Branching processes in random environment: sudden death versus slow extinction

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joint work with

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1. Galton-Watson process

- Offspring generating function

$$f(s) := \mathbf{E}s^{\xi} = \sum_{k=0}^{\infty} \mathbf{P}(\xi = k)s^{k}$$

$A := \mathbf{E}\xi = f'(1)$ - the expected offspring size per particle

- $T = \min\{k : Z_k = 0\}$ - the extinction moment

Kolmogorov (1938) If $A = 1$ and $\sigma^2 := Var\xi < \infty$ then

$$\mathbf{P}(T > n) = \mathbf{P}(Z_n > 0) \sim \frac{2}{\sigma^2 n}, \qquad n \to \infty.$$

Lamperti, Ney (1968) For any $t \in (0, 1]$

$$\left\{ \frac{2Z_{nt}}{\sigma^2 nt} \middle| T > n \right\} \xrightarrow{d} Exp(1) + Exp\left(\frac{1}{1-t}\right)$$

where $Exp(a)$ is an exponentially distributed random variable with parameter a. The summands are independent.

What happens if $T = n$?

One can show that for any $t \in (0, 1)$

$$\left\{ \frac{2Z_{nt}}{\sigma^2 nt(1-t)} \middle| T = n \right\} \xrightarrow{d} Exp(1) + Exp(1).$$

The summands are independent.

Yaglom (1948) If $A < 1$ then

$$\lim_{n\to\infty} \mathbf{E}(s^{Z_n} \mid T > n) = F(s), \qquad F(1) = 1.$$

$\Rightarrow$ Simple but useful observation: For any $t \in (0,1)$

$$\lim_{n\to\infty} \mathbf{E}(s^{Z_{nt}} \mid T > n) = \lim_{n\to\infty} \mathbf{E}(s^{Z_{nt}} \mid T = n) = s\frac{F'(s)}{F'(1)}.$$

T.Harris (1948) If $A > 1$ and $\mathbf{E}\xi\log(1+\xi) < \infty$ then

$$\lim_{n\to\infty}\mathbf{P}\left(\frac{Z_n}{A^n} < x \,\middle|\, T > n\right) = \mathbf{P}(W < x)$$

where $\mathbf{P}(0 < W < \infty) = 1$ and

$$\lim_{n\to\infty}\mathbf{E}(s^{Z_n}\,|\,n < T < \infty) = G(s).$$

$\Rightarrow$ One can show that for any $t \in (0,1)$

$$\lim_{n\to\infty}\mathbf{E}(s^{Z_{nt}}\,|\,n<T<\infty) = \lim_{n\to\infty}\mathbf{E}(s^{Z_{nt}}\,|\,T=n) = s\frac{G'(s)}{G'(1)}.$$

2. Inhomogeneous Galton-Watson processes

- Offspring generating function for the individuals of the n-th generation

$$f_n(s) := \mathbf{E} s^{\xi^{(n)}}, \quad f_n'(1) := \mathbf{E}\xi^{(n)}, \quad n = 0, 1, ...$$

- An **inhomogeneous** Galton-Watson process:

$$Z_0 = 1, Z_{n+1} = \sum_{j=0}^{Z_n} \xi_j^{(n)} \quad or \quad \mathbf{E}\left[s^{Z_{n+1}} \middle| Z_n \right] = \left(f_n(s) \right)^{Z_n}$$

- $\xi_j^{(n)} \stackrel{d}{=} \xi^{(n)}$ are i.i.d.

3. Galton-Watson processes in random environment

- Offspring generating functions $f_n(s) := \mathbf{E}s^{\xi^{(n)}}$ in generations $n = 0, 1, ...$ are RANDOM and I.I.D.

$\Rightarrow f_n'(1) := \mathbf{E}\xi^{(n)}$ are I.I.D. RANDOM variables;

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- Offspring generating functions $f_n(s) := \mathbf{E}s^{\xi^{(n)}}$ in generations $n = 0, 1, \ldots$ are RANDOM and I.I.D.

  $\Rightarrow f_n'(1) := \mathbf{E}\xi^{(n)}$ are I.I.D. RANDOM variables;

- $$Z_{n+1} = \sum_{j=1}^{Z_n} \xi_j^{(n)} \quad or \quad \mathbf{E}\left[s^{Z_{n+1}} \,\middle|\, Z_n; f_0, f_1, \ldots\right] = \Big(f_n(s)\Big)^{Z_n}.$$

- $\xi_j^{(n)} \stackrel{d}{=} \xi^{(n)}$ are i.i.d. **given** $f_0, f_1, \ldots$

$$\mathbf{E}_f\Big[Z_n\Big] = e^{n\times\frac{1}{n}\sum_{j=0}^{n-1}\log f_j'(1)} \approx (LLN) \approx e^{n\times\mathbf{E}\log f'(1)}$$

