On the Martingale Property of Exponential Local Martingales

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Outline

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Description of the main result

Diffusion Y:

$$dY_t = \mu(Y_t)\,dt + \sigma(Y_t)\,dW_t$$

with the state space $J = (l, r)$, $-\infty \le l < r \le \infty$

Possibly exits J (in a continuous way). Exit time: ζ

$$\begin{aligned} Z_t &= \mathcal{E}\left(\int_0^{\cdot} b(Y_s)\,dW_s\right)_{t\wedge\zeta} \\ &= \exp\left\{\int_0^{t\wedge\zeta} b(Y_s)\,dW_s - \frac{1}{2}\int_0^{t\wedge\zeta} b^2(Y_s)\,ds\right\} \end{aligned}$$

Z nonnegative local martingale $\Longrightarrow$ supermartingale
Z martingale $\Longleftrightarrow$ $\mathsf{E}Z_t = 1$, $t \in [0, \infty)$

Input: functions $\mu, \sigma, b\colon J \to \mathbb{R}$ (Borel, weak local integrability conditions)

Output: deterministic necessary and sufficient conditions for Z to be a true martingale in terms of μ, σ, and b

Where applies?

1. Girsanov's measure change
2. Characterization of different types of no-arbitrage
3. Characterization of when a stock price is a true martingale under ELMM
4. Constructing pathological (counter)examples

Why considering the possibility to exit J?

Needed e.g. for 2–4 above

Literature

Sufficient conditions

Girsanov, Gikhman–Skorokhod, Liptser–Shiryaev, Novikov, Kazamaki, ...

Necessary and sufficient conditions

Engelbert–Senf, Blei–Engelbert, ...

Blei and Engelbert (2009): criterion for $\mathcal{E}(M)$ to be a true martingale for a strong Markov continuous local martingale M

Outline

Terminology

Y exits the state space J at r means "with positive probability"

$s\colon (l,r)\to\mathbb{R}$ scale function of diffusion Y, $\rho := s'$

r is good if

$$s(r)<\infty \text{ and } \frac{(s(r)-s)b^2}{\rho\sigma^2}\in L^1_{\text{loc}}(r-)$$

Similar terminology for l

Auxiliary diffusion (with the same state space $J=(l,r)$):

$$d\widetilde{Y}_t = (\mu+b\sigma)(\widetilde{Y}_t)\,dt + \sigma(\widetilde{Y}_t)\,d\widetilde{W}_t$$

$\widetilde{s}\colon J\to\mathbb{R}$ scale function of diffusion $\widetilde{Y}$, $\widetilde{\rho} := \widetilde{s}'$

Useful facts

1. "r is good" means

$$s(r) < \infty \text{ and } \frac{(s(r) - s)b^2}{\rho\sigma^2} \in L^1_{\text{loc}}(r-)$$

or, equivalently,

$$\widetilde{s}(r) < \infty \text{ and } \frac{(\widetilde{s}(r) - \widetilde{s})b^2}{\widetilde{\rho}\sigma^2} \in L^1_{\text{loc}}(r-)$$

2. If one of the diffusions Y and $\widetilde{Y}$ exits J at r and the other does not, then r is bad

These facts are often helpful in the application of the theorem below to specific situations

Main result

Theorem Z martingale $\iff$ ((a) or (b)) and ((c) or (d))

(a) $\widetilde{Y}$ does not exit J at r

(b) r is good

(c) $\widetilde{Y}$ does not exit J at l

(d) l is good

Theorem above together with Fact 2 on the previous slide imply

Corollary Suppose Y does not exit J. Then

$$Z \text{ is a martingale} \iff \widetilde{Y} \text{ does not exit } J$$

Outline

Introduction

Formulation of the main result

Examples

Example: funny

Fix $\alpha > -1$ and define diffusion Y by

$$dY_t = |Y_t|^\alpha \, dt + dW_t, \quad Y_0 = x_0 \in J := \mathbb{R}.$$

Let Z be the local martingale given by

$$Z_t = \exp\left\{ \int_0^{t\wedge\zeta} Y_s \, dW_s - \frac{1}{2} \int_0^{t\wedge\zeta} Y_s^2 \, ds \right\}.$$

Our results imply the following classification:

$\alpha \in (-1, 1]$:	Z martingale, not u.i.
$\alpha \in (1, 3]$:	Z strict local martingale
$\alpha > 3$:	Z u.i. martingale

Example: driftless SDE

$dY_t = \sigma(Y_t)\,dW_t, \quad Y_0 = x_0 \in J := (0,\infty).$
We stop Y after it reaches 0

Corollary Y is a martingale $\iff x/\sigma^2(x) \notin L^1_{\text{loc}}(\infty-)$

Reduction to our setting

$$Y_t = x_0 \mathcal{E}\left(\int_0^\cdot \frac{\sigma(Y_s)}{Y_s}\,dW_s\right)_{t\wedge\zeta}$$

Corollary (CEV model) Let $\sigma(x) = x^\alpha$, $\alpha \in \mathbb{R}$. Then

Y is a martingale $\iff \alpha \le 1$,
Y is a strict local martingale $\iff \alpha > 1$

On my website

A. Mijatović and M. Urusov (2010). On the martingale property of certain local martingales. To appear in *Probability Theory and Related Fields*.

A. Mijatović and M. Urusov (2010). Deterministic criteria for the absence of arbitrage in diffusion models. To appear in *Finance and Stochastics*.

A. Mijatović, N. Novak, and M. Urusov (2010). Martingale property of generalized stochastic exponentials. *Preprint*.

The talk covers a part of the first paper.
The second paper: applications in finance.
The third paper: a generalization of the first one.

Thank you for your attention!

Papers available at

http://www.uni-ulm.de/mawi/finmath/people/urusov.html