Advances in Mathematical Finance: In Honor of Dilip Madan*, M. Fu, R. A. Jarrow, J.-Y. Yen, and R. J. Elliott (eds.), Birkhäuser (2007), pp. 147–172
- E. Eberlein, J. Liinev: The Lévy swap market model. Applied Mathematical Finance 14 (2) (2007) 171–196
- E. Eberlein, W. Kluge, P. J. Schönbucher: The Lévy Libor model with default risk. Journal of Credit Risk 2 (2) (2006) 3–42
- E. Eberlein, N. Koval: A Cross-Currency Lévy Market Model. Quantitative Finance 6 (2006) 465–480

References (cont.)

- E. Eberlein, W. Kluge, A. Papapantoleon: Symmetries in Lévy term structure models. International Journal of Theoretical and Applied Finance 9 (6) (2006) 967–986
- E. Eberlein, W. Kluge: Exact pricing formulae for caps and swaptions in a Lévy term structure model. Journal of Computational Finance 9 (2) (2006) 99–125
- E. Eberlein, A. Papapantoleon: Symmetries and pricing of exotic options in Lévy models. In *Exotic Option Pricing and Advanced Lévy Models*, A. Kyprianou, W. Schoutens, P. Wilmott (eds.), Wiley (2005), pp. 99–128
- E. Eberlein, W. Kluge: Valuation of floating range notes in Lévy term structure models. Mathematical Finance 16 (2005) 237–254
- E. Eberlein, A. Papapantoleon: Equivalence of floating and fixed strike Asian and lookback options. Stochastic Processes and Their Applications 115 (2005) 31–40
- E. Eberlein, F. Özkan: The Lévy Libor Model. Finance and Stochastics 9 (2005) 327–348
- E. Eberlein, J. Jacod, S. Raible: Lévy term structure models: no-arbitrage and completeness. Finance and Stochastics 9 (2005) 67–88

References (cont.)

- E. Eberlein, E. A. von Hammerstein: Generalized hyperbolic and inverse Gaussian distributions: limiting cases and approximation of processes. In *Seminar on Stochastic Analysis, Random Fields and Applications IV*, Progress in Probability 58, R. C. Dalang, M. Dozzi, F. Russo (eds.), Birkhäuser Verlag (2004) 221–264
- E. Eberlein, G. Stahl: Both sides of the fence: a statistical and regulatory view of electricity risk. Energy & Power Risk Management 8 (6) (2003), 34–38
- E. Eberlein, F. Özkan: Time consistency of Lévy models. Quantitative Finance 3 (2003) 40–50
- E. Eberlein, J. Kallsen, J. Kristen: Risk management based on stochastic volatility. Journal of Risk 5 (2) (2003) 19–44
- E. Eberlein, F. Özkan: The defaultable Lévy term structure: ratings and restructuring. Mathematical Finance 13 (2003) 277–300
- E. Eberlein, K. Prause: The generalized hyperbolic model: financial derivatives and risk measures. In *Mathematical Finance-Bachelier Congress 2000*, H. Geman, D. Madan, S. Pliska, T. Vorst (eds.), Springer Verlag (2002), 245–267

References (cont.)

- E. Eberlein, S. Raible: Some analytic facts on the generalized hyperbolic model. In *Proceedings of the 3rd European Meeting of Mathematics*, Progress in Mathematics 202, C. Casacuberta, et al. (eds.), Birkhäuser Verlag (2001) 367–378
- E. Eberlein: Recent advances in more realistic risk management: the hyperbolic model. In *Mastering Risk 2*, C. Alexander (Ed.), Prentice Hall-Financial Times (2001), 56–72
- E. Eberlein: Application of generalized hyperbolic Lévy motions to finance. In *Lévy Processes: Theory and Applications*, O.E. Barndorff-Nielsen, T. Mikosch, and S. Resnick (eds.), Birkhäuser Verlag (2001) 319–337
- E. Eberlein, S. Raible: Term structure models driven by general Lévy processes. Mathematical Finance 9 (1999) 31–53
- E. Eberlein, U. Keller, K. Prause: New insights into smile, mispricing and value at risk: the hyperbolic model. Journal of Business 71 (1998) 371–405
- E. Eberlein, J. Jacod: On the range of options prices. Finance and Stochastics 1 (1997) 131–140
- E. Eberlein, U. Keller: Hyperbolic distributions in finance. Bernoulli 1 (1995) 281–299