Standard classification: $a := \mathbf{E}\log f'(1)$ - the expected **log** of the offspring size per particle

$$a < 0 \text{ - subcritical,}$$
$$a = 0 \text{ - critical,}$$
$$a > 0 \text{ - supercritical.}$$

INCOMPLETE - For BPRE $\mathbf{E}\log f'(1)$ **may not exist!**

Associated random walk

- $X_i := \log f'_{i-1}(1)$
- $S_0 = 0, \qquad S_n = X_1 + X_2 + \cdots + X_n, \quad n \geq 1.$

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New classification (Afanasyev, Geiger, Kersting, V. (2005)): A BPRE is called

- supercritical if $\lim_{n\to\infty} S_n = +\infty$ with probability 1;
- subcritical if $\lim_{n\to\infty} S_n = -\infty$ with probability 1;
- critical if $\limsup_{n\to\infty} S_n = +\infty$ and $\liminf_{n\to\infty} S_n = -\infty$ (both with probability 1);

Why such a classification?

$$\mathbf{P}_f(Z_k > 0) = \mathbf{P}_f(Z_k \geq 1) \leq \mathbf{E}_f\left[Z_k\right] = e^{\sum_{j=0}^{k-1} \log f_j'(1)} = e^{S_k}.$$

$\Rightarrow$

$$\mathbf{P}_f(Z_n > 0) = \min_{0\leq k\leq n} \mathbf{P}_f(Z_k \geq 1) \leq e^{\min_{0\leq k\leq n} S_k} \to 0$$

with probability 1 in the subcritical and critical case.

Basic assumption:

Doney condition:

$$\mathbf{P}(S_n > 0) \to \rho \in (0,1)$$

as $n \to \infty$. $\Rightarrow$ The critical case.

FROM NOW ON WE CONSIDER THIS CASE ONLY!!

- For any symmetric random walk $\rho = 1/2$.
- If $b^2 = VarX_1 < \infty$ then by the central limit theorem

$$\mathbf{P}(S_n > 0) = \mathbf{P}\left(\frac{S_n}{b\sqrt{n}} > 0\right) \to 1/2, \qquad n \to \infty.$$

Quenched approach: the study the behavior of characteristics of a BPRE for typical realizations of the environment $f_0, f_1, ...$

$$\mathbf{P}_f(Z_n > 0) = \mathbf{P}_f(T > n) = \mathbf{P}(T > n | f_0, f_1, ...)$$

is a random variable! on the space of realizations of the environment $f_0, f_1, ...$

$\mathbf{P}_f(Z_n \in dx)$ **is a random law!**

$\mathbf{P}_f(Z_{n-1} \in dx \,|\, T = n)$ **is a random conditional law!**

Annealed approach: the study the behavior of characteristics of a BPRE performing averaging over possible scenarios $f_0, f_1, ...$ on the space of realizations of the environment:

$$\mathbf{P}(Z_n > 0) = \mathbf{E}\mathbf{P}_f(Z_n > 0)$$

is a number !

Dyakonova, V., (2004)

In the critical case and **for the quenched approach** we have the following convergence in distribution

$$\frac{\mathbf{P}_f(Z_n > 0)}{e^{\min_{0\le k\le n} S_k}} \xrightarrow{d} \zeta$$

where $\zeta \in (0,1]$ with probability 1.

$\Rightarrow$ if $\sigma^2 = VarX < \infty$

$$\mathbf{P}_f(Z_n > 0) \approx \zeta e^{\min_{0\le k\le n} S_k} = \zeta e^{\sigma\sqrt{n}\times\frac{1}{\sigma\sqrt{n}}\min_{0\le k\le n} S_k} \approx \zeta e^{-|g|\sigma\sqrt{n}}$$

where g is a Gaussian random variable:

$$P(g < x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy.$$

Annealed approach (averaging over scenarios):

Afanasyev, Geiger, Kersting, V., (2005) Under Doney's condition

$$\mathbf{P}(Z_n > 0) = \mathbf{E}\mathbf{P}_f(Z_n > 0) \approx \frac{l(n)}{n^{1-\rho}}, \qquad n \to \infty$$

for a slowly varying function $l(n)$. In particular, if $VarX < \infty$ then

$$\mathbf{P}(Z_n > 0) \approx \frac{C}{\sqrt{n}}, \qquad n \to \infty$$

Comparing this with

$$\mathbf{P}_f(Z_n > 0) \approx \zeta e^{-|g|\sqrt{n}}$$

we see that $\mathbf{E}\mathbf{P}_f(Z_n > 0)$ **does not reflect the typical behavior of the survival probability!**

"Bill Gates phenomenon"

99 people - 200$ per month

1 person "Bill Gates" - 100 000 000$ per month

"Typical" - - 200$ per month

"Average"-

$$(200 \times 99 + 100\,000\,000 \times 1)/100 \sim 1\,000\,000\$$$

Basic notation:

- $T = \min\{k : Z_k = 0\}$ - the extinction moment;

FROM NOW ON!!!

$$f_n(s) = \frac{q_n}{1 - p_n s}, \, n = 0, 1, 2, \ldots$$

Associated random walk

- $X_i := \log f'_{i-1}(1)$ - the log of the offspring expectation in generation $i-1$;
- $S_0 = 0, \qquad S_n = X_1 + X_2 + \cdots + X_n, \quad n \geq 1.$
- $\tau(m,n)$- the left-most point on the interval $[m,n]$ at which the minimal value of the random walk $\{S_k,\, m \leq k \leq n\}$ is attained.

Theorem 1. (Vatutin, Dyakonova (2005))

For any $t \in (0,1)$, any $x > 0$ as $n \to \infty$

$$\mathbf{P}_f\left(\frac{Z_{nt}}{e^{S_{nt}-S_{\tau(0,nt)}}} \le x \middle| T>n;\, nt<\tau(0,n)\right) \xrightarrow{p} \int_0^x ye^{-y}dy.$$

Non rigorously: if $T>n$, $nt<\tau(0,n) \Rightarrow$

$$\ln Z_{nt} \asymp S_{nt} - S_{\tau(0,nt)},\ n \to \infty.$$

BOTTLENECKS!!! earlier the time-point $\tau(0,n)$ of the global minimum of S_k within the interval $[0,n]$.

CONDITIONING $\{T > n\}$

Theorem 2. (Vatutin, Dyakonova (2005))

For any $t \in (0,1)$, any $x > 0$ as $n \to \infty$

$$\mathbf{P}_f\left(\frac{Z_{nt}}{e^{S_{nt}-S_{\tau(0,n)}}} \leq x \middle| T>n;\ \tau(0,n) \leq nt\right) \xrightarrow{p} 1 - e^{-x}.$$

Non rigorously: if $T>n$, $\tau(0,n) \leq nt$ $\quad\Rightarrow$

$$\ln Z_{nt} \asymp S_{nt} - S_{\tau(0,n)}.$$

$\Rightarrow$ **NO tragic bottlenecks AFTER the moment** $\tau(0,n)$**.**

Hence, we have to the left of the time-point $\tau(0, n)$ a similarity with the **ordinary critical processes** conditioned on the event $\{T > n\}$

while to the right of $\tau(0, n)$ a similarity with the **ordinary supercritical processes** conditioned on the event $\{T > n\}$!

CONDITIONING $\{T = n\}$

Theorem 1*. (Vatutin, Kyprianou (2008))

For any $t \in (0,1)$, any $x > 0$ as $n \to \infty$

$$\mathbf{P}_f\left(\frac{Z_{nt}}{e^{S_{nt}-S_{\tau(0,nt)}}} \leq x \middle| T=n;\ nt<\tau(0,n)\right) \xrightarrow{p} \int_0^x y e^{-y} dy.$$

Non rigorously: if $T=n$, $nt<\tau(0,n)$ $\Rightarrow$

$$\ln Z_{nt} \asymp S_{nt} - S_{\tau(0,nt)}, \qquad n \to \infty.$$

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Non rigorously: if $T=n$, $nt<\tau(0,n)$ $\Rightarrow$

$$\ln Z_{nt} \asymp S_{nt} - S_{\tau(0,nt)}, \qquad n \to \infty.$$

Thus, if $nt < \tau(0,n)$ then the limiting distributions conditioned on $\{T > n\}$ and $\{T = n\}$ coincide!

Theorem 2*. (Vatutin, Kyprianou (2008))

For any $t \in (0,1)$ as $n \to \infty$

$$\mathbf{P}_f\left(\frac{Z(nt)}{e^{S_{nt}-S_{\tau(nt,n)}}} \le x | T = n; \tau(0,n) \le nt\right) \xrightarrow{p} \int_0^x y e^{-y} dy.$$

Non rigorously: if $T = n$, $\tau(0,n) \le nt$ $\Rightarrow$

$$\ln Z_{nt} \asymp S_{nt} - S_{\tau(nt,n)}.$$

Again, **BOTTLENECKS** but now at the points $\tau(nt,n)$ of **prospective** minima of S_k within the interval $[0,n]$.

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THE PROCESS DIES SLOWLY IN A NATURAL WAY.

AVERAGING?

"Bill Gates" phenomenon once again: quenched against annealed

Vatutin, Wachtel, (2009)

If X_1 is stable with parameters $\alpha \in (0,2), |\beta| < 1 \Rightarrow$

$$\lim_{n\to\infty} \mathbf{P}(Z_{n-1} > e^{xn^{1/\alpha}} | T = n) = \mathbf{P}\left(\Lambda > x\right)$$

where Λ is a proper strictly positive random variable.

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SUDDEN DEATH - A CATASTROPHE

If, however, $VarX_1 < \infty \Rightarrow$ then

$$\lim_{N\to\infty} \lim_{n\to\infty} \mathbf{P}(Z_{n-1} > N | T = n) = 0$$

$\Rightarrow$

NO CATASTROPHE

THANKS